

# Physical optics

Robert Williams  
Wood





## PHYSICAL OPTICS

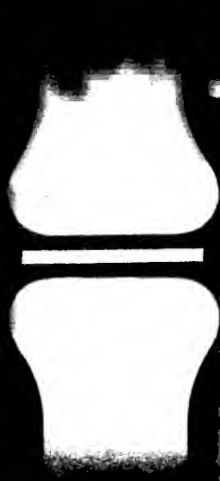


Fig. 1.



Fig. 2.



Fig. 3.

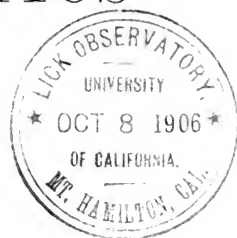


Fig. 4.



Fig. 5.

# PHYSICAL OPTICS



BY

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## PREFACE.

THE present volume was commenced at a time when Preston's *Theory of Light* was practically the only advanced text book on the subject in English, suitable for general class work. This work, while excellent in every respect, could scarcely be said to represent our present knowledge of the subject. Anomalous dispersion and the relation existing between absorption and dispersion was barely mentioned, and of course the recent remarkable discoveries in the field of magneto-optics were not recorded. In the meantime two very excellent books have appeared, the English translation of Drude's *Lehrbuch der Optik*, which cannot be surpassed, and Schuster's *Theory of Optics*, which, while extremely interesting and suggestive, omits all mention of the Laws of Radiation, Fluorescence and Phosphorescence, and the whole subject of the "Optics of moving media."

In the present volume especial stress has been laid on the experimental side, and it is the author's hope that the perhaps too frequent references to experiments with which he has been more or less directly associated, will not be taken as an indication of a lack of perspective.

No pretence at originality in the mathematical treatment is made: the work has been compiled to a great extent from lecture notes, and many plagiarisms doubtless occur. The excellent theoretical treatment, based upon the electro-magnetic theory, given by Drude, has been followed very closely, and it is hoped that this acknowledgment may serve in place of the numerous quotation marks which would otherwise be necessary. Various other standard text-books have been drawn upon freely, especially the very comprehensive work of Verdet.

Too much space has perhaps been given to the theory of dispersion, and the incorporation of the somewhat lengthy development of the dispersion formula by elementary methods, based upon the elastic-solid theory, may appear superfluous. The advantage of this treatment lies in the fact that it does not involve the use of imaginary

quantities, which are always a little troublesome to the student at first; in addition to this it appears to be a little more intelligible, the reciprocal actions between the vibrating atom and the ether being more readily grasped by the mind than the somewhat vaguer conception of displacement currents in the ether and their action upon charged electrons. The electro-magnetic treatment follows.

The illustration of the book has been greatly facilitated through the courtesy of Mr. Wm. Francis, who has furnished blocks of many plates and figures from the *Philosophical Magazine*.

I am under very great obligation to my friend Professor J. S. Ames, who has made many valuable suggestions from time to time, cleared up many doubtful points, and read the manuscript during the process of its preparation.

R. W. WOOD.

BALTIMORE, *May 2nd*, 1905.

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## CHAPTER I.

### THE NATURE OF LIGHT.

**Older Theories.**—The foundations of our present knowledge respecting the nature of light were laid during the latter part of the 17th century, although the modern wave-theory did not take definite form until over a century later. The important discoveries which may be said to mark the beginning of the science of optics, may be summed up in a few words.

In 1666 Sir Isaac Newton effected the prismatic decomposition of white light into its component colors, and proved that no further color change resulted from subsequent refractions. He moreover recombined the spectrum colors, and formed from them white light. This was a great step in advance in one way, for it had been thought previously that color was produced by refraction, manufactured by the prism so to speak, whereas Newton showed that the colors were originally present in the white light, the function of the prism being merely to separate them or sort them out, which it accomplished in virtue of its power of deviating rays of different colors through different angles. Curiously enough this discovery, which we are taking as marking the beginning of our definite knowledge about light, is one which we shall demolish in the last chapter of this book, for our present idea regarding the action of the prism more nearly resembles the idea held previous to Newton's classical experiments: we now believe that the prism actually manufactures the colored light, and what is more to the point, we have a pretty definite idea regarding the manner in which it manufactures it, in which respect we may consider ourselves in advance of Newton's contemporaries.

The importance of Newton's discovery is not to be underestimated on this account, and his conception of the nature of white light will be held to throughout the greater part of this book, for it represents perfectly all of the experimental facts with which we are acquainted, and the treatments of nearly all of the optical phenomena which we are to study are greatly simplified by its use.

Newton elaborated what is known as the corpuscular theory of light, and clung to it tenaciously to the last, the weight of his opinion retarding in no small degree the development of the wave-theory, which was first clearly expressed in 1678. On the corpuscular theory light was regarded as a flight of material particles

emitted by the source, the sensation of sight being produced by their mechanical action upon the retina. The rectilinear propagation followed at once from the second law of motion, whereas the early supporters of the wave-theory were unable to account for it, as every known form of wave motion bent freely around the edges of obstacles.

In 1676 it was demonstrated by Römer, a Danish astronomer, that light required a finite time for its propagation, travelling across space with a velocity which he estimated at 192,000 miles per second. Now the impact of corpuscles moving at such a speed might well be expected to exert a pressure, and attempts were at once made to establish the materiality of light by detecting this pressure, all of which were failures however. At the present time we know that light does exert a pressure, though a very small one, but this pressure can be shown to be the necessary consequence of the impact of waves, so that it is as strong evidence of the truth of the wave-theory, as it would have been of the emission-theory had it been discovered in the days of Newton.

A wave-theory of light was first expressed in definite form by Huygens in 1678, and twelve years later he satisfactorily explained reflection, refraction, and the phenomenon of double refraction in uniaxal crystals, which was discovered by Bartholinus in 1670. Although he discovered the phenomenon of polarization, which would have practically been the death blow to the emission theory, had its nature been understood, he was wholly unable to account for it. We must remember, however, that he had longitudinal waves in mind, *i.e.* waves in which the direction of the vibration was parallel to the direction of propagation, and polarization would be as difficult to account for by such a theory as by the corpuscular one. He was moreover unable to offer any satisfactory explanation of the rectilinear propagation of light, or the formation of shadows, and his theory fell into disrepute.

Strangely enough Newton himself made the discovery which, if handled in the proper manner, would have established almost beyond a doubt the validity of the wave-theory.

He devised a method for studying the dependence of the colors of thin films, first observed by Boyle and Hooke, upon the thickness of the film. These colors, however, he sought to explain on the emission hypothesis. Grimaldi in 1665 was engaged with the study of diffraction, or the bending of light around the edges of obstacles. Admitting sun-light through two small apertures into a darkened room, he observed what he thought to be a darker region at the point where the two diverging beams overlapped. As he was merely looking for evidence of the non-materiality of light, he regarded his experiment as conclusive and pursued the subject no further. The apparent destructive interference of light, which Grimaldi thought that he had observed, was without doubt an effect due to contrast.

True interference was first observed by Dr. Young at the beginning of the 19th century nearly 150 years later, whose justly celebrated experiments established almost beyond question the validity of the wave-theory.

Young, however, at first regarded the waves as longitudinal, which assumption, though erroneous, did not affect the validity of his reasoning concerning the formation of interference fringes and the colors of thin plates. Fresnel commenced his optical studies in 1814 and introduced, for the first time, the conception of transverse waves, a conception which he found necessary for an explanation of polarization. Rectilinear propagation he accounted for by a most ingenious method of dividing the wave front up into zones, often wrongly attributed to Huygens, and showing that the disturbances coming from the collective zones, produced zero illumination within the shadow according to the well known principles of interference. This was a very bold hypothesis, for it necessitated an ether having the properties of an elastic solid, a condition difficult to reconcile with the free and unobstructed motion of the planets through it. This "elastic solid" theory, however, came to be generally accepted, and can still be used to advantage in treating many optical phenomena, for it is more easily intelligible than the modern electro-magnetic theory. Light, on this theory, is regarded as a transverse displacement of a medium called the ether, having properties similar to those of an elastic solid, the displacement being propagated from point to point, according to the well known laws which govern wave motion. There are many objections, one of which is the difficulty regarding the longitudinal disturbance, which always accompanies the transverse one, in the case of a solid. No existence of any such longitudinal disturbance in the ether has ever been found.

Various hypotheses have been made to get around the difficulty. The phenomena of light cannot well be reconciled with the presence of any longitudinal disturbance which is propagated with finite velocity. It has been gotten rid of in the theory by considering the ether as incompressible, which gives to the longitudinal disturbance an infinite velocity. Lord Kelvin made a still bolder suggestion in 1888; he showed that if a "contractile ether" be assumed, the velocity of the longitudinal wave is infinitely small. In a solid if  $\epsilon$  is the elasticity or resistance opposed to a shearing strain,  $k$  the resistance to compression, and  $d$  the density, it can be shown that the velocity of the transverse wave is  $\sqrt{\frac{\epsilon}{d}}$ , while

that of the longitudinal is  $\sqrt{\frac{k + \frac{4}{3}\epsilon}{d}}$ . In an incompressible fluid  $k$  would

be infinitely large, and we should have an infinite velocity for the longitudinal impulse. To give us zero velocity for this disturbance,  $k + \frac{4}{3}\epsilon$  must equal zero, that is  $k$  must be negative, or there must be a negative resistance to compression. This can only be true in a medium in which the conditions are such that it would shrink if left to itself, and it is hard to imagine a stable ether endowed with such properties. Lord Kelvin gets over the difficulty by showing that the instability disappears if we regard the ether as rigidly supported at its boundaries. The condition may be illustrated by considering the case of a mass of small soap bubbles, such as is formed by blowing into a soap solution. The mass in this condition offers a resistance to compression in virtue of the enclosed air. Suppose we could spirit the air away:

**Simple Periodic Motion.**— If a particle moves along a straight line in such a way that its distance  $y$  from a fixed point satisfies the equation

$$y = a \sin (\omega t - a),$$

in which  $t$  is the time and  $a$  and  $\omega$  are constants, its motion may be defined as simple harmonic, or as Schuster prefers to call it, simple periodic motion. A particle which is acted upon by a force which varies directly with its distance from a fixed point will, if displaced and released, execute a motion represented by the above equation, if no other forces, such as friction for example, come into play. Forces of this nature are assumed to be called into play by the displacements, mechanical or electrical, which constitute light, and we will accordingly begin by establishing the above equation and interpreting its meaning.

Let the force corresponding to displacement  $y$  be

$$\Psi = -py,$$

in which  $p$  is a constant, viz. the force corresponding to unit displacement: the minus sign is given since the direction of the force is opposed to that of the displacement. If  $m$  is the mass of the particle we have for the acceleration,

$$-\frac{py}{m} = -k^2y \left( \text{writing } k^2 = \frac{p}{m} \right); \quad \therefore \frac{dv}{dt} = -k^2y.$$

Now  $v = \frac{dy}{dt}$  and we therefore have  $\frac{dv}{dt} = \frac{d^2y}{dt^2} = k^2y$ , the integral of which is  $y = a \sin (kt - a)$ .

This can also be shown in the following way.

The work done on a mass is measured by its kinetic energy and is represented by the product of the force and the distance through which it acts. Since the force varies in the present case, we have for the work done on the particle, displaced a distance  $a$ , and moved back by the force to position  $y$ ,

$$- \int_a^y py dy,$$

the minus sign being given because the path traversed is in the negative direction of  $y$ .

Equating this to the expression for kinetic energy gives us

$$\frac{1}{2}mv^2 = - \int_a^y py dy = \frac{1}{2}p \int_a^y 2y dy = \frac{p}{2}(a^2 - y^2);$$

$$\therefore v^2 = \frac{p}{m}(a^2 - y^2), \quad v = \pm k\sqrt{a^2 - y^2};$$

$$\therefore \frac{dy}{dt} = k\sqrt{a^2 - y^2}, \quad \frac{dy}{\sqrt{a^2 - y^2}} = k dt,$$

whence

$$kt = \int_0^y \frac{dy}{a\sqrt{1 - \frac{y^2}{a^2}}} = \sin^{-1} \frac{y}{a},$$

or

$$y = a \sin kt. \quad \dots\dots\dots(1)$$

Suppose the time to increase from  $t=0$  to  $kt=\frac{\pi}{2}$ , then  $y=a$ , the maximum value which it can attain, or in other words the amplitude of the vibration.

$$\text{For } kt = \frac{\pi}{2}, \quad y = a, \quad \text{For } kt = \pi, \quad y = 0,$$

$$\text{" " } = \frac{3\pi}{2}, \quad y = -a, \quad \text{" " } = 2\pi, \quad y = 0,$$

the particle having performed one complete vibration.

If the time occupied is called  $T$ , we have  $kT = 2\pi$ , or

$$k = \frac{2\pi}{T}, \quad T = \frac{2\pi}{k} = 2\pi \sqrt{\frac{1}{\frac{p}{m}}} = 2\pi \sqrt{\frac{m}{p}}.$$

Substituting for  $k$  in equation (1) gives us

$$y = a \sin \frac{2\pi}{T} t.$$

**Wave-Motion.**—In the above discussion we have investigated the motion of a single particle. We will now consider what happens when the particle is bound to other particles by forces which tend to keep the particles at a fixed distance, such, for example, as an attractive force and a repulsive force, the latter increasing more rapidly than the former as the particles approach. Such a medium would be capable of transmitting transverse waves, and we can imagine a sort of atomic ether consisting of extremely minute particles bound together by forces as above specified. This conception need not be taken as expressing our views regarding the constitution of the ether by any means, but as we shall make use of a medium of this nature in the elementary deduction of the dispersion formula, we may as well take it for our type of medium, in the preliminary study of wave-motion.

Suppose our particles to be arranged in a row (Fig. 1) and held at a fixed distance apart, say by a spiral spring in which they are imbedded. If one of them is

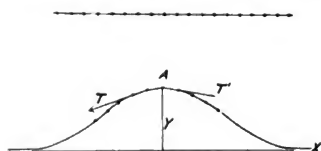


FIG. 1.

displaced, it is drawn back by the vertical component of the forces  $T$  and  $T'$  due to the increased tension of the spring. Suppose the particles displaced as shown in Fig. 1, the radius of curvature at  $A$  being  $R$ . The force acting on  $A$  along  $y$  will be  $2T \sin \alpha$ , where  $\alpha$  is the angle

subtended at the centre of curvature by the element  $ds$  of the medium, i.e. the distance between the two particles adjoining  $A$ ;  $2T$  represents the sum of the tangential forces  $T$  and  $T'$ , which are assumed equal for small values of  $\alpha$ .

$$\text{We have then} \quad F = 2T \sin \alpha = T \cdot 2\alpha = \frac{T}{R} ds \quad \left( \text{since } 2\alpha = \frac{ds}{R} \right).$$

Now the curvature  $\frac{1}{R} = -\frac{d^2y}{dx^2}$  for small displacements, and if  $\rho$  is the density per unit length,  $\rho ds$  is the mass, and we have

$$\rho ds \frac{d^2y}{dt^2} = -F = T \frac{d^2y}{dx^2} ds, \quad \text{or} \quad \frac{d^2y}{dt^2} = \frac{T}{\rho} \frac{d^2y}{dx^2}.$$

This equation has for its solution

$$y = f\left(x - \sqrt{\frac{T}{\rho}} t\right) + f'\left(x + \sqrt{\frac{T}{\rho}} t\right),$$

or

$$y = f\left(t - \frac{x}{\sqrt{\frac{T}{\rho}}}\right) + f'\left(t + \frac{x}{\sqrt{\frac{T}{\rho}}}\right),$$

in which  $f$  and  $f'$  are arbitrary functions.

In the case which we are considering, if the row of particles is displaced as figured, and released, a wave motion will spread out in both directions, with a velocity equal to  $\sqrt{\frac{T}{\rho}} = V$ .

If now the particle  $A$  vibrates in simple periodic motion  $y = a \sin 2\pi \frac{t}{T}$ , we have one of our wave disturbances represented by

$$y = a \sin \frac{2\pi}{T} \left(t - \frac{x}{V}\right), \quad \text{or} \quad y = a \sin 2\pi \left(\frac{t}{T} - \frac{x}{\lambda}\right) \quad \text{since} \quad VT = \lambda.$$

This equation represents as well a plane-wave travelling along the  $x$ -axis; its amplitude is  $a$ , its periodic time  $T$  and its wave-length  $\lambda$ .

We may get the form of the wave by giving to  $t$  any fixed value, for example  $t = 0$ , when our equation becomes

$$y = a \sin 2\pi \frac{x}{\lambda}.$$

We can plot the curve in the following way.

We will plot the ordinates ( $y$ ) for values of  $x$  equal to multiples of  $\frac{\lambda}{12}$  (Fig. 2). Divide the circumference of a circle into 12 equal parts,

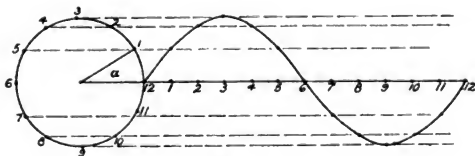


FIG. 2.

and call the radius unity. For  $x = 1 = \frac{\lambda}{12}$ , the  $\cos$  of  $\alpha$  is the ordinate of point 1 on the circumference of the circle. The same holds for the other points, therefore we have only to draw lines parallel to  $x$

through the points on the circle and mark their intersection with ordinates erected at 1, 2, 3, etc. The points thus determined lie on the wave.

**Absence of Back-Wave.**—If a point in a medium is made to vibrate in simple periodic motion, it sends out waves in both the positive and negative direction. Now when a wave meets a point in a medium, the point is made to execute periodic motion, and the wave beyond the point can be regarded as due to its motion. In this case, however, the moving point only sends out a disturbance in one direction, though its motion is identical with that of the point sending out waves in both directions. As we shall in the next chapter make use of this conception of a point thrown into vibration by a wave as a source of other waves, it is of some importance to distinguish between a secondary source of this nature and an actual source of light.

Let the curved line in Fig. 3 represent a wave travelling towards the right. We know that this wave will be propagated with its type unchanged, and that the medium behind it will come to rest the moment the wave has passed. If, however, we distort the medium into

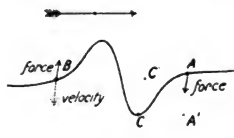


FIG. 3.

the shape figured, and then release it, we shall have a wave travelling in both directions. The difference between the two cases will become at once apparent if we consider the velocities as well as the displacements of the particles. Consider the first case, that of the moving wave: the particle at *A* is acted on by a force drawing it downward, and being

at rest initially it moves in consequence. The particle at *B* is acted upon in the opposite direction by an equal force. It, however, is *not at rest*, for it is moving in a downward direction with a velocity represented by the dotted arrow, for the wave has just passed by it, and it is returning to its position of equilibrium: this velocity just compensates the force due to the distortion of the medium and the particle comes to rest. In the second case both *A* and *B* are at rest initially, and both move the moment the restraint is removed, and we have a wave moving in both directions. We can in the same way see how the vibration of *A* by the passage of the wave through it fails to give a back-wave. It moves let us say to *A'*, which it will do in time  $\frac{T}{4}$ . In the meantime the point *C* has returned to *C'*, and its

velocity just compensates the force due to the displacement of *A*, which in a medium initially at rest would result in a back-wave.

**Wave-Front.**—We may define the wave-front as the continuous locus of the points of the medium which are about to be disturbed. Thus defined the wave-front marks the limit which the disturbance has reached at the instant considered. A more general definition, however, and one which we shall find more useful is the following. *The wave-front is the continuous locus of points which are in the same phase of vibration, or a surface of equal phase.* If this surface is plane, we speak of the waves as plane-waves, and since in isotropic media the rays are perpendicular to the wave-front, the rays are in this

case parallel. The waves coming from sources of light situated at infinity (*e.g.* the stars) are plane.

If the source is at a finite distance the wave-fronts are spherical, if the velocity of propagation is independent of the direction, as is the case in isotropic media. By means of mirrors or lenses it is possible to transform a spherical wave-front into a plane one, but we possess no means of starting a plane-wave directly. We can perhaps get a better case of what this would involve in the following way.

Consider a vibrating particle attached to an elastic string: waves will run along the string and the wave-front will be a point (Fig. 4*a*). Attach a number of strings to a rod vibrating in a direction parallel to its length (Fig. 4*b*), and the wave-front will be a straight line if we regard the strings as forming a continuous medium (Fig. 4*b*).

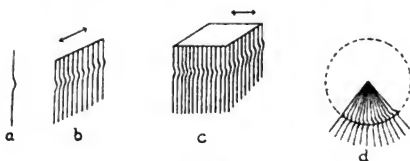


FIG. 4.

There is no such thing in nature as a linear wave of light, for the reason that such waves can only occur in a medium of two dimensions. The conception of such a wave is often made use of in elementary treatments of diffraction, as the problems are much simplified by restricting the disturbance to two dimensions.

If now our strings are attached to a vibrating plane, the continuous locus of equal phase is obviously a plane, parallel to the moving plane, since the waves all start at the same instant, and travel with equal velocities. To realize this condition in optics it would be necessary to arrange a plane source of light, over the surface of which the vibration was uniform, *i.e.* the phases of all the vibrating particles would have to be the same, a condition which obviously cannot be realized. By attaching the strings to a vibrating point, and arranging them so that they stretch out in all directions, we represent roughly the conditions under which we obtain a spherical wave. It should be observed, however, that in the case of a to-and-fro motion of the point, there are two directions in which transverse waves will not be given out, these directions coinciding with the direction of motion of the point. We have this circumstance occurring in certain optical phenomena, as we shall see later on (certain facts connected with the Zeeman effect for example).

**Frequency and Wave-Length.**—The length of the light wave depends as we have seen upon two factors, the velocity and the frequency or time of vibration. Since the velocity in refracting media is usually less than the velocity in ether, the wave-length is reduced when the disturbance enters such a medium, for the frequency remains the same. The wave-length and frequency obviously depend upon the nature

of the source. Flames colored by metallic salts may emit light of definite frequencies, such as the sodium flame, the light of which consists chiefly of two yellow radiations, commonly designated as the *D* lines.

Light in which we have but a single wave-length is said to be monochromatic. It must be remembered, however, that strictly monochromatic light involves an infinite train of waves, such as would emanate from a particle the vibrations of which were subject to no sudden or gradual changes of phase. Absolutely homogeneous or monochromatic light is something that has no actual existence, though we are accustomed to speak of light which the spectroscope shows as a single narrow line, as monochromatic.

The color depends upon the wave-length, but the color cannot always be taken as an indication of wave-length, as certain colors can be imitated by the simultaneous action upon the retina of two trains of waves, either of which acting alone would give rise to a totally different color from that perceived when both act together.

For example, a yellow scarcely distinguishable from the yellow of the sodium flame, can be produced by a mixture of red and green light in the proper proportions. A screen can be easily prepared which transmits red and green only and in about the right proportions to produce the sensation of "subjective yellow" as it is called.

Canada balsam, boiled down until it will solidify on cooling, is stained with "brilliant-green" and naphthalene yellow, in the same proportions used for making dichromatism prisms (see page 351) and a small quantity pressed out between two warm glass plates until the color of the transmitted light is yellow. Examination with a small spectroscope reveals the fact that in reality no yellow light is transmitted, only red and green. We have then the important distinction that while wave-length determines color, color does not necessarily determine wave-length.

Lord Rayleigh recommends a mixture of an alkaline solution of litmus with chromate of potash. If a window, backed by well lighted clouds, is viewed through such a solution and a prism it presents a most splendid appearance, for the red and green images are widely separated, the region where they overlap being colored with the compound yellow. A screen capable of transmitting only the yellow region is difficult to prepare. A mixture of bichromate and permanganate of potash answers fairly well, and can be made to match the color of the first screen. A sodium flame is invisible through the first and easily visible through the second. Both together are practically opaque even with very intense white light. (See Appendix A.)

The different radiations present in a source may be separated by a prism or diffraction grating, as we shall see, and we obtain in this way what is known as a spectrum of the source: not all of the radiations in the spectrum affect the eye, for, as we know by experiment, there are regions beyond the red and violet which we cannot see. The longer waves in the infra-red spectrum can be recognized by their heating power, or by their action on phosphorescent substances; the ultra-violet or short waves can be detected by photography or by their action in causing fluorescence.

The length of the light wave can be measured with great precision by methods which will be described later on. The comparative lengths of the waves in the different parts of the spectrum are shown in Fig. 5. The shortest waves, discovered by Schumann and subsequently accurately measured by Lyman, are powerfully absorbed by air, and have to be investigated in vacuo. The longest waves

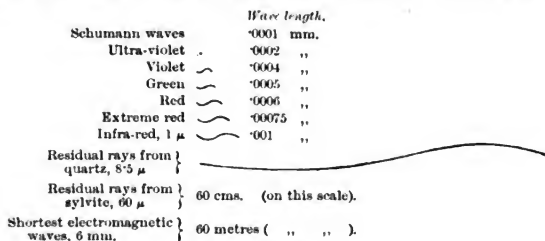


FIG. 5.

thus far detected in the radiations from a source of light were found by Rubens and Aschkinass, by a method which will be discussed later. For the sake of comparison, the shortest electromagnetic wave ever obtained (6 mms.) would, on the above scale, have a length of 60 metres.

**Sources of Light for Experimental Purposes.**—It will, perhaps, be well in the introductory chapter to describe briefly a number of sources of light, which will be found useful in experimental work pertaining to the subject of Physical Optics. As a source of white light, the sun is to be preferred when great intensity is required. Next to this comes the electric arc, the type most suitable for experimental work being a lamp in which the positive carbon is horizontal. If great intensity is not necessary the Nernst filament will be found very serviceable. It ranks next to the arc in intrinsic intensity, requires no attention, and has the added advantage of narrowness. It may thus be used in many cases as a substitute for an illuminated slit. If an electric current is not available, a Welsbach lamp, surrounded by a sheet iron chimney furnished with a small vertical slit will be found an excellent substitute.

As sources of monochromatic light we possess various colored flames and vacuum-tubes, from the spectrum of which we can pick out a monochromatic radiation by screening off the wave-lengths which are not desired. A simple form of apparatus for accomplishing this is described in Mann's *Manual of Optics*. It is easily constructed, not expensive, and can be made without the services of a skilled mechanic (Fig. 6). Light from a slit, *S* made parallel by a lens *L*, traverses a glass prism, after which it is reflected back through the prism and collimating lens, the convergent beam

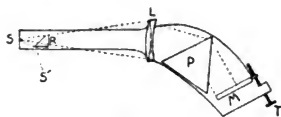


FIG. 6.

being then deviated to one side by a small right-angled prism, the focused spectrum falling upon a screen provided with a vertical slit *S*. By turning the mirror *M* by means of adjusting screws, any desired portion of the spectrum may be passed out through the side slit. It is possible with this instrument to obtain fairly monochromatic light from a source giving a continuous spectrum, or to pick out the highly homogeneous radiations which are emitted by metallic vapors, brought to a state of luminescence by the electrical discharge in vacuum-tubes. The sodium flame is the most generally used source of monochromatic light. Its radiations, however, consist of two wave-lengths, consequently its spectrum consists of two lines in the yellow very close together. To separate one of these from the other is a matter of considerable difficulty, and for most purposes the complete radiation will be found equally satisfactory. The most satisfactory flame can be obtained by winding a piece of asbestos paper around the top of a Bunsen burner (fastening it with wire) and saturating it with strong brine.

Monochromatic red light can be obtained by saturating an asbestos cylinder with a solution of chloride of lithium, and a satisfactory green by means of a small fragment of metallic thallium, fused to a loop of platinum wire. The bead should be mounted so as to barely touch the outer edge of the flame, otherwise it will rapidly evaporate. For long continued work, however, the most satisfactory light is the mercury arc, from the radiation of which we can pick out by means of color screens, or the simple spectroscope described above, any one of the numerous bright lines. The following screens have been recommended for use with this form of lamp. The solutions are made with water, and contained in cells made by cementing glass plates to annular strips cut from heavy brass tubing.

Methyl violet 4 R. (Berlin aniline fabrik) very dilute, and nitrosodimethyl-aniline transmits wave-length 365. Methyl violet + chinin sulphate (separate solutions), the violet solution is made strong enough to blot out wave-length 4359. This screen transmits 4047 and 4078, also faintly 3984.

Cobalt glass + Aesculin solution transmits 4359.

Guinea green B extra (Berlin) + Chinin Sulphate transmits 4916.

Neptune green (Bayer, Elberfeld) + Chrysoidine. Dilute the Chrysoidine sufficiently to just transmit 5790 and 5461, then add Neptune green until the yellow lines disappear.

Chrysoidine + Eosine transmits 5790. The chrysoidine should be dilute and the eosine added until the green line disappears.

Mercury arc lamps are now made of fused quartz by Heraeus of Hanau, Germany. They are most satisfactory in every respect, though rather expensive. The ultra-violet radiations are so intense that the air becomes immediately charged with ozone, and glasses should always be worn, as an exposure of only a minute or two of the naked eye to the light results in a very serious and painful inflammation. The glasses need not be dark, as the harmful rays are absorbed by ordinary transparent glass. Cadmium lamps can also be obtained of quartz. These, however, require a certain amount of preliminary heating.

An instrument is made by R. Fuess, of Steglitz, near Berlin, which is a combined spectroscope and monochromatic illuminator. The author has found this instrument most useful. It is very easily calibrated, and will furnish a beam of approximately monochromatic light from sun or arc light, the width of the band being not much wider than the distance between the sodium lines. The transmitting slit can be removed in an instant, and the insertion of the eye-piece in its place transforms the instrument into a very good spectroscope.

**Velocity of Light.**—The first determination of the velocity of light was made by a Danish astronomer Römer in 1676. From observations made on the eclipses of Jupiter's satellites he showed that the inequalities noted in their times could be explained by the finite velocity of propagation of light. Since the time of rotation of the satellites around the planet is constant for each satellite, they will enter the shadow of the planet at regular intervals, and the times of the eclipses can be predicted with the greatest accuracy. Römer found, however, that the intervals between successive eclipses of a given satellite varied gradually if the observations extended over a year. The eclipses were found to occur earlier or later than the calculated time, according as the earth and Jupiter were on the same, or opposite sides, of the sun. The discrepancy was obviously due to the time taken by light to travel across the earth's orbit. Calculation showed that the velocity of light was about 192,000 miles per second.

The second determination was made in 1728 by Bradley, who discovered the phenomenon known as the aberration of light. He observed that the apparent position of the stars shifted slightly from time to time, and finally came to the conclusion that this small apparent motion could be explained by taking into account the earth's motion in its orbit, together with the fact that light is propagated with a finite velocity. The phenomenon of aberration will be more fully discussed in the chapter on the relative motion of matter and ether.

**Fizeau's Method.**—Galileo had made an unsuccessful attempt to determine the velocity of light, by placing two observers at a great distance apart, each furnished with a lamp. One observer uncovered his lamp and the second observer watched for the flash and removed the screen from his lamp at the moment it appeared. The first observer was to determine the velocity by noting the time elapsing between the uncovering of his own lamp and the appearance of the distant light.

This method failed obviously, owing to the enormous velocity of light. In 1849 Fizeau made an experimental determination of the velocity



FIG. 7.

of light by means of a revolving disc furnished with a toothed rim. The method is essentially as follows: A beam of light was introduced into the tube of a telescope by means of a collimator fitted in to its side, and was focused by means of a reflecting plate upon the rim of

the toothed wheel (see Fig. 7). This point was at the principal focus of the object glass of the telescope; consequently the light, after passing between the teeth of the wheel, was made parallel by the objective.

After traversing a distance of three or four miles, it fell upon a second lens, which brought it to a focus upon a concave spherical mirror, the centre of curvature of which coincided with the centre of the lens. The light was thus returned as a parallel beam over the same path, and entered the eye-piece at *E*, passing through the reflecting plate. If the toothed wheel is rotated the beam of light will be made intermittent, and if the speed be great enough the light which passes through the space between the teeth will, upon its return, be cut off by the adjacent tooth, which in the meantime has advanced into the position previously occupied by the space. On looking into the telescope the observer sees at first a bright star, which diminishes in intensity as the speed of rotation is increased, finally disappearing entirely. Further increase in speed causes the reappearance of the star, the light passing through a given space, falling upon the next adjacent space upon its return. Fizeau experienced great difficulty in determining accurately the speed at the moment when the eclipses occurred. The image of the distant star was never bright, and the light reflected from the teeth of the wheel caused a general illumination of the whole field. To obviate this difficulty Young and Forbes, in repeating the experiment, bevelled the teeth so that the light reflected from them fell upon the blackened sides of the telescope. The teeth were also blackened so as to diminish their reflecting power as much as possible. In 1874 Cornu repeated the experiment with certain modifications. To avoid the difficulty of determining the exact moment at which the star was eclipsed, he made use of an electrical chronograph, arranged so as to record every hundred revolutions. Seconds were marked by a clock, and tenths of a second by means of a vibrating spring. By means of a key the observer could record any instant at which he wished to know the velocity. The speed and its rate of change could be determined at every instant from the record of the chronograph. Instead of attempting to determine the moment of complete extinction, Cornu compared the brilliancy of the image with a light of fixed intensity. On increasing the speed the intensity of the image sank, and the speed of the wheel was recorded at the moment at which it was equal to the intensity of the standard light. After extinction the star reappeared and the speed was recorded at the moment when it regained its former brightness. The speed corresponding to complete extinction was the mean of these two. Cornu's final result for the velocity was 300,330 Kms. per sec. in air, or 300,400 in vacuo.

**Foucault's Method.**—Wheatstone had suggested that a revolving mirror might be employed in the determination of the velocity of light, and his suggestion was taken up by Arago, but it remained for Foucault to carry out the experiment in a form capable of giving accurate results. The arrangement of his apparatus is shown in Fig. 8. Sunlight after transmission through an aperture at *S* and an achromatic lens *I* falls upon a mirror *R*, which can be rotated at high speed. A concave mirror *M* fixed at a distance of several

metres returns the light to the revolving mirror. If the mirror  $R$  is at rest the light returned by it after reflection from the inclined plane mirror comes to a focus at  $a$ .

The axis of the mirror  $R$  is at the centre of curvature of the mirror  $M$ , consequently the cone of rays, which converges upon  $M$ , is returned over the same path, and the rotation of  $R$  will not affect the position of the image at  $a$ . This, however, is only true if the mirror is in the same position when the rays meet it a second time, as will be readily seen by considering the passage of a ray from  $S$  to  $a$ . If the mirror turns through an appreciable angle while the light is traversing the distance  $2RM$  the image will be shifted to a point  $a'$ .

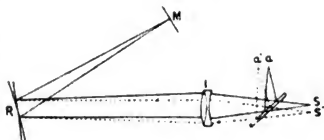


FIG. 8.

The revolving mirror was driven by an air turbine, the speed being determined by a stroboscopic method. The displacement of the image amounted to only 0.7 mm., which gave for the velocity of light 298,000,000 metres per second.

**Michelson's Experiments.**—Foucault's method was improved by Michelson, who placed the lens between the two mirrors (Fig. 9). The lens was 8 inches in diameter and had a focal length of 150 feet. The revolving mirror was placed 15 feet inside the principal focus, and the mirror  $M$  at a distance of 2000 feet. Deflections of the image amounting to 133 mms. were obtained, which made it possible to dispense with the oblique reflecting plate, and observe the image directly, with an eye-piece placed to one side of the slit. The speed of the mirror was determined by means of a tuning fork, one of the prongs of which carried a light mirror, which reflected the light from the revolving mirror into the eye-piece. When the fork vibrated the spot of light was drawn out into a band, which broke up into a number of moving images as soon as the mirror was set in rotation. A single stationary image was obtained only when the mirror made as many turns per second as the frequency of the fork; this condition was easily secured by regulating the air pressure at the turbine. The mean result for the velocity of light (reduced to the velocity in vacuo) was  $299,910 \pm 50$  kilometres per second. No indication of the phenomenon alleged to have been observed by Young and Forbes was observed. A difference of velocity between the red and blue radiations, as large as their experiment indicated, would have resulted in a drawing out of the image into a spectrum 10 mms. in length.

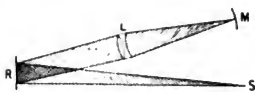


FIG. 9.

Experiments were also made on the velocity of light in bisulphide of carbon, a tube three metres in length being interposed between the mirrors. The ratio of the velocity in this fluid to the velocity in air was found to be 1.758, while the ratio indicated by the refractive index

is 1.64. This discrepancy will be explained in a subsequent chapter. Professor Michelson also experimented with lights of different colors, and found that red light travelled 1 or 2 per cent. faster than green light in the carbon bisulphide.

**Newcomb's Experiments.**—A series of experiments were made by Newcomb at Washington in 1880-82, with an apparatus of slightly different type. Sunlight entered the slit at *S* (Fig. 10), and, after reflection from a mirror at the elbow joint, passed through the telescope lens and fell upon the revolving mirror *m*, from which it was reflected along the line *z* to the distant mirror. The object glass of the receiving telescope was immediately below that of the sending telescope, the light entering it being received from the lower part of

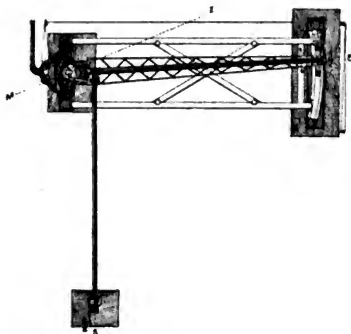


FIG. 10.



FIG. 11.

the revolving mirror. This consisted of a rectangular prism of steel (Fig. 11), the surfaces of which were nickel plated, driven by means of an air blast. The speed was regulated by means of a slight counter blast directed against the lower fan wheel. By employing two lenses in the manner indicated the diffused light from the strongly illuminated upper portion of the mirror did not enter the receiving telescope. The mirror could be driven in either direction, by interchanging the direct and counter blasts; the displacement measured could thus be doubled.

The quantity measured directly was the angular deviation of the return image, and not its linear displacement; this was accomplished by swinging the observing telescope, the eye-piece end moving along a graduated arc, the divisions of which were read by means of a pair of microscopes.

Newcomb's final result was, for the velocity in vacuo,

$$v = 299,860 \pm 30 \text{ kilometers.}$$

**Group Velocity.**—An important distinction exists between the velocity of a group of waves and the velocity of a single wave. We can get a very good idea of what is meant by group velocity

by throwing a stone into a quiet pond, and watching the circular waves which spread out. If the attention be fixed on a single wave crest at the center of the group, it will be seen presently to lead the group, the waves ahead of it appearing to die out, and in a few seconds its amplitude will become so small that the eye can no longer be kept on it. There are just as many waves in the group, however, as there were before, and a little further observation will reveal the fact that, as the waves in front die out, new ones appear in the rear. The group is obviously moving forward with a velocity less than that of the individual waves.

The explanation of the phenomenon was first given by Stokes, who regarded the group as formed by the superposition of two infinite trains of waves, of slightly different wave-length, which advanced in the same direction but with different velocities.

Lord Rayleigh was the first to draw attention to the bearing of group velocity upon optical problems. In his article on "The Velocity of Light" (*Nature*, 1881), he called attention to the fact that, in all experiments made for the purpose of determining the velocity of light, it is the group-velocity, and not the wave-velocity, which is actually measured. What is actually determined is the velocity with which some peculiarity impressed upon the wave train moves forward. Since it is impossible in the case of light to pick out and watch a single wave, the best that we can do is to measure the speed with which a block, cut out of a wave-train, advances. If the medium is free from dispersion, *i.e.* if waves of all possible lengths are propagated with the same velocity, the group-velocity and wave-velocity will be the same, the group being propagated without alteration.

This will be made clear by reference to Fig. 12. In the lower diagram we have two superposed trains of waves, moving in the direction of the arrow. The resultant disturbance is indicated in the upper diagram. The longer waves (dotted line) are out of step with the shorter (solid line) at *A* and *C*, and the resultant is zero at these points. At *B*, where there is agreement of phase, the resultant amplitude is double that of the single waves. If now the velocities of the two sets of waves are equal, it is evident that the group shown in the upper diagram will move forward without alteration with the wave-velocity. If, however, the shorter waves move at the higher speed it is evident that they will presently get out of step at *B*, and into step at *C*, which now becomes the center of the group. The group thus advances with a velocity greater than that of the individual waves. If the reverse is the case the amplitude to the left of *B* increases as the group advances, the amplitude to the right of *B* diminishing, *A* becoming eventually the center of the group. In this case the group-velocity is less than the wave-velocity.

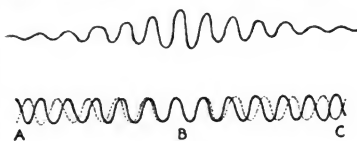


FIG. 12.

P.O.

B

We will now derive an expression for the group-velocity. Let the longer wave  $\lambda'$  (dotted line in Fig. 13) move with a velocity  $V' > V$ , the velocity of the shorter wave  $\lambda$ . Let  $T$  be the time required for the point marked  $V'$  to overtake the point marked  $V$ . When this event has occurred the center of the group, defined as the point of maximum resultant amplitude, and originally at  $B$ , will have moved back a distance of one wave-length. Now the crest  $V'$  is approaching the crest  $V$  with a velocity  $V' - V$ , therefore  $(V' - V)T = \text{distance } V'V = \lambda' - \lambda$ . If we write  $dV = V' - V$  and  $d\lambda = \lambda' - \lambda$ , we have

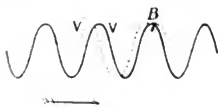


FIG. 13.

$$T = \frac{1}{\frac{dV}{d\lambda}}.$$

During the time  $T$  the  $\lambda$ -wave train moves forward a distance  $VT$ , therefore the center of the group as defined above has moved a distance

$$X = VT - \lambda = \left( V - \lambda \frac{dV}{d\lambda} \right) T,$$

and the group-velocity is given by dividing this quantity by  $T$ . Calling  $U$  the group-velocity, we have

$$U = V - \lambda \frac{dV}{d\lambda}.$$

If the medium is free from dispersion  $\frac{dV}{d\lambda} = 0$ , and  $U = V$ .

We shall have occasion to use the formula for group-velocity in studying the action of a prism on white light, which will be taken up in the next to the last chapter.

The difference between  $U$  and  $V$  only comes into play in determinations of the velocity of light in strongly dispersive media, the correction to be applied amounting to 7.5 % in the case of bisulphide of carbon.

Michelson, employing the revolving mirror method, which has been shown by Rayleigh to yield  $U$  and not  $V$ , found the velocity of light in air 1.77 times greater than in this fluid, while determinations made by measuring the refractive indices gave the value 1.64. If we increase 1.64 by 7.5 % we obtain the value 1.76, which is in close agreement with the value observed by Michelson.

It is worthy of remark that determinations of the velocity of light by observations made on the aberration of light from the stars, give us  $V$ . Römer's method, however, yields  $U$ , and the close agreement between the values obtained by these two astronomical methods indicates that light is propagated across interstellar space without dispersion.

Independent evidence that waves of all lengths travel with the same velocity in the free ether, is furnished by the variable star Algol, which shows no color sequence when increasing in brightness, as would be the case if waves of different lengths travelled with different velocities.

**The Doppler-Fizeau Principle.**—Doppler, in 1842, called attention to the change in the pitch of a sound, which resulted when the source was moving towards or away from the observer, and applied the principle to luminous disturbances radiated from bodies in motion, explaining the colors exhibited by certain stars as due to their proper motion. The acoustical phenomenon is most frequently heard when travelling in a railroad train. If a whistling locomotive is passed, the drop in the pitch is very noticeable, especially if the locomotive is moving rapidly in the opposite direction. Doppler's application of the principle to stellar phenomena was unsound, and Fizeau appears to have been the first to show that the effect would manifest itself as a slight shift in the position of the bright or dark lines in the spectrum. If the source of light is moving towards the observer, the frequency of the disturbance as it passes the observer is increased, and the wave-length diminished: the spectrum lines are therefore shifted towards the violet: the reverse is true when the source is moving away in the line of sight. By photographing the spectrum of a star alongside of a comparison spectrum, it is possible to determine, not only whether the star is moving towards or away from us, but also the velocity with which it approaches or recedes. The principle has had wide applications in astro-physical research, and the rapidly accumulating data regarding stellar velocities, will, at some future date, in all probability furnish the key to the solution of that greatest problem of astronomy, the nature of the motion of the multitude of suns which make up the universe.

Double stars have been discovered by the Doppler effect, the components of which no telescope will show separated, and their time of revolution about their common center of gravity determined. Such stars are called spectroscopic binaries. The first was discovered at the Harvard Observatory by Pickering. Observations of a number of spectra of this star, taken at different times, showed that the lines became double at stated intervals, an effect which could only be accounted for by assuming the source of light to consist of two bodies which alternately approached and receded, in other words two bodies revolving around their common center of gravity.

Keeler applied the principle to the study of the rings of Saturn, and showed that each portion of the ring was rotating at the speed which an isolated satellite would have at the same distance from the planet.

The effect was first obtained in the laboratory by Belopolski in 1901 (*Astro. Phys. J.*, 13, pg. 15-24), who reflected a beam of light from a system of moving mirrors, subsequently analyzing the light with a spectroscope. The displacement of the spectrum lines was of the calculated order of magnitude, which was, however, an exceedingly small quantity. The minimum velocity capable of modifying the wave-length to such a degree that the spectroscope will note the change is a kilometer or perhaps half a kilometer a second. The change of wave-length resulting from reflection from a moving mirror is double the change resulting from the motion of the source with the same velocity. Belopolski made use of multiple reflections from two systems of mirrors, mounted on the rims of a pair of opposed wheels, which could be revolved at high speed. In this way he was able to

obtain a shift of the spectrum lines which, though small, was easily measurable.

The motions of the molecules of a luminous gas modifies slightly the wave-length of the emitted light. Since the molecules are moving in all possible directions, with all sorts of velocities, the result is that the spectrum lines appear slightly broadened, the broadening increasing with the temperature. The subject has been fully treated by Lord Rayleigh (*Phil. Mag.* (5), 27, page 298, 1889).

The change in the period  $T$ , of a source of light moving with a velocity  $v$ , is given by the equation

$$T' = T \left( 1 \pm \frac{v}{c} \right),$$

in which  $T$  is the actual period of the vibration,  $T'$  the period of the radiation, and  $c$  the velocity of light. It is to be carefully observed that when the source is in motion, the frequency of the vibration in the source differs from the frequency with which the waves pass by the observer, the former being unaffected by the motion.

## CHAPTER II.

### RECTILINEAR PROPAGATION OF LIGHT.

**Huygens's Principle.**—One of the objections which was first urged against the wave theory of light was its failure to account for the rectilinear propagation of luminous disturbances, and the formation of shadows. Waves of sound and water waves were observed to bend around the corners of obstacles, and it was perhaps naturally argued that if light consisted of a wave-motion, it should behave in a similar manner. The objection was partially answered by Huygens, though it remained for Fresnel to give the complete explanation.

Huygens's conception of the manner in which wave-motion was propagated was as follows: He regarded every vibrating point on the wave-front as the center of a new disturbance: these secondary disturbances, travelling with equal velocity, are enveloped by a surface identical in its properties with the surface from which the secondary disturbances start, and this surface forms the new wave-front.

For example, in Fig. 14, consider  $O$  a luminous point, and  $AB$  a portion of the spherical wave front. Adjoining points  $a, b, c, d$ , etc., on this wave-front are vibrating in unison and can be regarded as centers of new disturbances, which spread out around them as indicated by the dotted lines. It is evident that these secondary waves are enveloped by the spherical surface  $A'B'$ , and this surface is the new wave-front. If the luminous point is at a great distance, and we are dealing with a plane-wave, we have the condition shown in the lower figure.

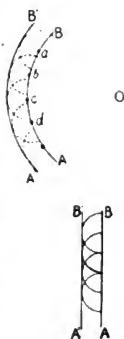


FIG. 14.

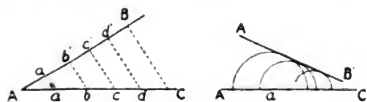


FIG. 15.

on the reflecting or refracting surface, as they are collectively or

This view of wave propagation is known as the Huygens principle. It can be applied to the calculation of the position of a reflected or refracted wave-front, by regarding the points

successively struck by the incident wave, as individual centers of new disturbances. For example, consider a wave-front  $AB$  descending in an oblique direction on a reflecting surface  $AC$ . The points  $a, b, c, d$  of the surface will be struck in succession by the points  $a', b', c', d'$  of the wave front, consequently they will become successively the centers of secondary disturbances, as indicated in Fig. 15, which are enveloped by the plane surface  $A'B'$ . This is the reflected wave-front, and we shall see later on that it makes the same angle with the reflecting surface as the incident wave.

**Rectilinear Propagation.**—Assuming Huygens's conception of the mechanism of wave-propagation to be correct, how are we to account for the rectilinear propagation of light? Suppose we have a luminous body at  $O$  (Fig. 16) and an opaque screen, a coin for example at  $A$ . We know that no light penetrates into the conical region behind the coin (neglecting for the present a phenomenon known as diffraction).

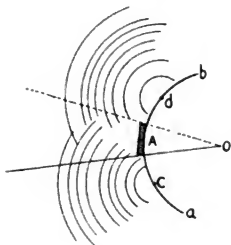


FIG. 15a.

But if all points on the wave-front are acting as independent sources, we should naturally expect them to be effective in illuminating the region behind the obstacle; in other words, we should expect the luminous waves given off by the points  $c$  and  $d$  to have some effect in the space behind the coin. Why does not the entire

wave appear luminous to an eye behind the obstacle if every point of it is giving off radiant energy in the form of secondary wavelets?

The answer given by Huygens was that these secondary waves produced no appreciable effect at a point unless they were *at that point* enveloped by a common tangent plane, or that the only effective portion of a secondary wavelet was the small point at its apex which touched the plane tangent to all of them. Huygens regarded the impulses as coming at irregular intervals, and his explanation of the rectilinear propagation of light amounted simply to the assumption that only one point on the secondary wavelet was effective in producing light.

Fresnel was the first to give a really satisfactory explanation. Making use of the principle of interference discovered by Young, by which two luminous vibrations may destroy one another, he arrived at the somewhat startling conclusion that the absence of light in the shadow of a body was due to destructive interference between the secondary wavelets. This explanation not only accounted for the darkness behind the obstacle, but explained perfectly the slight bending of the rays around the edges, a phenomenon known as diffraction, which had been previously explained by assuming the edge to exert a modifying action on the luminous rays which passed close to it.

It was no longer necessary to assume that only a minute portion of the secondary wave was operative in producing light, which as a matter of fact is contrary to experimental evidence, as can be shown by allowing a plane-wave to fall on an opaque screen perforated with a

very small aperture. The point on the wave-front not cut off by the screen acts as a center of a disturbance, which spreads out into the space behind the screen as shown in Fig. 16, and a card placed in the position shown will be illuminated over an area many times greater than the aperture.

Waves of sound behave in a similar manner, and it is actually possible to photograph the secondary wavelet. The accompanying photograph is one of a series made by the author to illustrate certain features of wave-motion. The method by which the pictures were made will be discussed later on.<sup>1</sup>

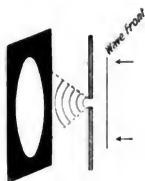


FIG. 16.

For the present it will be merely necessary to state that in every case the sound photographed is the crack of an electric spark, which gives of course a single pulse, instead of a train of waves. The series shown in Fig. 17 was made to illustrate the principle of Huygens.

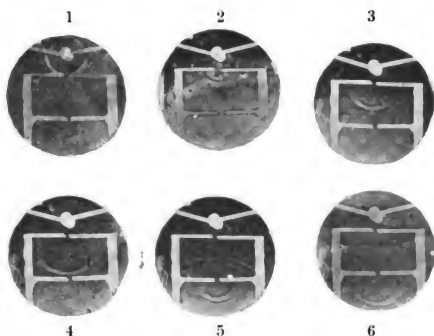


FIG. 17.

The spark which started the wave was arranged to snap directly above a screen provided with a narrow slit. A short distance below this first slit a second was mounted, and it will be seen that the two narrow apertures become in succession the centers of secondary waves which diverge precisely as if the source of the sound, that is to say the spark, were situated in the aperture itself. In No. 1 of the series, the wave, which started at the point *A*, has just encountered the first screen. The aperture which appears in the photograph as a break in the horizontal white line becomes the center of a new hemispherical wave, the gradual development of which is shown in Nos. 2, 3, and 4. In No. 5 the secondary wave has collided with the second screen and been reflected, the aperture in this screen becoming in its turn the origin of a new secondary wavelet. These pictures show that if all but a small part of the original wave is screened off, this small

<sup>1</sup> Wood, "Photography of Sound Waves," *Philosophical Magazine*, August 1899.

part becomes a complete wave, and again if a small portion of this secondary wave is allowed to pass through a small aperture, it becomes in turn a complete wave.

Before considering in detail Fresnel's explanation we must make an assumption regarding the nature of the secondary wavelet, which is based on the circumstance that no disturbance is radiated *backward*. An opaque screen which absorbs all of the energy falling on it, has no effect whatever on the vibration of the medium between it and the luminous source.

From this we infer that the secondary wavelet is only propagated forward, and lies wholly in front of the plane tangent to the wave-front at the center of the wavelet. We are also justified both by theory and experimental evidence in assuming that the effect of the secondary wavelet is greatest on the line which is normal to the

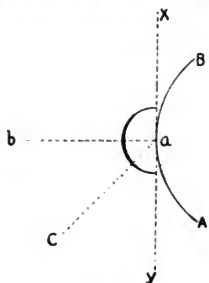


FIG. 18.

tangent plane at the point of tangency. This will be better understood by reference to Fig. 18, where  $AB$  is the wave-front,  $a$  the center of any secondary wavelet, and  $xy$  the tangent plane behind which we assume that the secondary wavelet never spreads. The effect of the wavelet is greatest along the line, or in the direction  $ab$ , less along  $ac$ , and falls off continuously, having the value 0 in the direction  $ax$ . This may be summed up by saying that the effect of the secondary wavelet decreases with increasing obliquity. The reason for the absence of a back-wave has been given (page 8).

We will commence the investigation of Fresnel's treatment of the subject by examining the effect of a linear wave on a point  $P$  in front of it

Let  $AB$  be the wave-front, which we may consider moving as a whole up and down parallel to itself. Thus all the particles on  $AB$  move together, and the secondary waves leave them at the same moment.

Draw a perpendicular from  $P$  to the wave front, meeting it at  $C$ , which point is called the pole of the wave with reference to the point  $P$ .

Lay off on  $AB$  points  $M_1, M_2, M_3, M_4$ , etc., so that the path from  $M_1$  to  $P$  is half a wave-length longer than the path from  $C$  to  $P$ , and  $M_2$  half a wave-length further from  $P$  than  $M_1$ , and so on. If secondary wavelets start simultaneously from these points and move with the same velocity, the disturbance from  $C$  will reach  $P$  first, since  $CP$  is the shortest path, and the wavelet from  $M_1$  will reach  $P$  half a wave-length behind the one coming from  $C$ , since we have so located  $M_1$  on the wave-front that the path  $PM_1$  is half a wave-length longer than

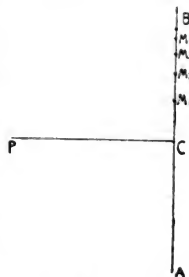


FIG. 19.

*PC*. This means that the crest of the wavelet from  $M_1$  reaches  $P$  at the same moment that the trough of the wavelet from  $C$  is passing through it, or the waves are in opposite phase and would destroy each other if both were equal. In the same way other wavelets coming from points lying between  $C$  and  $M_1$  will reach  $P$  with phases opposite to those coming from corresponding points between  $M_1$  and  $M_2$ . The same will be true for wavelets coming from points between  $M_2M_3$  and  $M_3M_4$ .

To determine the effect of the whole wave at  $P$  we determine the total effect or resultant of all the secondary wavelets, paying attention to their phases as well as their amplitudes. The effect at  $P$  of each of the elementary arcs into which we have subdivided  $AB$  we consider as proportional to its length, and inversely proportional to its distance from  $P$ . As we recede from  $C$  the effect will also diminish on account of the increasing obliquity.

We will now determine the relative lengths of the arcs into which we have subdivided  $AB$ .

Let the distance from the pole of the wave to  $P$  be  $b$ , then the distance of  $M_1$  from  $P$  is  $b + \frac{\lambda}{2}$  and the length of the arc  $CM_1$  is

$$\sqrt{\left(b + \frac{\lambda}{2}\right)^2 - b^2} \text{ or } \sqrt{b^2 + 2\frac{b\lambda}{2} + \frac{\lambda^2}{4} - b^2} \text{ or } \sqrt{b\lambda},$$

if we neglect  $\frac{\lambda^2}{4}$ , which is very small in comparison to  $b\lambda$ .

The path  $PM_2 = b + 2\frac{\lambda}{2}$  or  $b + \lambda$ ; therefore

$$\begin{aligned} CM_2 &= \sqrt{(b + \lambda)^2 - b^2} \\ &= \sqrt{b^2 + 2b\lambda + \lambda^2 - b^2} = \sqrt{2b\lambda}, \end{aligned}$$

neglecting  $\lambda^2$ , which is small.

Therefore

$$CM_1 = \sqrt{b\lambda}, \text{ and } CM_1 = \sqrt{b\lambda},$$

$$CM_2 = \sqrt{2b\lambda}, \quad M_1M_2 = \sqrt{2b\lambda} - \sqrt{b\lambda} = \sqrt{b\lambda}(\sqrt{2} - \sqrt{1}),$$

$$CM_3 = \sqrt{3b\lambda}, \quad M_2M_3 = \sqrt{3b\lambda} - \sqrt{2b\lambda} = \sqrt{b\lambda}(\sqrt{3} - \sqrt{2}).$$

The arcs thus decrease rapidly in length in the neighborhood of the pole. The length of any arc at distance  $R$  ( $G$  in the diagram) is determined as follows, since the small right angle at  $G$  is similar to the right triangle  $PCG$ .

$$G : \frac{\lambda}{2} = R : \sqrt{R^2 - b^2} \text{ or } G = \frac{\lambda}{2} \frac{R}{\sqrt{R^2 - b^2}},$$

a quantity which decreases with increasing  $R$ , approaching the value  $\frac{\lambda}{2}$  as a limit. The arcs far removed from the pole decrease slowly in length, approaching the limiting value  $\frac{\lambda}{2}$ .

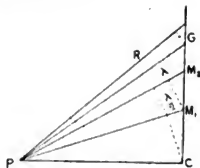


FIG. 20.

Besides decreasing in length, the elements become less and less effective owing to increasing distance and obliquity.

Remembering that adjoining elementary arcs send disturbances of opposite sign to  $P$ , we see that the effect of all is represented by a series of alternately positive and negative members, which at first decrease rapidly, then more slowly approaching 0 in value, since the very remote arcs are inoperative owing to their obliquity. Calling the effect due to the central arc  $l$ , and that due to the following ones  $m, m', m'',$  etc., the whole effect is represented by the series

$$l - m' + m'' - m''' + m^{iv},$$

the value of which is a fraction of the value of the first member.

Therefore the effect of the entire wave at  $P$  is less than that due to the first element acting alone. If we were to screen off all of the wave except the first element the illumination at  $P$  would be greater than that due to the whole wave, a surprising conclusion which, as we shall soon see, can be verified by experiment.

The wave-length of light is so small that with  $P$  at a distance of 10 cms. from the wave-front  $M_1$  would be scarcely more than 0.2 mm. from the pole of the wave. At a short distance from the pole the arcs would become very nearly equal and opposite in their effect, consequently the effective portion of the wave reduces itself to a comparatively small area around the pole, and if we screen off this region we shall have darkness at  $P$  owing to the destructive interference between the disturbances coming from the outlying elementary arcs, or a shadow will exist behind the screen.

**Effect of a Plane-Wave on an Exterior Point.**—Thus far we have been considering wave-motion in two dimensions only, a hypothetical case. Let us now find an analogous treatment for waves moving in space, which is the condition under which we observe them in our experiments.

Consider a plane-wave (Fig. 21) moving towards  $P$ , an exterior point: we require the effect at this point of all the secondary wavelets emanating from the wave-front. Draw a perpendicular from  $P$  to the wave front, intersecting it at  $C$ , the pole of the wave with respect to  $P$ . Around  $C$  describe circles on the wave-front such the first is half a wave-length further from  $P$  than  $C$  is, the second 2 half wave-lengths, etc. The rings thus formed on the wave-front will be analogous to the elementary arcs into which

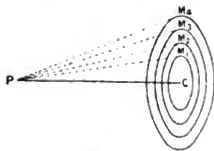


FIG. 21.

we divided the linear wave, that is to say, the secondary disturbances coming from any circle will reach  $P$  half a wave-length ahead of those coming from the circle encircling it.

We regard the effect of the disturbances coming from each ring as proportional to its area and as decreasing with increasing distance and obliquity as before. Let us now investigate the areas of the rings.

The radii of the circles are obviously equal to the distances  $CM_1, CM_2,$  on the linear wave, namely  $\sqrt{b\lambda}, \sqrt{2b\lambda}, \sqrt{3b\lambda},$  etc., and the areas or  $\pi r^2,$

$$\pi b\lambda, 2\pi b\lambda, 3\pi b\lambda.$$

Neglecting the square of  $\lambda$  as we have done, we find the area of the central circle and each surrounding zone to be equal or  $\pi b\lambda$ .

For a zone at distance  $R$  from  $P$  we have its width given by  $\frac{\lambda}{2} \frac{R}{\sqrt{R^2 - b^2}}$ , as in the case of the element of the linear wave.

Its circumference is  $2\pi\sqrt{R^2 - b^2}$ , and its area, or the product of these two quantities, is  $\pi\lambda R$ .

The effect due to the disturbances coming from a single one of the zones will be proportional to its area and inversely proportional to its distance. The slight increase in the area of the zones as we recede from the center of the system is compensated by the increased distance, so that, other things being equal, we could regard the successive zones as producing equal and opposite effects at the point. The zones, however, become less and less effective as we recede from the center owing to the increased obliquity. We can therefore represent the resultant effect by a series of terms of alternate sign which decrease slowly at first, and then more rapidly, eventually becoming zero, thus:

$$S = m_1 - m_2 + m_3 - m_4 + \text{etc.}$$

The sum of this series is usually stated as being equal to one half of the first term plus one half of the last term; the method usually adopted to prove this consists in balancing the second term against half of the first and half of the third, and so on. Schuster has shown that this treatment is too arbitrary, no reason being given why the balancing is not effected in some other way, for example by considering the second term balanced by three quarters of the first and one quarter of the third, which would make the resultant outstanding effect approximately equal to one quarter of that due to the first member acting alone. Schuster shows in what cases the addition of the series can be effected in the manner indicated. He first writes the series in the two following forms:

$$S = \frac{m_1}{2} + \left(\frac{m_1}{2} - m_2 + \frac{m_3}{2}\right) + \left(\frac{m_3}{2} - m_4 + \frac{m_5}{2}\right) + \left(\frac{m_{n-2}}{2} + m_{n-1} + \frac{m_n}{2}\right) + \frac{m_n}{2},$$

$$S = m_1 - \frac{m_2}{2} - \left[\left(\frac{m_2}{2} - m_3 + \frac{m_4}{2}\right) + \left(\frac{m_4}{2} - m_5 + \frac{m_6}{2}\right) + \left(\frac{m_{n-2}}{2} - m_{n-1} + \frac{m_n}{2}\right)\right] \\ - \frac{m_{n-1}}{2} + m_n.$$

Suppose first that each term of the original series is greater than the arithmetical mean of the two adjacent terms. From the above equations we see that

$$m_1 - \frac{m_2}{2} + m_n - \frac{m_{n-1}}{2} < S < \frac{m_1}{2} + \frac{m_n}{2},$$

for in both equations the terms bracketed are all small negative quantities, and the value of  $S$  lies somewhere between the two quantities given above.

If  $m_1$  is very nearly equal to  $m_2$  and  $m_n$  nearly equal to  $m_{n-1}$ , the two limits lie close together, and we may write

$$S = \frac{m_1}{2} + \frac{m_n}{2}.$$

If the series is such that each term is *less* than the mean of its neighbors,  $S$  lies between the same limits (transposed).

If in the first  $p$  terms of the series each term has a greater value, and in the remaining part a smaller value than the arithmetical mean of the terms between which it stands, we may break up the series into two, and obtain the sum

$$S = \frac{m_1}{2} \pm \frac{m_p}{2} \mp \frac{m_{p+1}}{2} + \frac{m_n}{2}.$$

It is thus clear that the expression for  $S$  given above will be the correct summation only, if the series can be broken up into a small number of separate series for each of which the value of a term is either smaller or greater than the arithmetical mean of the terms between which it stands, so that the sum of all such values may be neglected.

The problem therefore reduces to a determination of the effect due to one half of the central zone.

The secondary wavelets from this zone unite into a disturbance the phase of which is midway between those of the wavelets from the center and rim, for we may divide the zone into a series of concentric rings of equal area, the effects of which at the point are equal in amplitude, and of phases ranging over half a complete period. The phase of the resultant is therefore a quarter of a period behind that due to the element at the center. We must consequently consider that the secondary waves start with a phase one quarter of a period ahead of that of the primary wave. Lord Rayleigh, in his article on Wave-Theory (*Encyclopedia Britannica*), shows that, in calculating the magnitude of the effect due to the central zone, we must take into account the phases of the secondary disturbances. The amplitude of the resultant can be shown to be less than if all its components had the same phase in the ratio  $2 : \pi$ .

Assume the amplitude on the wave-front to be unity, and consider that the secondary wave from a small element of its surfaces produces a resultant effect represented by  $kds$ . If  $r$  is the radius of the zone its resultant effect will be  $2/\pi k\pi r^2$ . Now  $r^2 = b\lambda$ , and the amplitude due to the whole zone is therefore  $2kb\lambda$ . The whole wave will produce an amplitude one half as great, which we may equate to unity, since we have assumed unit amplitude on the wave-front, and a plane-wave is propagated without loss of amplitude. From this we find that  $k$ , the factor which represents the effect of the secondary wave, is equal to  $1/b\lambda$ . That the amplitude due to the secondary wave should vary inversely as the distance  $b$  is to be expected, but it may not be at once obvious why it should vary inversely with the wave-length. There is no mystery about the matter however. If we keep the distance  $b$  fixed and increase the wave-length, we are obliged to increase the size of the zone, if the conditions are to remain as before, that is, if the zone is to

produce the same effect at the point. The secondary disturbances are now coming from a larger area, while only producing the same resultant effect, consequently the effect produced by any small element of surface  $ds$  will be proportionally smaller.

Let us now put the theory to experimental test.

Suppose we screen off all of the wave-front except the central circle of the zone series. There is now no encircling zone to partly neutralize it, and the illumination is greater than that due to the entire wave. This can be accomplished by placing a screen provided with a small circular aperture at such a distance from the point  $P$  that the area of the aperture is equal to the area of the central zone, when the amplitude at once becomes double, and the illumination four times that due to the unobstructed wave. It is of course apparent that the actual size of the zones on the wave-front in a given plane depends on the distance of the point  $P$ . As this distance increases the zones widen out. On a wave-front distant about 5 feet from the point, the zones would be of the size shown in Fig. 22, so that if our small circular aperture was of the size of the central circle in the figure, the illumination at a point on the normal 5 feet behind the aperture would be greater than if the screen were not present. And now comes a very curious fact: suppose we increase the size of our aperture until it contains another zone. The disturbances coming from this ring will be out of phase with those coming from the central circle, and will entirely destroy them. Thus by increasing



FIG. 22.

the size of the hole we can reduce the illumination to zero. The experiment can be performed with very simple apparatus, provided one has a dark room of sufficient length. A pin hole in a piece of thin sheet metal illuminated with arc or sun light makes a suitable source of light. A first class iris diaphragm, such as is furnished with the best photographic objectives, furnishes us with an aperture the size of which can be varied at will. The diaphragm should contract to a diameter of 3 mms. or less, and the outline of the opening should be circular and not polygonal, as is the case with the cheaper kinds provided with but few wings. Suppose the smallest aperture to have a radius of 1.5 mms.: we require the distance of the point so situated that only the central circle of the zone system is exposed. The formula  $CM_1 = \sqrt{b\lambda}$  shows us that, if we put  $\lambda = .0005$  mm., the distance of the point is 4.5 meters. This is for plane-waves, or with our source at a great distance. For the condition of source and point

at equal distances from the aperture we substitute  $\frac{\lambda}{4}$  for  $\frac{\lambda}{2}$  in the original formula, for now there will be a path difference on both sides of the screen; in other words, the vibration at the edge of the aperture will be slightly behind that at the center. The distance now increases to 9 meters. Clearly we shall need a long room for our experiment, for the source must be 9 meters behind the screen, or our total optical path must be 18 meters. We can, however, reduce this by one half by using a small reflector of silvered glass, an excellent arrangement being to so arrange things that the diaphragm and the illuminated point are

close together. To accomplish this we place it at a distance of 9 meters from the source and mount our mirror 4.5 meters behind it, reflecting the light back to a point a little to one side. If we hold a sheet of paper here we shall see a little point of light. Put a little drop of white paint on a bit of glass, and mount it in such a position that it lies in the center of the small spot of light. This forms our illuminated point. Now, increase very slowly the size of the diaphragm and the light gradually fades away, the drop of paint presently becoming invisible. Twice as much light comes through the hole as before, yet the point is in darkness. The law of the conservation of energy tells us, of course, that no light has been destroyed. It has simply gone somewhere else, and where it has gone to does not concern us at present. The fact that it no longer manifests itself at the point in question is sufficient.

Let us now try the converse of this experiment by substituting for the aperture a small circular disc of the same diameter. According to Huygens theory, if placed over the central zone, it should cut off the illumination at the point entirely. On the Fresnel theory we simply remove the first member of the series, and the effect is that due to the remainder of the series, namely half the second member, or the illumination is unaffected by the interposition of the circular disc, and this is precisely what we find to be the case. By increasing the size of the disc we cut off another zone, still without influencing the illumination, and this may be continued, not indefinitely, but until, owing to the increasing obliquity, the effect of the zones begins to diminish appreciably. We thus see that the center of the shadow of a circular body may, under certain conditions, be as brightly illuminated as the surrounding field, a proposition due to Poisson.

Fresnel's memoir on diffraction was presented to the French Academy and reported on by Poisson, who raised the objection that if the treatment were applied to the case of a circular disc (a case which had not been treated by Fresnel), it would lead to the conclusion that the illumination along the axis of the disc would be the same as if the disc were absent, which was supposed to be a *reductio ad absurdum*. In this case it is clear that the illumination will be represented by the above-mentioned series, with as many members removed as there are zones covered by the disc, which will be as before one half of the first exposed zone, and if we assume the zones to produce equal effects, the illumination should be the same as without the disc. As a matter of fact, the experiment had already been recorded by Deslisle, but it had been forgotten, and was rediscovered by Arago and Fresnel, who observed the bright spot in the center of the shadow of a circular disc.

This experiment is easily performed with a small disc of metal, a copper cent for example. If a coin is used, a new one should be selected, the edge of which is smooth and undented. It should be supported by means of three fine threads, which can be attached to the coin with wax. Using the same source of light as before, with the coin mounted at a distance of three or four meters, we shall find, if we explore the region behind the coin with a low power eye-piece, that there is a brilliantly illuminated region along the axis of the geometrical shadow. The illumination is faint in the immediate vicinity

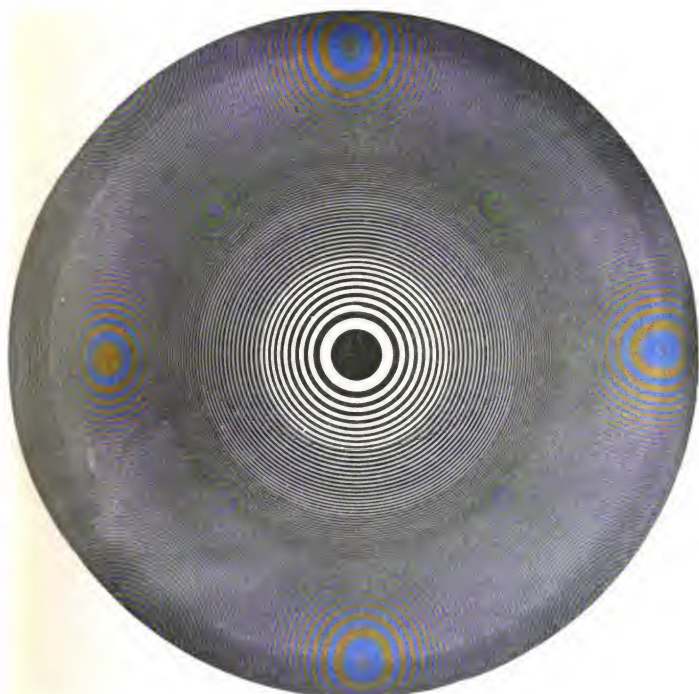


PLATE I.

*To face p. 30.*

of the coin, owing to the obliquity of the secondary waves, but at a distance of several meters behind the coin it is nearly as bright as if the coin were absent. If the eye is brought into coincidence with the luminous spot it will be found that the light comes from the edge of the coin, which appears brilliantly illuminated. If the eye is moved a little to one side the ring breaks up into two spots of light situated on opposite sides of the coin. We are now getting the light which bends into the shadow radially, without the great reinforcement due to agreement of phase.

**Zone-Plate.**—A remarkable verification of Fresnel's theory is furnished by what is known as a zone-plate. If we describe on a large sheet of white paper circles, the radii of which are proportional to the square roots of the natural numbers, we shall have very nearly an exact drawing of the zone system, the neglected terms containing the square of  $\lambda$  introducing a very slight error. If now we blacken the alternate rings with ink, and take a greatly reduced photograph of the whole on glass, we shall obtain a device which will enable us to screen off the alternate zones on the wave-front. Suppose we intercept a plane-wave with such a plate and consider the illumination at a point so situated behind the plate that the central circle of the plate corresponds in size and position to the first zone on the wave-front. The black rings stop all the secondary disturbances from the alternate or odd zones, which previously neutralized those coming from the even ones, consequently all the secondary disturbances coming from that portion of the wave-front covered by the plate reach the point in the same phase, and the illumination will be very intense. The whole surface of the zone-plate will send light to the point, the action being very similar to that of a convex lens. The distance of the illuminated point from the zone-plate we may speak of as its focus, and we readily see that the smaller the zones the shorter the focal length.

A very good zone-plate can be made by making a photographic negative of Plate I., which is from a very carefully executed drawing. It is well to make several plates of different focal lengths. That they have properties similar to lenses may be well shown by holding one of suitable focus, say half a meter, between the eye and a distant lamp. If the central zone is brought over the flame the whole plate fills up with light like a lens. By combining a zone-plate with a low-power eye-piece we can form a telescope which will give a fairly sharp image of a brilliant object, such as in an incandescent lamp.

Lord Rayleigh, in his article on Wave-Theory in the *Encyclopedia Britannica*, called attention to the fact that if it were possible to provide that the light stopped by the alternate zones could be allowed to pass, but with a reversal of phase, a fourfold intensity in the illumination at the focus would result. In this case the secondary disturbances from all the zones, both odd and even, would reach the point in the same phase. This can be accomplished in two ways,<sup>1</sup> first by making the zones of a thin film of gelatine on glass, the thickness of the film being such as to retard the waves one half wave-length; secondly, by forming the zones of metallic silver on the hypotenuse surface of a right angle

<sup>1</sup> Wood, "Phase Reversal Zone-Plates," *Phil. Mag.*, June 1898.

prism. In this case the light from the odd zones is reflected metallically from the silver, while that coming from the even zones has been reflected from the air surface (total internal reflection). Reflection under these two conditions introduces a phase change of almost exactly a half wavelength, and the light at the focus is quite as brilliant as with the gelatine plates.

The zone-plate has many peculiar properties. It has multiple foci and can act at the same time both as a convex lens and a concave lens; but these properties can be discussed to better advantage after we have studied diffraction.

**Law of Regular Reflection deduced from the Principle of Interference of Elementary Waves.**—The construction given by Huygens for the reflected and refracted waves is incomplete, just as was his construction for rectilinear propagation, for he was obliged to assume that only a single point on the secondary wavelet was operative in producing illumination. In applying the Fresnel principle of interference to reflection from a plane surface, we regard each point on the surface of the mirror, as it is struck by the incident wave, as the center of a secondary wavelet, and determine the collective effect of these wavelets at any point, just as we did when we regarded points on the wave-front as centers of disturbance. We know that the light, radiating from a point and reflected from a plane mirror, which is effective in illuminating a given point, comes from a point on the mirror so situated that lines joining it with the source of the light and the point illuminated, make equal angles with the normal. Let us see if we can construct a system of zones on the surface of the mirror in such a way that the effects coming from all will be essentially reduced to that due to a small area surrounding a point situated as described. The problem is somewhat more complicated than the one which we have just solved, for the centers of the secondary wavelets on the surface of the mirror are not vibrating in unison as are those on the wave-front. The reason of this is obvious, for different points on the mirror are struck by the wave-front at different times, and the secondary disturbances therefore do not start simultaneously, and will not be in agreement of phase except in the case of a plane-wave incident normally.

Let  $A$  (Fig. 23) be a point on the reflecting surface  $MN$ , and  $O$  the illuminating point. The amplitude at  $A$  will be inversely proportional to the distance  $OA$ , and if a secondary wavelet starts from  $A$  its effect at  $P$  will be inversely proportional to  $AP$ . The effect at  $P$  due to a secondary wavelet coming from  $A$  started by a disturbance coming from  $O$ , will be inversely proportional to the product  $OA \times AP$ . We will investigate first the effect at  $P$  of secondary wavelets coming from a linear strip of the mirror  $MN$ . We will consider  $A$  so situated that  $OA$  and  $AP$  make equal angles with the normal at  $A$ , and take points  $A'$  and  $A''$  such that the path

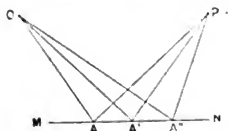


FIG. 23.

$$OA'P - OAP = \frac{\lambda}{2} \quad \text{and} \quad OA''P - OA'P = \frac{\lambda}{2}$$

Now let  $A_1$  be any point on  $MN$  and let

$$OAP = b, OA_1P = b + \delta = R, \text{ and } AA_1 = s.$$

Consider  $A_1$  near  $A$ , and  $R$  as a function of  $s$ , and develop by M'Laurin's theorem,

$$\text{When } s = 0, \frac{dR}{ds} = 0, b + \delta = b + \frac{s^2}{2} \left( \frac{d^2R}{ds^2} \right) + \dots,$$

neglecting terms containing higher powers of  $s$  than the square.

This gives us  $s = \sqrt{\frac{2\delta}{\frac{d^2R}{ds^2}}}$ , from which we have, by substituting for

$\delta, \frac{\lambda}{2}, \lambda, \frac{3\lambda}{2}$ , etc., the following values for the elementary arcs

$$AA' = \sqrt{\frac{\lambda}{\frac{d^2R}{ds^2}}}, \quad A'A'' = \sqrt{\frac{\lambda}{\frac{d^2R}{ds^2}}} (\sqrt{2} - 1), \quad A'A''' = \sqrt{\frac{\lambda}{\frac{d^2R}{ds^2}}} (\sqrt{3} - \sqrt{1}).$$

The elementary arcs therefore decrease rapidly in length in the neighborhood of  $A$ .

The effect of this linear strip is then represented by a decreasing series of alternately positive and negative terms, or the effect of  $MN$  on  $P$  reduces to a fraction of the arcs adjoining  $A$  on either side. We now divide the reflecting plane into narrow strips parallel to  $MN$ , and what holds for the strip  $MN$  holds for all the others, each one reducing to a small element adjoining the point so situated that the sum of its distances from  $O$  and  $P$  is a minimum, the same as in the case of the point  $A$ . We now require the locus of these points. If  $O$  and  $P$  were at the same height above the surface it would obviously be a straight line, perpendicular to  $MN$ , passing through a point midway between the projections of  $O$  and  $P$  on  $MN$ .

If  $O$  and  $P$  are not at the same height above the surface, which is the more general case, we find the locus in the following manner.

Let  $ABCD$  be the reflecting plane (Fig. 24),  $O$  the luminous point at height  $h$  above the plane,  $P$  the illuminated point at height  $k$ , and  $L$  the distance between  $M$  and  $N$ , the projections of  $O$  and  $P$  on the mirror. Arrange a coordinate system  $xy$  on the plane of the mirror, with its origin at  $M$  and its  $x$  axis coincident with  $MN$ .

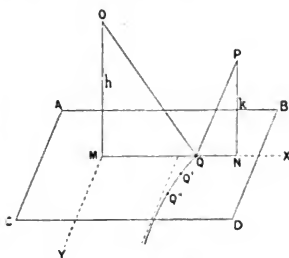


FIG. 24.

We require the locus of points  $Q, Q', Q''$ , etc., so situated that the distances  $OQP, OQ'P$ , etc., shall be a minimum.

$$OQ = \sqrt{x^2 + y^2 + h^2}, \text{ and } OQ' = \sqrt{(l-x)^2 + y^2 + k^2}.$$

We require

$$\sqrt{x^2 + y^2 + h^2} + \sqrt{(l-x)^2 + y^2 + k^2}, \text{ a minimum.}$$

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C

By differentiating and equating to zero, we obtain

$$\frac{x}{\sqrt{x^2 + y^2 + h^2}} = \frac{l - x}{\sqrt{(l - x)^2 + y^2 + k^2}}, \quad \frac{x^2 + y^2 + h^2}{x^2} = \frac{(l - x)^2 + y^2 + k^2}{(l - x)^2},$$

or  $1 + \frac{y^2 + h^2}{x^2} = 1 + \frac{y^2 + k^2}{(l - x)^2}, \quad (y^2 + h^2)(l - x)^2 = x^2(y^2 + k^2).$

Solving for  $x$ , we get  $x = \frac{l\sqrt{y^2 + h^2}}{\sqrt{y^2 + h^2} + \sqrt{y^2 + k^2}}$

as the equation of the locus of points so situated that the sum of their distances from  $O$  and  $P$  is a minimum.

Letting  $y = 0$ , we obtain the value for  $x$  on  $MN$ ,

$$x = \frac{lh}{h + k}.$$

This can be verified by geometry, for the triangles  $OMQ$  and  $PNQ$  are similar. Let  $MQ = x$  and  $NQ = (l - x)$ .

Then  $\frac{h}{x} = \frac{k}{l - x}$ , or  $x = \frac{hl}{h + k}.$

Furthermore, if  $O$  and  $P$  are at the same height above the mirror, or  $h = k$ , we have  $x = \frac{l}{2}$ , or the curve is asymptotic to a line parallel to the  $y$  axis, and bisecting  $l$ . This asymptotic curve is the axis of a narrow strip, made up of the effective portions of the linear elements into which we divided the mirror, or we have reduced the effect of the whole mirror to that of a strip.

We now lay off on this curve points so situated that the sum of their distances from  $O$  and  $P$  differs continuously by  $\frac{\lambda}{2}$ , and by drawing short lines through these points, parallel to  $MN$ , we divide the strip into elementary areas. The effect of each one of these elements at  $P$  is represented by its area divided by the product of its distances from  $O$  and  $P$  (Fig. 25).

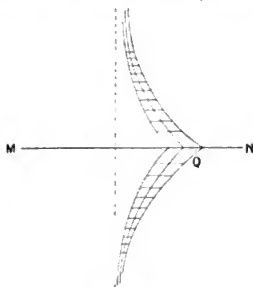


FIG. 25.

It can be shown that the areas near  $MN$  are of the order of magnitude  $s\sqrt{h\lambda}$ , where  $s$  is the length of the effective strip of the original linear element, while the areas far removed from  $MN$  are of the order  $s'\sqrt{\lambda y}$ , and their respective effects at  $P$  are

$$\frac{s\sqrt{h\lambda}}{hk}, \quad \frac{s'\sqrt{\lambda y}}{y^2},$$

the latter quantity being small in comparison with the former.

The whole effect is thus represented by a diminishing series of alternately positive and negative diminishing terms, and is equivalent

to a fraction of the first term. The effect of the entire reflecting surface thus reduces itself to a small area surrounding the point so located on the line  $MN$  that the sum of its distances from  $O$  and  $P$  is a minimum, which is in accordance with the law of regular reflection.

**The Zone System on the Reflecting Surface.**—In the foregoing discussion we have determined the effect of the secondary disturbances from a reflecting surface by dividing the surface into rectilinear strips, and ascertaining the collective effect of these strips. This gives us no clue to the form of the zones on the surface, which we will now investigate.

Let  $O$  (Fig. 26) be the luminous source and  $P$  the illuminated point, and  $A$  a point so situated on the surface of the mirror  $XY$  that  $OA$  and  $AP$  make equal angles with the normal, and lie in the same plane. The boundary of the first zone will then be the locus of points such as  $M$ , so situated that

$$OM + MP = OA + AP + \frac{\lambda}{2}.$$

Now a point moving in such a manner that the sum of its distances from  $O$  and  $P$  is a constant quantity traces an ellipsoid of which  $O$  and  $P$  are the foci. The boundary of the first zone is the section of the reflecting surface with this ellipsoid. We may, however, use a slightly different construction, which simplifies the problem of calculating the form and area of the zones.

Let  $Q$  be a point situated below the reflecting surface in such a position that  $PQ$  is normal to, and bisected by, the surface. Then  $MQ = MP$  and  $AQ = AP$ , and we have the locus of the points bounding the first zone as the intersection of the surface, with the ellipsoid traced by a point moving so that  $OM + MQ = OA + AQ + \frac{\lambda}{2}$ , of which

$O$  and  $Q$  are the foci. The major axis of this ellipsoid is  $OQ + \frac{\lambda}{2}$ . The minor axis we determine as follows. Let  $BE$  be one half the minor

axis, then  $OB = \frac{OQ + \frac{\lambda}{2}}{2}$  and  $OE = \frac{OQ}{2}$ .

$BE^2 = \frac{(OQ + \frac{\lambda}{2})^2}{4} - \frac{OQ^2}{4}$ . Squaring the numerators, cancelling, and neglecting the term containing the square of  $\lambda$ , which is small, we have  $BE = \frac{1}{2}\sqrt{OQ\lambda}$ . This is very small in comparison to  $OQ + \frac{\lambda}{2}$  the major axis, or the ellipsoid is very eccentric and can be regarded as a cylinder at its intersection with the reflecting surface.

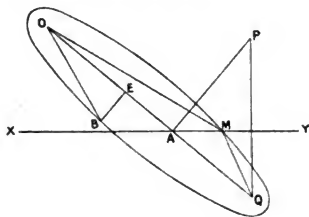


FIG. 26.

The section cut by the reflecting surface is an ellipse, the minor axis of which is equal to the diameter of the cylinder. It can be shown that the radius of the cylinder is approximately represented by

$$r = \sqrt{\frac{OA \cdot AQ}{OQ}} \lambda. \quad (\text{Verdet. Light.})$$

The major axis is given by  $2 \frac{r}{\cos \alpha}$ , where  $\alpha$  is the angle of incidence.

By making the path  $OM + MQ$  increase half a wave-length at a time we shall obtain a series of coaxial ellipsoids, the intersections of which with the reflecting surface form the zone system. The areas of the ellipses can be shown to be very nearly equal, forming a decreasing series similar to the circular zones on the plane-wave front. If we describe such a series of ellipses on a flat mirror and make the alternate zones opaque, the flat surface will concentrate parallel rays incident at the proper angle, much like a concave mirror. Such a plate can be made by photographing our drawing of the circular zone system at an angle of  $45^\circ$ . The negative when placed on a piece of silvered glass gives very sharp focal images for light incident at the same angle. If the elliptical zone system is printed on the hypotenuse surface of a right angle prism, in metallic silver, our phase difference of half a period between the adjacent zone results from the reflection occurring under different conditions. Such prints can be made by a method given in the paper referred to in the previous section.

**Reflection and Refraction by Unpolished Surfaces.**—One of the most interesting and instructive applications of the Fresnel construction is to the diffuse reflection and refraction which occurs when light strikes unpolished or matt surfaces such as paper, plaster of Paris or ground glass. We have explained regular reflection by showing that there is destructive interference between the wavelets arriving at any point from the surface of the mirror, and that the illumination is practically due to disturbances coming from a small region surrounding the point so situated that straight lines joining it to the source of light and the illuminated point make equal angles with the normal. An unpolished surface destroys all phase relation between the particles on the wave-front. The secondary wavelets start from the elevated portions of the surface first, since these portions are struck first by the incident wave, and the reflected wave front, instead of being plane, is pitted and corrugated in an irregular manner. It is impossible to arrange any zone system on such a surface, for there are all possible phase differences irregularly distributed over the reflected wave front, consequently each point on the surface acts as an independent luminous source, sending light out in all directions. We can apply the Fresnel theory to reflection of this sort in the following way.

Suppose we have a plane surface  $XY$  (Fig. 27) and a luminous point  $S$ , and are considering the effect at  $P$ , which we will suppose to be the point to which a ray  $SA$  would be reflected. We have, however, at  $A$  an elevation of height  $H$ , and the secondary wavelet will leave

the point  $B$  sooner than it would have left the point  $A$  were the elevation absent. We can see that the effect at  $P$  will be the same in either event, provided the difference between the path  $SBP$  and  $SAP$  is small in comparison to the wave-length. At normal incidence it is obvious that this path difference will be  $2H$ , therefore a surface having elevations on it of such magnitude that twice their height is not small in comparison to the wave-length, will not reflect regularly at normal incidence. With a given roughness long waves may be

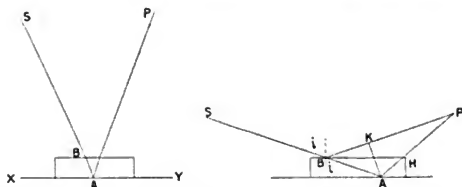


FIG. 27.

regularly reflected, and short waves irregularly. It can be seen from the right-hand figure that the difference of path becomes less as the angle of incidence increases, being in the case figured  $BA - BK$ , which is less than  $H$ , and that at grazing incidence it will become 0. It can be shown geometrically that the path difference is represented for all incidences by  $2H \cos i$ , the value of which must not exceed a small fraction of a wave-length if regular reflection is to occur.

$$SB = SA - AB = SA - \frac{H}{\cos i},$$

$$\text{and } BP = AP + BK = AP + AB \cos(\pi - 2i) = AP - \frac{H}{\cos i} \cos 2i,$$

$$SA + AP - (SB + BP) = \frac{H}{\cos i} (1 + \cos 2i) = 2H \cos i.$$

Since the path difference decreases as the angle of incidence increases, it is obvious that for a given roughness we shall get regular reflection when the incidence angle is so great that  $\rho\lambda = 2H \cos i$ , where  $\rho$  is a small fraction; therefore if we gradually increase the incidence angle, the long waves will be reflected first, and then the shorter. Smoked glass, which at perpendicular incidence will show no image of a lamp at all, will at nearly grazing incidence give an image of surprising distinctness, which is at first reddish, becoming white as the angle increases.

Let us next consider the effect of a matt surface on refraction. Here the phase differences are due to retardations of the portions of the wave-front encountering the elevations, on those portions encountering the depressions. With a given degree of roughness the retardation will be greater when the substance has a high refractive index, or more accurately when the difference between the refractive indices of the media bounding the rough surface is large. When the

retardation between two adjacent paths is larger than a small fraction of a wave-length, we have diffuse transmission. If we take a sheet of ground glass and wet the surface, the glass transmits more direct light than it did before, since we have lessened the difference between the refractive indices of the bounding media. If we substitute benzole for water the glass becomes still more transparent, and by bringing up the refractive index of the benzole by an addition of Canada balsam, we can cause the ground surface to disappear entirely.

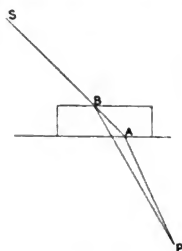


FIG. 28.

Let us now examine the effect of the angle of incidence on the transmission (Fig. 28).

The path difference expressed in time between two disturbances travelling over the paths  $SAP$  and  $SBP$  can be found as follows. The time occupied over the route  $SAP$  (which would be the one followed were there no elevation on the surface) will, if  $v$  and  $v'$  be the velocity of propagation in the upper and lower media, be  $\frac{SA}{v} + \frac{AP}{v'}$ , while the time over  $SBP$  or *via* the elevation will be  $\frac{SB}{v} + \frac{BP}{v'}$ , and the difference in time will be the difference between these two quantities.

The regularity of the transmission will not be affected if this time difference

$$\frac{SA}{v} + \frac{AP}{v'} - \left( \frac{SB}{v} + \frac{BP}{v'} \right)$$

is small in comparison to the time of a complete vibration. To change this time difference into a path difference, we substitute for  $\frac{v}{v'}$  the quantity  $n$ , which is the relative refractive index between the two media, which gives  $SA + nAP - (SB + nBP)$ .

If this quantity is small in comparison to the wave-length, the regularity of the transmission will be unaffected.

This path difference can be shown to be

$$H \frac{n^2 - 1}{\sqrt{n^2 - \sin^2 i} + \cos i},$$

where  $H$  is the height of the elevation and  $i$  the angle of incidence. This quantity has its smallest value when  $i = 0$ , when the path difference becomes  $H(n - 1)$ , or the regularity of transmission decreases as the angle of incidence increases, the opposite of what we found in the case of reflection.

If the refractive index of the substance is 1.5, then  $H(1.5 - 1)$  or  $H/2$  must be small in comparison to the wave-length of light, if the light is to be regularly transmitted at perpendicular incidence. Inequalities can then exist, the heights of which are, say, not greater than  $\frac{1}{4}\lambda$ , which is four times as great a discrepancy as we could have on a reflecting surface.

Summing up we have (for perpendicular incidence) for regular Reflection  $2H = \rho\lambda$  and for Transmission  $H/2 = \rho\lambda$ .

That the long waves are regularly transmitted and reflected by rough surfaces can be very beautifully shown in the following way. Grind two pieces of plate glass together with fine emery and water, until a matt surface is obtained which, when dry, barely shows the outline of a gas flame by transmission. When held close to the eye, the flame may even be invisible, but it is better to get the surface so that it just begins to transmit regularly. Continued grinding is all that is necessary, as the emery gets finer during the process.

Clean and dry the plate, put a drop of water on one side, and lay a smaller plate of clear glass on the drop. The water will increase the transmitting power of a portion of the plate. If we view a gas flame by reflection, holding the plate close to the eye, we shall see two images, one reflected from the smooth upper surface, the other from the rough glass-water surface: the former will be white, the latter distinctly red: if the dry surface barely shows a trace of the flame by transmission, it will fail to show it by reflection, except at large angles of incidence.

A paper by Lord Rayleigh on "Polish," will be found of great interest in connection with this subject. (*Proc. Royal Inst.* xvi. p. 563, and *Nature*, lxiv. p. 385, 1901.)

## CHAPTER III.

### THE REFLECTION OF LIGHT FROM PLANE AND CURVED SURFACES.

WHEN light strikes the boundary surface, separating two media of different optical densities, some of the energy is reflected back into the first medium, and some crosses the boundary and is transmitted through, or absorbed by, the second medium. We have shown in the previous chapter that if the surface is smooth to within one-eighth ( $\frac{1}{8}$ ) of a wave-length, we shall have regular reflection, and the law of reflection from a plane-mirror has been demonstrated by the Fresnel theory of destructive interference.

As a matter of fact, we are practically unable to make a surface so perfect that absolutely no light is diffused. Admit a ray of sunlight into a dark room and reflect it from the most perfect mirror attainable; were diffuse reflection not present the mirror itself would be invisible, which is never the case. The percentage of diffused light decreases as the angle of incidence increases, as has been shown in the previous chapter, regular reflection taking place even on matt surfaces at grazing incidence.

In studying the reflection of light from plane and curved surfaces we shall investigate not only the direction of the reflected rays, but also the form of the reflected wave-fronts.

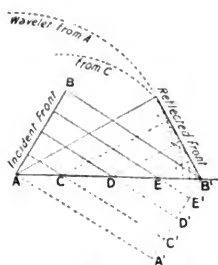


FIG. 29.

**Reflection of a Plane-Wave from a Plane-Mirror.**—Here the incident rays are parallel to one another and normal to the wave-front. We have seen that the reflected wave-front will be the surface enveloping the secondary wavelets given off by the various points on the mirror's surface as they are struck in succession by the incident wave. The general method of constructing the reflected wave-front is shown in Fig. 29. Let  $AB'$  be the surface of the mirror, and  $AB$  the incident-wave

front, the rays being indicated by arrows. At the moment figured, a secondary disturbance is about to leave the point  $A$ . This secondary disturbance will have spread out all around  $A$ , to a distance equal

to  $BB'$  at the moment when the point  $B$  on the wave-front encounters the surface. The secondary wavelets from points  $C, D, E$ , etc., intermediate between  $A$  and  $B$ , will have lesser radii. To construct them draw  $A'B'$  parallel to  $AB$ . This will give us a subsequent position of the wave-front, assuming the mirror not present. Join these two wave-fronts by perpendicular lines, which represent rays, which cut the mirror's surface at  $C, D, E$ , etc. The wavelet around  $C$  must obviously have a radius equal to  $CC'$ , while that around  $D$  has a smaller radius,  $DD'$ , and so on for all the other points. If we describe these spheres (circles in the diagram) we shall find that they are enveloped by a plane surface, which makes the same angle with the mirror's surface as the incident wave. This can be proven by similar triangles, Rt. triangle  $ABB' =$  Rt. triangle  $AFB' =$  Rt. triangle  $AB'A'$ . (Hypotenuse in common and  $AF = BB' = AA'$  by construction.) Therefore, their homologous angles are equal. The rays being normal to the wave-front will make equal angles with the normal to the surface of the mirror.

We can apply this same method to the construction of the wave-front after reflection from a surface of any form. In brief, we draw the wave-front before it encounters the reflecting surface and also in some subsequent position, behind the mirror, which it would occupy at a later moment were the mirror not present. Join these two fronts by normal lines (rays) and describe around the points at which they cut the reflecting surface circles whose radii are equal to the respective distances of the points from the wave-front in its second or imaginary position. The envelope of these circles shows us the position of the reflected wave-front, at the time at which the incident wave would have reached its imaginary position were the mirror absent.

Let us now apply this method to the construction of the reflected wave-front, when a spherical wave encounters a plane-mirror. Let  $O$  be the luminous point around which we construct the circular section of the spherical wave intersecting the mirror at  $A$  and  $B$  (Fig. 30). Completing the wave-front below

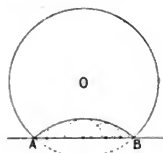


FIG. 30.

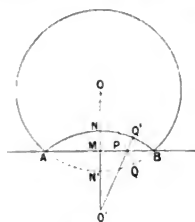


FIG. 31.

the surface of the mirror, describe around points on the mirror's surface circles whose radii are equal to their distances from the wave-front below the mirror measured in a normal direction,—that is, along the rays or radii of the original wave. The envelope of these circles is itself a circle of the same radius as the original wave, with its centre of curvature at the same distance below the mirror as the luminous point is above. The reflected wave is thus a portion of a sphere with its center below the mirror. This can be proven as follows:

Draw  $OO'$  perpendicular to the mirror's surface (Fig. 31) making  $OM = O'M$ , and with  $O'$  as a center draw the arc  $ANB$  representing

the reflected wave-front. Draw rays  $OPQ$  and  $O'PQ'$ .  $OP = O'P$  (homologous sides of equal Rt. triangles),  $\therefore PQ = P'Q'$ , since  $O'Q' = OQ$  (radii of equal circles).

This shows us that a wavelet drawn around  $P$  with a radius  $PQ$  touches the spherical arc  $ANP$  at  $Q$ , and since this is true for all points on the mirror's surface, it follows that a spherical surface of the same radius as that of the incident wave is the envelope of all the secondary wavelets. The rays being normal to the wave-front, it follows that after reflection they come apparently from the point  $O'$ , which we call the image of  $O$  in the mirror.

**Reflection of Sound Waves.**—A striking analogy exists between sound and light; we can show nearly all of the phenomena of reflection, refraction, and diffraction by means of sound waves. An electric spark is the center of a spherical sound wave, which expands at the rate of about three hundred and thirty-one (331) meters per second, and at the same time is the center of spherical light waves, which expand at the rate of three hundred thousand (300,000) kilometers per second. We have no means of directly showing the form of the wave-front of the *luminous* disturbance. We can calculate its form before and after reflection and show the agreement between these forms and the rays as actually observed. but we cannot actually show the wave-front. In the case of the *sonorous* disturbance, however, the wave-front can not only be seen, but photographed. We are dealing with a spherical shell of condensed air and, by a suitable optical contrivance which will be described in the next chapter, we can study at our leisure the changes which the wave-front undergoes.

The author has recently prepared an extensive series of photographs of sound waves for the purpose of illustrating optical phenomena.

The case that we have just considered, namely, the reflection of a spherical wave from a plane surface, is shown in Fig. 32. The sound



FIG. 32.

wave is started by an electric spark which has just passed between two brass balls, seen in line, one behind the other at the center of each picture. The wave of condensed air is illuminated and photographed by the light of a second spark occurring a moment later. By properly regulating the time interval between the two sparks a progressive series of views is obtained showing the wave-front at different stages of its development.

The form of the reflected wave or echo is seen to be identical with the form of the light wave as calculated by Huygens's principle.

**Reflection by Ellipsoidal Mirror.**—If a spherical wave start at one focus of an ellipsoid of revolution, the reflected wave will be spherical in form, and will collapse to a point at the other focus, or rays of

light issuing from one focus come accurately together at the other focus. A surface capable of bringing rays of light accurately to a focus, either by reflection or refraction, is said to be Aplanatic, consequently an ellipsoidal mirror is aplanatic for rays issuing from a point situated at either focus. This can be shown by the following construction.

Around one focus of an ellipse describe a circle which falls just outside of the furthest extremity of the ellipse. Draw a number of radii to the circle, and around the points where the radii cut the ellipse, describe circles with radii equal to the distances from the respective points to the outer circle (measured along the radii). The circles will be enveloped by another circle (the reflected wave-front), the center of which is at the other focus of the ellipse.

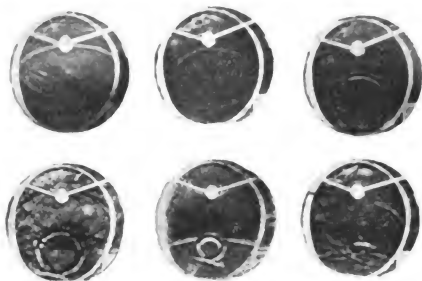


FIG. 33.

In Fig. 33 we have a series of photographs showing a sound-wave starting at the focus of an elliptical mirror. The expanding sphere is seen to have been changed by reflection into a contracting sphere, which shrinks to a point at the other focus. The same thing can be shown by making a shallow, flat-bottomed, elliptical dish of wood, filling it with mercury, and touching the surface of the fluid at one focus of the ellipse.<sup>1</sup>

**Reflection from a Parabolic Mirror.**—If we construct the reflected wave-front in the case of a wave starting at the focus of a parabolic mirror, we shall find that the reflection transforms the spherical wave into a plane wave. The reflected rays being normal to the wave-front are parallel, and are consequently projected in a narrow beam out of the mirror. This is the principle on which the naval search lights are constructed.

Let  $O$  be the focus of the parabolic section of the mirror. Construct the imaginary spherical wave-front (unreflected)  $EF$ , and around points  $A$ ,  $C$ , etc., on the parabola construct circles, or secondary wavelets, with radii equal to the distances of the points from the imaginary wave-front measured along the radii of the circle  $EF$ .

<sup>1</sup> By placing the dish in the sunlight, and receiving the reflected light on a screen, the experiment can be shown to a class.

The wavelets will be enveloped by a straight line, the section of a plane. It is easy to see that this line is straight, or that the reflected wave-front is accurately plane. Every point on the parabola is equidistant from the focus and the directrix;  $\therefore OA = AB$  and  $OC = CD$ . Around  $A$  and  $C$  we have circles with radii equal to  $AE$  and  $CF$  respectively. Now,  $OE = OF$ , being radii of same circle, and  $DG = OF$  and  $BH = OE$ ;  $\therefore DG = BH$  or the enveloping line is everywhere equidistant from the directrix, and consequently parallel to it. The reflected wave is, therefore, a true plane. The projection of a truly

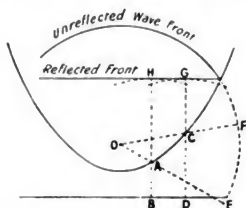


FIG. 34.

plane-wave from a paraboloid mirror is practically never realized, since the source of light is always of finite size—that is, the waves do not all start from the focus.

In Fig. 35 we have the reflection of a sound-wave from a parabolic mirror. The converse of this case is also true. Plane-waves entering a parabolic mirror are transformed by reflection into converging spherical waves which shrink to a point at the focus of the paraboloid.



FIG. 35.

This means that parallel rays, or rays coming from an object situated at a great distance, are brought accurately to a focus by a mirror of this form, or the parabolic mirror is Aplanatic for parallel rays.

**Reflection by Hyperboloid.**—A spherical wave originating in one focus of an hyperboloid is, by reflection, changed into a sphere whose center is at the other focus. The reflected rays appear, therefore, to come from this point, each focus of the hyperboloid being the virtual image of the other. The proof of this is left to the student.

**Reflection from Spherical Surfaces.**—If the reflecting surface be a portion of a sphere the effects are more complicated, except in the special case of waves starting at the center of curvature. The rays do not all meet at a point, as in the cases which we have considered, but envelope a surface known as the *Caustic*. An example of a caustic is the cusped line of illumination seen on the tablecloth when the light of a lamp strikes the inner surface of a silver napkin ring. We have seen that a concave paraboloid brings parallel rays accurately to a focus. A concave spherical mirror does not do this. Rays near the axis come to a focus approximately at a point, but as we recede from the axis we soon find the reflected rays falling

wide of the focus. This effect is known as Spherical Aberration. In constructing telescope mirrors, opticians strive to give the surface as nearly as possible the figure of a paraboloid. The nature of the wave-front in cases where caustics are formed is not at once apparent. The subject is usually treated by ray methods, and we shall accordingly begin by considering one or two examples geometrically, although the evolution of the wave-front, and the relation between the wave-front and the caustic, form a more interesting study.

**Reflection from Convex Spherical Surfaces.**—In studying reflection by ray methods we can regard a curved surface as made up of an infinite number of plane surfaces, for each one of which the law of equal angles holds. Let us take as the first case the reflection of light, radiating from a luminous point, at a convex spherical mirror.

When the light is incident on a small portion of a sphere in a nearly normal direction, we may regard the reflected rays as emanating from a point behind the mirror, the virtual focus, as will appear presently, if not at once apparent. If, however, we employ a large arc of the mirror this does not hold even approximately, and we require an expression for the position of this focus in terms of the angle of incidence.

Let the radius of curvature of the mirror be  $r$ , the distance of the radiating point from the center of curvature be  $b$ , and the angle of incidence be  $i$ ; find  $g$  the distance of the focus  $D$  from the center.

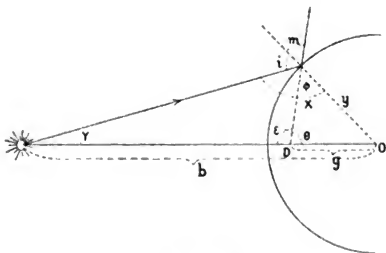


FIG. 36.

$$(1) \quad g : r = \sin \Phi : \sin \theta, \quad (\text{Sides of } \triangle \text{ in ratio of sines of opp. } \angle \text{s.})$$

$$b : r = \sin X : \sin Y.$$

Dividing  $\frac{g}{b} = \frac{\sin \phi \sin Y}{\sin \theta \sin X}$  and  $\sin \phi = \sin x$  (since  $X$  is the supplement of  $\Phi$ ).

$$\therefore \frac{g}{b} = \frac{\sin Y}{\sin \theta} = \frac{\sin (i - \beta)}{\sin (i + \beta)} \quad \text{since} \quad Y + \beta + X = X + i \text{ and} \\ \sin \theta = \sin (\beta + \phi) = \sin (\beta + i),$$

$$\frac{g}{b} = \frac{\sin i \cos \beta - \cos i \sin \beta}{\sin i \cos \beta + \cos i \sin \beta}.$$



front is rather intricate, and a construction that will enable us to follow it in its evolution is desirable. This can be accomplished by employing a second method.

$ABC$  is the mirror,  $AOC$  the plane-wave (Fig. 37). Around points on  $ABC$  as centers describe circles tangent to the wave. These circles will be enveloped by another surface,  $ADE$ , below the mirror (the orthogonal surface). If we erect normals on this surface, we have the reflected rays, and if we measure off equal distances on the normals, we have the reflected wave-front. By drawing the orthogonal surface we avoid the complication of having to measure off the distances around a corner. The orthogonal surface is an epicycloid formed by the rolling of a circle of a diameter equal to the radius of curvature of the mirror on the mirror's surface, and the normals can be erected by drawing the

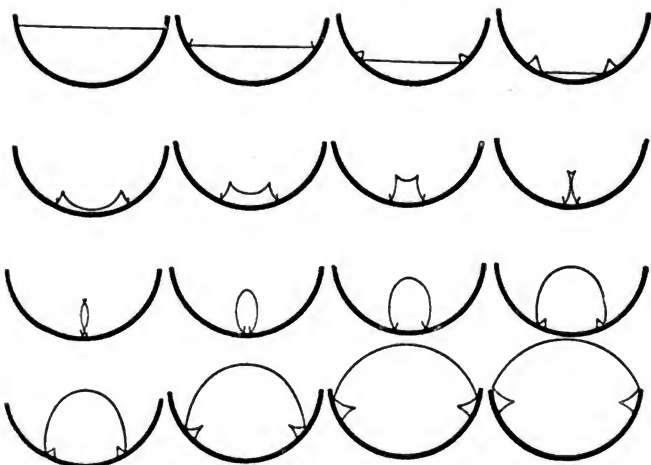


FIG. 38.

arc  $FG$  (the path of the center of the generating circle), and describing circles of diameter  $BE$  around various points on it. A line joining the point of intersection of one of these circles with the epicycloid, and the point of tangency with the mirror, will, when produced, give a reflected ray; for example,  $JK$  produced, for circle described around  $H$ . This construction once prepared, the series of wave-front pictures can be very quickly made. Three or four sheets of paper are laid under the construction and holes punched through the pile by means of a pin, at equal distances along each ray (measured from the orthogonal surface).

The center of the mirror and the point where its axis meets the surface are also indicated in the same manner. The sheets are now separated, and corresponding pin-holes are united on each sheet by a broad black line, which represents the wave-front. After a time it becomes necessary to consider double reflections, and to do this we

are compelled to construct twice-reflected rays (indicated by dotted lines), and measure around a corner each time.

The geometrically constructed fronts are shown in Fig. 38. These are diagrams taken at intervals on a kinetoscope film prepared by the author for illustrating the wave evolutions.

About one hundred constructions were made, in the manner just described, and photographed in succession on the film, which, when run through the machine, gives us the moving wave on the screen in a most graphic manner.<sup>1</sup>

A series of photographs of a sound-wave entering a hemicylindrical mirror<sup>2</sup> is shown in Fig. 39, and it will be seen that the forms are

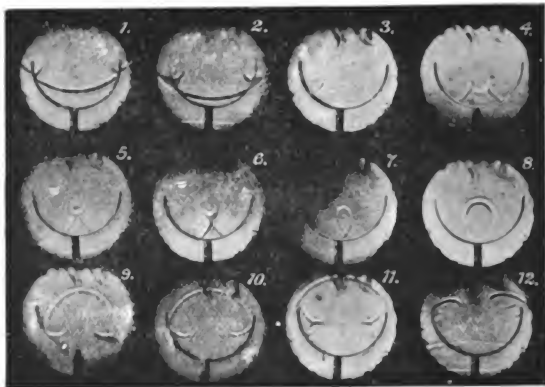


FIG. 39.

identical with the geometrical constructions. The reflected front is cusped, and in certain stages (No. 4) has a form not unlike a volcanic cone with a bowl-shaped crater.

In Fig. 40 we have a number of wave-fronts in different stages of reflection, and it is at once apparent that the cusp traces the caustic surface, indicated by a dotted line.



FIG. 40.

This gives us at once a *physical*, as distinguished from a *geometrical*, definition of a caustic, which is a surface traced by a moving cusp of the wave-front.

Let us examine a little more carefully the manner in which this cusped wave is propagated. A superficial examination of the forms

<sup>1</sup> "Photography of Sound-Waves, and Kinematographic Demonstration of Reflected Wave-Fronts." Wood, *Proceedings Royal Society*, Vol. 66.

<sup>2</sup> Cylindrical surfaces have been used instead of spherical for obvious reasons. The sectional view is of course the same in each case.

might lead one to imagine that the bowl of the crater collapsed to a point at the principal focus of the mirror. This can of course only be true in the case of a concave spherical wave, which is only given by a parabolic mirror. We shall find as a matter of fact, if we examine the geometrical construction, that the cusp of the wave, or the rim of the crater, which traces the caustic as we have seen, is continuously passing through a focus. In other words, the curvature of the crater increases as we go from the bottom to the rim, at which point the radius becomes zero. The inner edge is then continually passing through a focus and appearing on the outside, building up, as it were, the sides of the cone. These wave-fronts were drawn by constructing the orthogonal surface, which was shown to be, in section, an epicycloid formed by rolling a circle, whose diameter was equal to the radius of curvature of the mirror, around the outside of the mirror. The evolute of this curve is the caustic, itself an epicycloid, and the reflected wave-fronts form a family of parallel curves, which are the involutes of the caustic.

Though the caustic and orthogonal surface (evolute and involute) are similar epicycloids, the reflected wave-fronts, or parallels to the orthogonal surface, are not epicycloids. It may be well to point out here an error that sometimes appears in text-books on Optics, namely, the assumption that the wave-front (say in the case of a spherical wave refracted at a plane surface) is an hyperboloid in the second medium, because the caustic is the evolute of an hyperboloid. An hyperboloid wave will not propagate itself as an hyperboloid, nor an ellipsoidal wave as an ellipsoid (except in an anisotropic medium), the parallels to a conic being in general curves of the eighth degree. In the case above cited, we should speak of the wave-fronts after refraction as the parallels to an hyperboloid.

Let us suppose the wave to be just entering the mirror. The form of the portion which has already suffered reflection is a cusp extending around the upper edge of the hemisphere (Fig. 41). The upper branch of the cusp is concave upward, and is the portion of the wave which left the reflecting surface and has passed through a focus. The lower branch is concave downward, or in the direction of propagation, and represents the portion of the wave which has just left the surface and is on the way to its focus. The radius of curvature increases from zero as we go away from the cusp-point along either branch, as has been said before. This cusped wave moves down the mirror, the lower branch being continually replenished by consecutive portions of the incident wave as it encounters the mirror, the upper branch being continually added to by elements of the lower branch as they pass through their foci at the cusp.

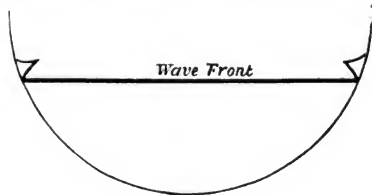


FIG. 41.

As has been said, the cusp traces the caustic surface, and since the wave is always coming to a focus on the cusp, the increased illumination along the caustic is accounted for.

The difference between a parabolic reflector and a spherical one is now clear. The former gives us a spherical wave which will collapse to a point, the latter an approximately spherical wave near the axis only, the rest of the wave being incapable of shrinking to a point.

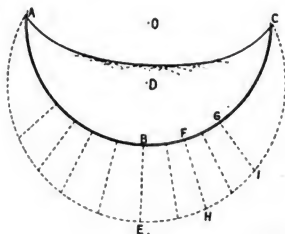


FIG. 42.

We will next consider the opposite case, starting a spherical wave at the principal focus of the concave, spherical mirror, and determine the form of the reflected wave, which we found to be plane in the case of the paraboloid. We will use the first method, constructing a

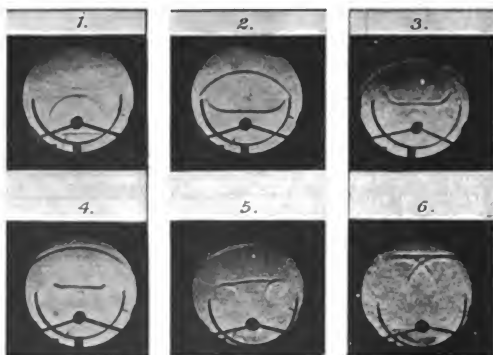


FIG. 43.

single reflected front only. Let  $ABC$  be the mirror, with its focus at  $D$ , where the wave originates. Draw dotted lines representing the wave in a subsequent position (Fig. 42), and around the points  $B$ ,  $F$ ,  $G$ ,  $H$  on the mirror describe arcs with radii equal to  $BE$ ,  $FH$ , and  $GI$  respectively. These arcs will be enveloped by the

reflected wave-front, which is approximately plane near the axis of the mirror, curling up at the edges, however, the whole resembling a shallow, flat-bottomed saucer.

Roughly sketch in a few normals to this wave, and determine its form in subsequent positions, and it will be found that the curved sides of the saucer run in to a focus around the edge of the flat bottom (a ring focus), disappearing for a moment and then reappearing on the under side, but turned over the other way. As the wave advances, the flat bottom contracts, and the cusps formed by the union of the turned-over sides with the bottom trace a caustic which has the form of a long, tapering funnel. Photographs of a sound-wave taken under these

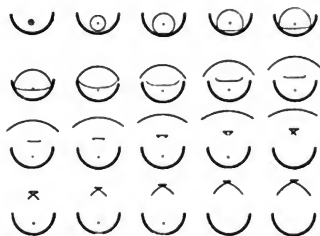


FIG. 44.

conditions, and diagrams from the kinematograph films, are shown in Figs. 43 and 44. The reflected wave-fronts, and the caustic traced by the cusps in this case, are shown in Fig. 44a.

A useful piece of apparatus can be made by silvering the outside of a hemispherical glass evaporating dish or half of a large, round-bottomed flask. The concave mirror thus formed should be mounted on a stand, and a two-candle "pea" electric lamp arranged so that it can be moved along the axis of the mirror.

If we place the lamp in the focus of the mirror, and hold a sheet of ground-glass in front of it at the proper distance, we can show the luminous ring formed by the passage of the sides of the saucer-shaped wave through a focus.

The illumination within the ring is due in part to unreflected light, and in part to the flat portion of the reflected wave.

If we move the lamp to a point midway between the principal focus and the surface of the mirror, we get a ring of intense brilliancy, with but very little light within it.

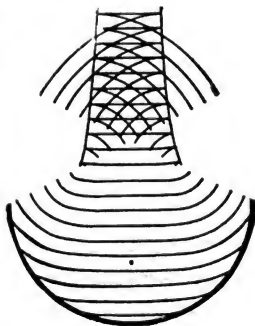


FIG. 44a.

The wave-front constructions for this condition are shown in Fig. 45, the distribution of energy being roughly shown by shading the reflected wave-fronts.

It will be noticed that the reflected wave-fronts in this case possess the singularity of a point of inflection where the wave changes from a contracting to an expanding wave. It is interesting to compare this with the cusp condition in the cases previously discussed.

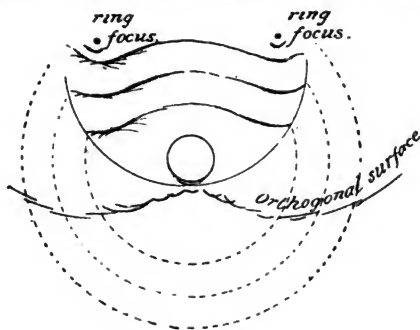


FIG. 45.

**Formation of Images.**—In the formation of a real image by a mirror, the rays radiating from a point are brought together again approximately at a point, or the reflected wave-fronts are converging spheres. A luminous object being made up of a collection of adjacent, radiating points, we have vast numbers of spherical waves entering the mirror from these points, and converging to points similarly situated with reference to one another. The formation of these images, and the study of their position and distance from the mirrors, belongs to geometrical optics. The influence of the form of the wave-front on the distinctness of the image may, however, be considered in connection with what has gone before. We have seen that the paraboloid and ellipsoid are the only surfaces that reflect spherical fronts; other

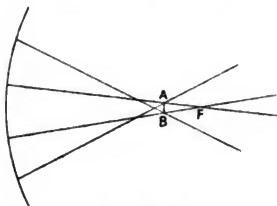


FIG. 46.

curved surfaces give, in general, caustics. If we use only a very small portion of the sphere, and receive the light normally, we have only the cusp of the caustic, which is very nearly a point. There is a certain amount of what is called longitudinal aberration, owing to the fact that the rays reflected from the periphery come to a point situated nearer the mirror than those reflected from the points near the axis. If the image be thrown on a screen, the bright spot formed by the central rays will be surrounded by a circular ring of light formed by the peripheral rays which have already come to a focus

and are diverging once more. The position of sharpest definition lies between the focal points of the marginal and central rays. The circle of light is smallest at this point, and is known as the circle of least confusion. The line  $AB$  in Fig. 46 shows the position of this point.

**Focal Lines.**—Suppose now that the incident light falls on the mirror in an oblique direction. This is the condition if we consider a small portion of the hemispherical, concave mirror far removed from the axis. The reflected wave will come to a focus in a line instead of a point; as we increase the distance of the screen from the mirror the line will decrease in length, increasing in width until it is transformed into a line at right angles to the first.

These lines are known as the primary and secondary focal lines respectively, and can be shown by holding a concave mirror in an oblique position and reflecting the light coming from a small, brilliant source on a screen placed at various distances from the mirror. We can best form an idea of how these lines are formed by considering the question first by a ray method and then by a wave-front method.

Let  $AB$  be the axis of the mirror, near the edge of which a bundle of rays parallel to the axis falls. Construct the reflected rays from a linear strip  $BC$  as shown in the sectional view (Fig. 47): we shall have a flat, converging fan coming to a focus at  $F'$ , then diverging and cutting the axis at  $F''$ . Now rotate the whole figure through several degrees around  $AB$  as an axis; the parallel sheet of incident rays will trace the rectangular incident bundle, the line  $BC$  will trace an approximately rectangular area of the mirror,  $F'$  will move through a short circular arc, approximately a straight line (the primary focal line), while the diverging fan will trace out wedge-shaped portions of space on each side of the axis, which have a common linear boundary at  $F''$  (the secondary focal line). The reflected rays between the two focal lines fill a space similar in shape to the sphenoid of crystallography. If any difficulty is found in forming a picture of this rotation figure in the mind, it can be removed by cutting out of cardboard a diagram representing a section of the mirror, incident and reflected rays as figured above, and mounting it on a knitting needle placed in coincidence with the axis  $AB$ . By rotating the needle through a small angle, the formation of the focal lines and the sphenoidal bundles of rays can be readily seen.

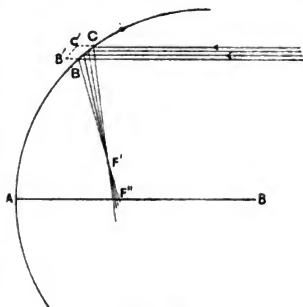


FIG. 47.

Let us next endeavour to explain the formation of focal lines by considering the form of the wave-front.

The curvature of the wave front as it leaves the mirror under these conditions is different along different meridians. If we cut a piece out of the side of a hen's egg we shall have something of analogous form.

Let  $AB$  be the direction of greatest curvature and  $CD$  that of least curvature (Fig. 48). To start with, suppose the curvature be equal along all lines parallel to  $AB$ , and suppose all lines parallel to  $CD$  to be straight. This will give us a cylindrical wave which will come to a linear focus at  $F_1$ , the length of the line being equal to the length of the cylindrical wave.

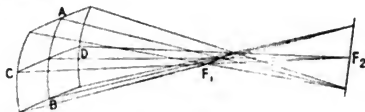


FIG. 48.

If we start with a square wave-front we shall find it contracting to a line as we approach  $F_1$ , and expanding beyond  $F_1$ , first as a horizontal rectangle, then a square, and finally a vertical rectangle. Now let us impress a slight curvature parallel to  $CD$ . The result of this will be that our square will now contract in both directions, only in one less rapidly than in the other, and the line at  $F_1$  into which it shrinks will be shorter than before, and instead of being straight will be slightly concave towards  $F_2$ . From here it can be regarded as an expanding wave in a vertical plane, and a contracting wave in a horizontal plane. It is easily seen that the line at  $F_1$  will now open out, first into a horizontal rectangle, as before, then a square, (as the two sides *closing in* become equal to the top and bottom *moving out*), then a vertical rectangle, and finally a vertical line at  $F_2$ , as the sides come together.

It is interesting to enquire as to the nature of the rectangular wave surface between  $F_1$  and  $F_2$ . From its nature we see that it must be concave towards  $F_2$  in the horizontal plane, and convex in the vertical, the surface resembling a small portion cut out from the inside of a thick cylindrical ring. We can, indeed, find surfaces of this form on our geometrically constructed wave-fronts.

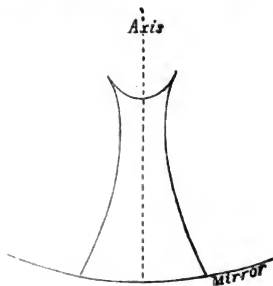


FIG. 49.

Consider the diagram shown in Fig. 49 (which will be recognized as the "volcanic cone" form), remembering that the complete wave-front is formed by the rotation of this figure around the axis of the mirror.

The bowl of the crater is concave along every meridian, but it is at once apparent that any portion of the outer slope has the required saddle-shape, being concave in horizontal planes and convex in vertical planes. From this it is evident that the outer wall of the volcanic cone, before it crosses the axis of the mirror, always represents the portions of the wave-front between the primary and secondary focal lines.

That this is true is evident, when we recollect that the first focal line is formed by the intersection of rays on the caustic surface, or,

regarded from the wave point of view, by the passage through their foci on the cusp of the wave, of adjacent elements of the wave-front. The second focal line lies on the axis of the mirror; consequently the wave-front between the lines is that portion of the surface which has passed through a focus on the cusp, but which has not crossed the axis.

It will be found that a small glass model of the wave-front, shown in cross section in Fig 49, is extremely useful in making the whole matter clear. It can be made by drawing down a large thin tube, melting the end down flat, and then sucking it in a little.

**Fermat's Principle.**—We sometimes find it stated that a ray of light in passing from one point to another by way of either a reflecting or refracting surface, chooses a path such that the time of transit is a minimum. This principle was stated by Fermat more than two centuries ago. It is true, however, only for plane surfaces. In the case of reflection from a plane surface the incident and reflected rays make equal angles with the normal, and we know from elementary geometry that this path is the shortest that can be traced from one point to the other by way of the surface. The same is true for convex surfaces, but for concave surfaces we find that in certain cases the path is a maximum instead of a minimum.

That the path is sometimes a maximum can be seen by the construction shown in Fig. 50. We will consider the passage of a ray

from the point *A* to the point *B* by way of the reflecting spherical surfaces *CDE* and *FGH*. Around the points *A* and *B* as foci we construct an ellipse which we will suppose to be tangent to the two reflecting surfaces at *D* and *G*. This ellipse is an aplanatic surface for rays issuing from either focus, consequently the time of transit of a ray from one focus to the other by way of the elliptical surface is the

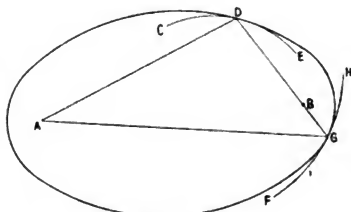


FIG. 50.

same for every point on the surface. Now the ellipse and the two spherical surfaces have common tangent planes at *D* and *G*, consequently *D* and *G* will be the points on the spherical mirrors so oriented that they can reflect rays from *A* to *B*. It is easy to see that the path *AGB* is shorter than any other path between *A* and *B* by way of the sphere which is exterior to the ellipse, while in the case of the other sphere the path actually pursued (*ADB*) is longer than any other path which we can draw from *A* to the surface and from thence to *B*. In this case we see that the path chosen by the ray is such as to make the time of transit a minimum. The conditions for a maximum or minimum may be expressed by saying that the variation of the time of transit with the change of path, ceases at the points for which the path is either a maximum or minimum, or  $\delta(AD + DB) = 0$ . This matter will be further discussed under refraction.

## CHAPTER IV.

## REFRACTION OF LIGHT.

In the preceding chapter we have discussed the forms and behavior of the wave-fronts reflected back into the first medium, when light falls upon the boundary between two media of different optical density. A portion of the energy, however, always passes into the second medium, except, perhaps, in the special case of total reflection, and even in this case mathematical analysis shows us that there is a disturbance beyond the boundary, though only penetrating to a distance of a few wavelengths. The energy crossing the boundary may either be absorbed by the second medium, or propagated according to the laws governing luminous disturbances in it.

In the present chapter we shall consider only the case of wave propagation in an isotropic medium, or one in which the velocity of propagation is independent of direction. Later on we shall investigate the refraction of light in bodies in which the velocity is different in different directions.

We will begin by considering the refraction of a plane wave at a plane surface.

**Refraction of Plane Wave at Plane Surface.**—Suppose a plane wave incident at an angle of  $45^\circ$  on a flat surface of glass, and assume the velocity in the glass to be less than the velocity in air, as we shall subsequently show it to be.

The various points on the glass surface become in succession centers of secondary disturbances as they are struck by the incident wave.

These secondary wavelets spread out in both media, and it has been shown by Huygens's construction that the reflected wave is the envelope of those spreading out in the first medium. If we apply the same construction to the second medium, supposing for the sake of simplicity that the velocity of the wave propagation in it is only one half as great

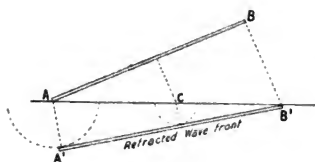


FIG. 51.

as in the first, the wavelets in the glass will have radii half as large as the corresponding wavelets in air, and the enveloping surface or

refracted wave-front is turned through an angle (Fig. 51). The rays, or normals of the wave, are therefore bent an equal amount.

It was determined in 1621 by Snell that in every case of refraction the incident and refracted rays make such angles with the normal to the surface, that the ratio of their sines is constant for any two given media. Snell's law of refraction we now know holds only for isotropic media. It can be easily deduced from Huygens's construction in the following way.

*Construction.*—The angle of incidence  $i$  is the angle between the incident ray and the normal to the surface. It is also the angle between the wave-front and the surface. The same is true for the angle of refraction. Let the velocity in air be  $v$ , and the velocity in glass be  $v'$  (equal to say  $\frac{v}{2}$ ), and let  $t$  equal the time required for the wave in air to traverse the distance  $BB'$  (Fig. 51). Then  $BB' = vt$ , and the radius of the secondary disturbance around  $A$  in the glass will be  $AA'$  or  $\frac{BB'}{2}$ . We have then  $\frac{BB'}{AB'} = \sin i$  and  $\frac{AA'}{AB} = \sin r$

or  $\frac{\sin i}{\sin r} = \frac{BB'}{AA'} = \frac{v}{v'} = \text{constant, in this case 2.}$

This constant is the relative refractive index between the media, and the above relation holds for every value of  $i$ , if the second medium is the one in which the disturbance travels at a lesser velocity. The refractive index is usually designated by  $\mu$ , and in the above case is of course 2, which is higher than is usually the case.

**Total Reflection.**—We have seen that in passing from a rare to a denser medium a refracted ray always exists, no matter how great the angle of incidence. This is not true if we reverse the conditions, for now the relative refractive index will be less than one, and we shall find that, if  $i$  exceeds a certain value,  $\sin r$  is greater than one. But no angle has a sine greater than one, therefore there can be no refracted ray. Let us apply Huygens's construction to the case. The secondary waves in the second medium will have radii greater than the corresponding ones in the first medium, since their velocity of propagation is greater. By dividing the radii of the reflected wavelets by  $\mu$ , we obtain the dimensions of the refracted wavelets. We shall find that, up to a certain value of  $i$ , these secondary disturbances will intersect the surface *within* the projection of the incident wave

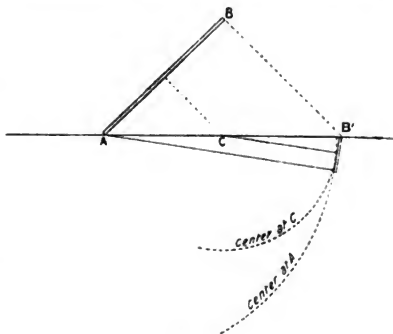


FIG. 52.

upon it; in other words, a tangent plane can be drawn from the point  $B'$  (Fig. 52). At a certain value of  $i$ , however, the secondary wavelet around  $A$  will intersect the surface at  $B'$ , and the same will be true for all of the other wavelets: (since, if the short leg of one of the right triangles divided by  $\mu$  gives us the hypotenuse, the same will be true of all the other similar right triangles). The tangent plane drawn from  $B'$  to these wavelets will be normal to the surface, and will touch the wavelets in a single point only (in the sectional diagram). The refracted ray therefore will travel along the surface.

The value of  $i$  for which this condition exists can be found by combining  $\sin i = \frac{AC}{AB}$  with  $\frac{AC}{\mu} = AB$ , which gives us  $\sin i = \mu$ ,  $\mu$  of course being the refractive index of the rarer medium with respect to the denser. If, as is customary, we consider  $\mu$  as the refractive index of the denser with respect to the rarer the last equation becomes

$$\sin i = \frac{1}{\mu}.$$

The angle determined by the above expression is known as the *Critical Angle*. If it be exceeded, the secondary wavelets cut the surface beyond the point  $B'$  and no tangent plane can be drawn, therefore no refracted ray exists. The energy in this case is totally reflected.

If in the formula  $\frac{\sin i}{\sin r} = \mu$  we assign various values to  $i$ , and solve for  $r$ , we shall find that every possible value of  $i$  between  $0^\circ$  and  $90^\circ$  gives a corresponding value of  $r$  if  $\mu$  is greater than one, which is always the case when the ray passes from a rare into a denser medium.

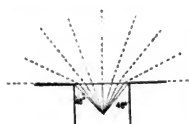


FIG. 53.

Take the case of rays of light entering the level surface of a pond, at all possible incidences from  $0^\circ$  to  $90^\circ$ . The zenith light passes straight down, the horizon light is refracted in a direction given by  $\frac{\sin 90^\circ}{\sin r} = 1.33$ , or  $\sin r = \frac{1}{1.33}$ , which gives for  $r$  a value slightly greater than  $48^\circ$ . In other words, no ray in the water makes an angle with the normal greater than about  $48^\circ$ . The light therefore which enters an eye under water consists of rays embraced by a cone of  $96^\circ$  angular aperture (Fig. 53) instead of  $180^\circ$ , as is the case when the eye in air is directed towards the zenith. If therefore, when submerged in water, the eye be directed towards the surface, the sky appears compressed into a circle of light subtending an angle of  $96^\circ$ , the appearance being precisely as if the water were covered with an opaque roof with a round hole directly over head. If, however, we are in diving armor, and look upward through the plate glass window of the helmet, the illusion of the hole vanishes, for now the horizon rays are refracted back into their original direction on passing into air once more, as is shown in (Fig. 54) the  $96^\circ$  cone, widening out to  $180^\circ$ .

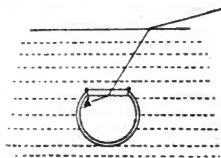


FIG. 54.

**Effect of Refraction on the Width of the Incident Beam.**—It is at once apparent, by reference to the diagrams for the construction of the refracted wave-fronts, that the width of the incident beam is increased in a direction parallel to the plane of incidence, when the rays pass from a rare to a dense medium, or a given portion of the wave-front is spread out over a larger surface. In passing from a dense to a rare medium the reverse is the case, the wave-front being compressed into a smaller area. This change in the width of the beam diminishes or enlarges (in one direction) the apparent size of objects seen under these conditions, producing distortion—a narrow line being broadened, and a circle drawn out into an ellipse. This effect is noticeable under certain conditions when objects are seen through prisms, and the matter will be more completely investigated later on.

**Refraction of a Wave by a Plane Parallel Plate.**—The application of Huygens's construction to the passage of a plane-wave through a glass plate bounded by parallel planes, shows at once that the emergent wave-front is parallel to the incident, no matter how great the angle of incidence. The direction of the ray is therefore unchanged, though each individual ray is shifted to one side by its passage through the plate. Inasmuch as the position of an object at a great distance depends solely on the direction of the parallel rays reaching the eye, it will not be changed by the interposition of a thick plate, at any angle. We can test this by viewing a very distant object through a thick piece of plate glass and turning the plate rapidly around a vertical axis to the right and left. Objects near the plate, however, will be found to shift their apparent position considerably as the plate is turned. If the two objects and the eye be in the same straight line it may seem at first sight as if the intervention of the oblique plate would in no way affect their apparent positions, for parallel rays from the distant object are unchanged in direction by passage through the plate, and the same is true of the rays from the near object. If, however, we remember that the ray is shifted laterally, the difficulty disappears, for the lateral shift, while it does not alter the apparent position of an object at infinity, displaces an object situated at a finite distance. This will be made clear by reference to Fig. 55.

Let  $A$  be a point not far from the plate. It is seen by an eye at  $E$  by means of the rays pursuing the path  $ABCE$ , and its apparent position is apparently  $A'$ . If the oblique plate be removed, the point  $A$  will be seen by the direct pencil of rays  $AE$ , and will appear in its true position  $A$ .

If we make the same construction for parallel rays coming from a distant point we shall find that the apparent position, or the *direction* from which the rays by which it is seen come, is unchanged.

A better way perhaps of looking at the problem is from the point of view of the wave front, and since the point  $A$  is near the plate, we require the refraction of a spherical wave at a plane surface.

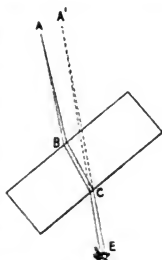


FIG. 55.

**Refraction of a Spherical Wave at a Plane Surface.**—Suppose a spherical wave originating at  $O$  (Fig. 56) to be refracted at the plane

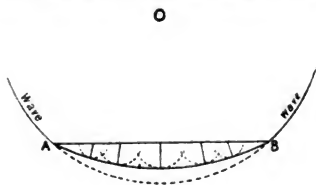


FIG. 56.

surface  $AB$ . If we construct the refracted wave-front by the method of Huygens, making the points on the refracting surface the centres of secondary wavelets whose radii are found by dividing their distances (measured along rays) from the wave in its unrefracted position, by the refractive index of the medium, we shall find that the incident

wave is flattened down into what at first sight appears to be a sphere of less curvature. Let us investigate the form of the refracted wave, which is sometimes erroneously stated to be an hyperboloid.

Suppose light diverging from  $O$  to be refracted at the surface  $AP$  (Fig. 57). Draw an incident ray  $OP$ , which is refracted in the direction  $PQ$ . Draw  $OD \perp$  to the surface, and produce it to  $O'$ , making  $OD = DO'$ . Draw a circle passing through the points  $O$ ,  $P$ , and  $O'$ , and produce  $PQ$  backwards until it intersects the circle at  $M$ , and the prolongation of  $OO'$  at  $J$ .  $\angle DOP =$  incidence  $\angle$ , also  $\angle OMP$ , both being measured by arc  $OP$ . Moreover  $\angle OMJ =$  incidence  $\angle$ , being equal to  $\angle OMP$ , since the normal at  $M$  bisects the  $\angle OMO'$ . We can now write

$$\frac{\sin i}{\sin r} = \frac{OJ}{O'M}$$

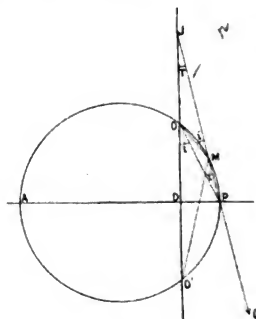


FIG. 57.

since  $\sin O'MJ = \sin i$  (supplementary angles) and the sides of  $\triangle$  in ratio of sines of opposite angles.

$$\therefore \mu = \frac{OJ}{O'M} = \frac{OJ}{OM};$$

$$\therefore \mu = \frac{O'J - OJ}{O'M - OM} = \frac{OO'}{O'M - OM} \text{ and } O'M - OM = \frac{OO'}{\mu} = \text{constant.}$$

This same relation holds no matter where  $P$  be taken, consequently the locus of  $M$  is an hyperbola having  $O$  and  $O'$  for foci, and the refracted ray  $PQ$  is normal to the hyperbola at  $M$ . The hyperbola is therefore the orthogonal surface of the refracted wave, since it is everywhere normal to the refracted rays, and the refracted wave-fronts are parallel curves, located by measuring off equal distances on the rays from the hyperbola. They will *not* be themselves hyperbolae, for the parallels to a conic are in general curves of the eighth degree. The evolute of the hyperbola is the caustic of the refracted wave, in this case virtual of course. After refraction then, the different elements

of the wave-front appear to come from points distributed along the caustic. If then we transfer our eyes from one position of the wave-front to another, the position of the radiant point in space will apparently alter. The same thing is true when the waves are refracted from a dense to a rare medium, the caustic in this case being the evolute of an ellipse, and the refracted waves parallels of an ellipse. The formation of a caustic under these conditions is shown in Fig. 58.

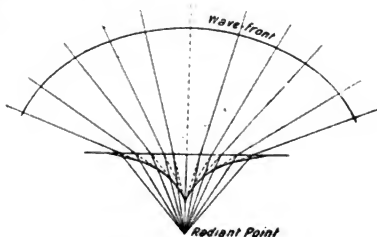


FIG. 58.

A small portion of the wave around the ray leaving the surface normally comes from the cusp of the caustic, which we may regard as a point, consequently this portion of the wave is approximately spherical.

The cusp of the caustic from which this portion comes is elevated above the true radiant point, consequently the refraction appears to bring the point nearer the eye. The bottom of a vessel of water consequently appears to be nearer than it really is.

**Fermat's Law.**—In the case of reflection we have seen that the path of a ray from one point to another by way of a reflecting surface is either a maximum or a minimum. The same is true in the case of refraction, as we shall now show. If the refracting surface is plane, the time of transit is a minimum, and we have what is known as the principle of least time. If the refracting surface is curved, the time may be either a maximum or a minimum, according to whether the refracting surface lies within or without the aplanaic surface, the same as in the case of reflection. Fermat's law may be deduced from Snell's law by the maximum and minimum method of the calculus.

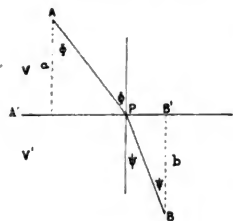


FIG. 59.

Let  $A$  be a luminous point at height  $a$  above the refracting surface  $A'B'$ , and  $B$  a point illuminated by a disturbance reaching it by way of any point on the refracting surface, for example, over the path  $APB$  (Fig. 59). The time of transit obviously changes with the position of  $P$ . We will start by assuming it to be a maximum or minimum, and see if the ordinary law of refraction follows.

From  $A$  and  $B$  drop perpendiculars on the refracting surface, of length  $a$  and  $b$  respectively: let the distance  $A'B' = p$ , then  $A'P = x$  and  $PB = p - x$ . Call the velocities in the two media  $v$  and  $v'$ , then the time along  $AP$  is  $\frac{AP}{v}$  and the time along  $PB$  is  $\frac{PB}{v'}$ . The whole time, which we require to be either a maximum or a minimum,

$$t = \frac{AP}{v} + \frac{PB}{v'}, \text{ or } t = \frac{\sqrt{a^2 + x^2}}{v} + \frac{\sqrt{b^2 + (p-x)^2}}{v'},$$

$$\frac{dt}{dx} = \frac{x}{v\sqrt{a^2 + x^2}} - \frac{p-x}{v'\sqrt{b^2 + (p-x)^2}} = 0.$$

Now  $\frac{x}{\sqrt{a^2 + x^2}} = \sin \Phi$ , and  $\frac{p-x}{\sqrt{b^2 + (p-x)^2}} = \sin \Psi$ ;

$$\therefore \frac{\sin \Phi}{v} = \frac{\sin \Psi}{v'}, \text{ or } \frac{\sin \Phi}{\sin \Psi} = \frac{v'}{v} = \text{Const.}, \text{ which is Snell's law.}$$

A second differentiation, or in this case a mere inspection of the figure, shows us that the time is a minimum.

We will now examine the case of refraction by a curved surface, following a demonstration by Czapski.

In Figure 60 let  $AB$  be a portion of an aplanatic refracting surface.

Though we have not yet discussed this surface, we can make use of it in the present case. It is the surface which will bring all rays emanating from  $O$  accurately together at  $O'$ , or change the expanding spherical waves into contracting spheres with centres at  $O'$ . The times of transit over all paths from  $O$  to  $O'$  by way of the aplanatic surface (a surface capable of bringing rays together at a point) are equal, or if  $n$  and  $n'$  be the refractive indexes of the

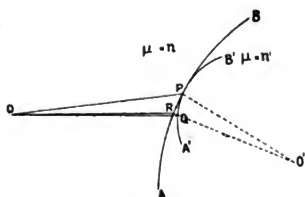


FIG. 60.

media the reduced path is  $n(OP) + n'(PO') = \text{Constant}$ . (By reduced path we mean the length of path in vacuo, which will contain the same number of waves as the number contained in the actual path.)

Now suppose that we have a refracting surface  $A'B'$  of greater curvature than the aplanatic, which it touches at  $P$ . The ray incident at  $P$  is obviously the one which passes through  $O'$ , and we are to ascertain whether the reduced path is greater or less than an hypothetical path through some other point on  $AB$ . Let this point be at  $Q$ , for which the reduced path will be  $n(OQ) + n'(QO')$ . The ray which reaches  $O'$  by way of the point  $R$  on the aplanatic surface has a reduced path  $n(OR) + n'(RO) + n'(QO')$  and the difference between them is

$$[n(OQ) + n'(QO')] - [n(OR) + n'(RO) + n'(QO')],$$

or

$$n(OQ - OR) - n'(RQ).$$

Now  $(OQ - OR) < RQ$  (sides of a triangle).

Therefore, since  $n < n'$ ,  $n(OQ - OR) < n'RQ$ , and the path by way of  $Q$  is less than the path by way of  $R$ ; but the latter is the same as the actual path by way of  $P$ , therefore that path is a maximum. In the same way we may show that if the refracting surface has a curvature less than the aplanatic, i.e. lies without it, the path will be a minimum.

**Refraction by a Prism.**—In the case of refraction by a prism we have to determine the deviation of a ray or wave-front, by passage through a medium bounded by two planes which make an angle with each other; this angle is called the angle of the prism. If the refractive index of the prism be greater than that of the medium in which it is immersed, as is usually the case, the deviation of the ray is always away from the vertex, that is, towards the base of the prism. This is obvious at first sight, except perhaps in the case in which the incident ray falls on the prism in the direction shown in Figure 61, for here the deviation at the first surface is *towards* the vertex, while that at the second is towards the base, the final direction depending on the relative magnitude of these two deviations. The angle of refraction at the second surface is greater than that at the first, and since the deviation increases as this angle increases, the deviation towards the base at the second surface is greater than the deviation towards the apex at the first.

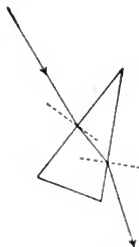


FIG. 61.

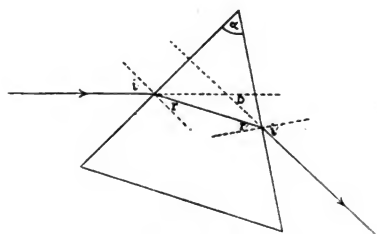


FIG. 62.

We will now derive an expression for the deviation. Let the angle of the prism be  $\alpha$  (Fig. 62) and let  $i$  and  $r$  be the angles of incidence and refraction at the first surface,  $i'$  and  $r'$  at the second. The deviation is obviously  $D$ , the angle between the emergent ray and the incident ray. The deviation at the first surface is  $i - r$ , at the second  $i' - r'$ , while

$$\begin{aligned} D &= (i - r) + (i' - r') \\ &= i + i' - (r + r'). \end{aligned}$$

But  $r + r' = \alpha$ , since  $\alpha$  + the two base angles of the prism = 2 rt.  $\angle$ s and  $(r + r') +$  the base angles = 2 right angles.

$$\therefore D = i + i' - \alpha.$$

This formula holds for the condition shown in the previous figure, except that in this case the negative sign must be prefixed to the angles  $i$  and  $r$ . Suppose the angle  $\alpha$  becomes zero, the prism then becoming a plane parallel plate. The deviation then becomes  $i + (-i') - 0$ , which is equal to zero (since in this case  $i = i'$ ).

**Refractive Index of a Prism.**—Let us now suppose the angle of incidence to be such that the ray passes through the prism parallel to the base. In this case  $i = i'$  and  $r = r'$ , and if we can measure  $D$  and know the angle  $\alpha$  we can easily determine the refractive index of the prism.

$$\text{We have } D = 2i - \alpha, \text{ or } i = \frac{\alpha + D}{2}, \text{ also } r = \frac{\alpha}{2}.$$

Substituting these values in  $\mu = \frac{\sin i}{\sin r}$  we have  $\mu = \frac{\sin \frac{1}{2}(a + D)}{\sin \frac{1}{2}a}$ .

We must now find some method of arranging the angle of incidence so that the path of the ray through the prism will be parallel to the base, *since it is only for this condition that the above formula holds.*

This adjustment is very easily made, for the deviation of the ray can be shown to be a minimum when the passage through the prism is symmetrical. There are several methods of proving this, the most direct and rational being the method of maxima and minima of the calculus. We must obtain an expression showing the change of deviation with the change of the angle  $r$ , and by equating this to zero derive the condition for a maximum or minimum. In other words

$$\frac{dD}{dr} = 0 \text{ and } \frac{d^2D}{dr^2} > 0 \text{ (condition for minimum),}$$

$$\frac{dD}{dr} = \frac{d(i + i' - a)}{dr} = 0.$$

We have  $\sin i = \mu \sin r$ , and  $\sin i' = \mu \sin r' = \mu \sin (a - r)$ .

Then  $i = \sin^{-1}(\mu \sin r)$  and  $i' = \sin^{-1}[\mu \sin (a - r)]$ ;

$$\therefore D = \sin^{-1}(\mu \sin r) + \sin^{-1}[\mu \sin (a - r)] - a,$$

$$\frac{dD}{dr} = \frac{\mu \cos r}{[1 - \mu^2 \sin^2 r]^{\frac{1}{2}}} - \frac{\mu \cos (a - r)}{[1 - \mu^2 \sin^2 (a - r)]^{\frac{1}{2}}} = 0.$$

Expressing the cos by the sin we have

$$\frac{\mu(1 - \sin^2 r)^{\frac{1}{2}}}{[1 - \mu^2 \sin^2 r]^{\frac{1}{2}}} - \frac{\mu[1 - \sin^2 (a - r)]^{\frac{1}{2}}}{[1 - \mu^2 \sin^2 (a - r)]^{\frac{1}{2}}} = 0.$$

Equating the above terms, multiplying the numerator by the denominator and cancelling, gives us,

$$(\mu^2 - 1) \sin^2 r = (\mu^2 - 1) \sin^2 (a - r),$$

$$\text{or } r = a - r \text{ and } r = \frac{a}{2} = r'.$$

By symmetrical passage  $r = r'$ ;  $\therefore D$  is either a maximum or minimum. A second differentiation gives

$$\frac{d^2D}{dr^2} = \frac{\mu^2 - 1}{\left(1 - \mu^2 \sin^2 \frac{a}{2}\right) \left(1 - \mu^2 \sin^2 \frac{a}{2}\right)^{\frac{1}{2}}}$$

If  $\mu > 1$  all the factors are positive and the whole expression is greater than 1, therefore  $D$  is a minimum.

There is another condition for which we can get a simple expression for  $\mu$ , namely: when either the incident or emergent ray is normal to the surface of the prism.

If  $i = 0$ ,  $r = 0$ , and  $r' = a$ ,  $D = i' - a$ .

Therefore

$$\mu = \frac{\sin (D + a)}{\sin a}.$$

**Magnifying Power of Prisms.**—When the prism is set at minimum deviation the widths of the incident and emergent beams are the same, otherwise not. For example: in Figure 63 when the incident beam falls normally on the first surface, and leaves the second surface at a large angle with the normal, the width of the beam has been contracted. If we view an object under these conditions, the eye being placed in the contracted beam, we shall find that it is magnified in the direction in which the beam has suffered contraction. A circular opening in a card backed by a sodium flame is a suitable object, and will be found to appear as an ellipse. If an achromatic prism is available (see page 87) a circular white object can be used, when the effect is very striking. If on the other hand the incident light makes a large angle with the normal, the emergent wave-front is expanded in width, and if the eye be placed in it, the object will appear decreased in size in this dimension, a circular card appearing as if turned edgewise. Brewster suggested that by using two achromatic prisms at right angles to each other, magnification might be shown in both directions, and the action of a telescope imitated.

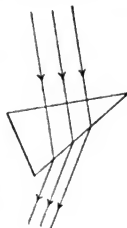


FIG. 63.

Lord Rayleigh has given a very neat demonstration of magnifying power, based on Fermat's law, which is applicable to telescopes as well as to the case just cited. It proves by a wave-front method that

the contraction of a beam of light, or the compression of a wave-front, causes magnification. Consider a wave-front of width  $AB$  (Fig. 64) refracted at the surface  $CP$ , and compressed thereby to width  $A'B'$ . By Fermat's law the time of transit over the path  $ACA'$  is equal to the time of transit over the path  $BPB'$ , being a minimum in each case. This we may express by saying

that  $\int \mu ds$  (the reduced path) is the same

along each ray. If from any cause  $B$  is retarded relatively to  $A$ , say an amount  $BE$ ,

$B'$  will be retarded an equal amount relatively to  $A'$ , namely  $B'E' = BE$ .<sup>1</sup> If this retardation be considered as represented by a rotation of the wave-front  $AB$  through angle  $\theta$  it will be measured by  $(AB)\theta$ . The wave-fronts  $AB$  in the two positions can be thought of as two separate fronts coming from two distant stars subtending an angle  $\theta$  at the point of observation. The retardation of  $B'$  must be of the same amount, consequently the rotation of the wave-front  $A'B'$  will be much greater than  $\theta$ , being measured by  $\Phi(A'B')$ . Since the retardations are equal we can write

$$\theta(AB) = \Phi(A'B') \quad \text{or} \quad \frac{\theta}{\Phi} = \frac{A'B'}{AB}.$$

<sup>1</sup> This is of course only true if  $AB$  and  $A'B'$  are in the same medium: the first surface of the prism, parallel to  $AB$  is not represented.

Now  $\Phi$  is the angle formed by the rotation of  $A'B'$  the compressed wave-front, consequently we may regard it in its two positions as two fronts coming from stars which subtend an angle  $\Phi$ , as much greater than  $\theta$  as  $AB$  is greater than  $A'B'$ .

The same reasoning can be applied to telescopes, the compression here being symmetrical, a plane wave of large area emerging from the eye-piece as a plane wave of small area, the magnifying power being equal to the ratio of the widths of the stream of light before and after entering the telescope.

**Refraction by a Lens.**—In the chapter on reflection it has been shown that a parabolic mirror transforms a plane wave into a contracting spherical wave, while an ellipsoidal mirror exerts the same action on spherical waves originating at one of the foci. It is possible to construct refracting surfaces having the same property. We will begin by computing the refracting surface, which shall be aplanatic for spherical waves.

Let  $O$  (Fig. 65) be the luminous point, and  $O'$  the conjugate focus

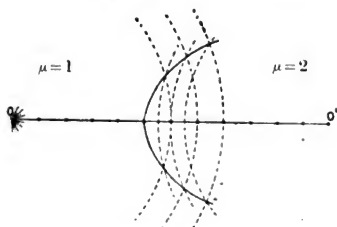


FIG. 65.

where the converging waves are to shrink to a point. By Fermat's principle the reduced paths along the different rays will be equal, and the disturbances will all reach  $O'$  in the same phase, resulting in intense illumination. Suppose  $O$  to lie in a medium of ref. index 1, practically in air, while  $O'$  is in a medium of ref. index 2 between which we require an aplanatic surface of separation. Let  $r$  and  $r'$  be

the distances of any point on the surface from  $O$  and  $O'$ , then  $r + \mu r' = \text{Constant}$ , the equation of a Cartesian oval. Choose a point on the line joining  $O$  and  $O'$  such that  $r = 5$  and  $r' = 7$ . The constant for this particular case will be 19. Now describe around  $O$  a circle of radius 6 and around  $O'$  a circle of radius of  $\frac{19-6}{2}$ . The intersection of these circles will give two more points on the aplanatic surface, which can be gradually built up by giving to  $r$  constantly increasing values.

The general form of the equation of a Cartesian oval is

$$\mu r + \mu' r' = \text{Constant, from which we get } \mu \frac{dr}{ds} + \mu' \frac{dr'}{ds} = 0.$$

In the case just considered the conjugate foci lie in different media. If they are to be in the same medium we require an intervening medium capable of effecting the required change in the form of the wave-front. We thus come to the Aplanatic Lens.

Spherical light waves, originating at a point in air, are to be transformed by a lens into converging spheres which come to a focus at another point also situated in air. Suppose the lens to be midway

between the two points and the curvature of its two surfaces the same. The spherical wave will be changed into a plane wave by the first surface, consequently the simplest way to construct the form of the lens will be to trace the surface aplanatic for a plane wave by the method given above, modifying it, however, in such a way as to make the sum of any path measured along a ray from the plane wave to the aplanatic surface, and the reduced path from this point to the focus, a constant. These aplanatic surfaces are, however, of very little practical importance, for they can only be reproduced approximately, and then only when the departure from a spherical surface is very slight. In the process of lens making the surfaces which are being ground together assume of their own accord a spherical form, since two surfaces, to fit together in all positions, must be of constant curvature. Lenses with spherical surfaces do not bring rays accurately to a point, or in other words do not give converging waves which are truly spherical. This results in what is known as spherical aberration, which has been treated sufficiently for the purposes of this book under *reflection*. A spherical surface may be made approximately aplanatic by local grinding, if the amount of material to be removed be not too great. This process is known as correcting the lens for spherical aberration, or figuring, and is largely a "cut and try" operation.

**Focal Length of a Lens.**—The distance from the middle of the lens to the focus varies with the distance of the source of light: if it be at infinity the waves are plane and the distance between the lens and the focus is called the focal length of the lens. This may be expressed in terms of the radii of curvature of the two surfaces and the refractive index of the glass, and the formula which we will now deduce can be used for determining the refractive index of a lens.

Let  $\mu$  = ref. index of lens,  $r$  = radius of surface  $AMB$ ,  $s$  = radius of surface  $ANB$  and  $y = AD$ .

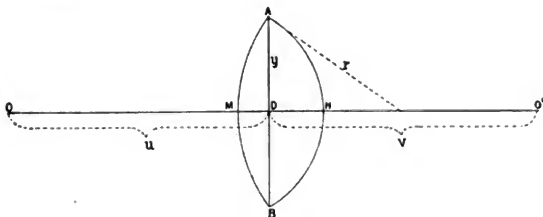


FIG. 66.

All disturbances from  $O$  reach  $O'$  at the same time ;

$$\therefore OA + O'A = OM + \mu(MN) + NO',$$

$$y^2 = 2rMD - MD^2 = 2sND - ND^2.$$

( $MD^2$  and  $ND^2$  can be rejected since  $MD$  and  $ND$  are small in comparison to  $r$  and  $s$ .)

# PHYSICAL OPTICS

$$\begin{aligned}\therefore MD &= \frac{y^2}{2r}, \quad ND = \frac{y^2}{2s}; \\ \therefore MN &= \frac{y^2}{2r} + \frac{y^2}{2s} = \frac{y^2}{2} \left( \frac{1}{r} + \frac{1}{s} \right). \dots\dots\dots(1)\end{aligned}$$

Denote  $OD$  by  $u$  and  $O'D$  by  $v$ , and we have

$$\begin{aligned}OA &= u + \frac{y^2}{2u} \quad \text{and} \quad O'A = v + \frac{y^2}{2v} \quad (\text{approximately}); \\ \therefore OA + O'A &= OO' + \frac{y^2}{2} \left( \frac{1}{u} + \frac{1}{v} \right). \dots\dots\dots(2)\end{aligned}$$

But  $OA + O'A = OM + O'N + \mu(MN)$ , and substituting from (2) we get

$$(\mu - 1)MN = \frac{y^2}{2} \left( \frac{1}{u} + \frac{1}{v} \right).$$

Substituting for  $MN$  the value given in (1),

$$\begin{aligned}\frac{y^2}{2} \left( \frac{1}{r} + \frac{1}{s} \right) (\mu - 1) &= \frac{y^2}{2} \left( \frac{1}{u} + \frac{1}{v} \right), \\ \frac{1}{u} + \frac{1}{v} &= (\mu - 1) \left( \frac{1}{r} + \frac{1}{s} \right) = \frac{1}{f},\end{aligned}$$

where  $f$  is the value of  $v$  when  $u = \infty$ , that is when the incident rays are parallel. The focal length of the lens is therefore  $f$ . Measure the radii of curvature of the two surfaces of a lens and its focal length, and determine  $\mu$ .

**Refraction of Light in Non-Homogeneous Media.**—The consideration of the laws of refraction in media in which the refractive index varies continuously from point to point leads us to a most interesting class of phenomena, the most common examples of which are the illusions known as Mirages.

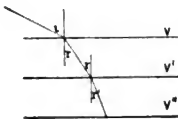


FIG. 67.

As an introduction to the subject it will be well to investigate the refraction of a ray of light by a number of media of different refractive indices arranged in horizontal strata of equal thickness. Let the velocities of light in the different strata be represented by  $v$ ,  $v'$  and  $v''$ , and let the angle of incidence on the

first layer be  $i$  (Fig. 67)—we require the deviation of the ray by the two boundaries.

We have  $\frac{\sin i}{\sin r} = \frac{v}{v'}$  for first boundary.  $\therefore \sin r = \sin i \frac{v'}{v}$ .

The incidence angle at the second boundary is obviously  $r$ , therefore we have

$$\frac{\sin r}{\sin r'} = \frac{v'}{v''} \quad \text{or} \quad \frac{\sin i \frac{v'}{v}}{\sin r'} = \frac{v'}{v''}, \quad \text{which gives} \quad \frac{\sin i}{\sin r'} = \frac{v}{v''},$$

showing that the direction of the ray in the third medium is the same as if the intervening medium were not present. Now suppose the number of layers to be increased indefinitely, and the thickness of

each to be reduced indefinitely. This gives us a medium of continuously varying refractive index, and we see that the direction of the ray at any point is the same as if the upper layers were removed, and the ray entered the flat surface of a medium of refractive index equal to that which the non-homogeneous medium has at the point in question. Suppose a ray to be travelling in a horizontal direction in a medium of this nature. As the ray is moving in a direction in which the refractive index does not change, it may seem at first sight as if there would be no change of direction. The discussion of the case by ray methods would lead to this conclusion, a result which plainly shows the danger of handling optical problems in this way. No matter how limited the width of the ray, the wave-front,—the motion of which constitutes the rays,—must have a finite size, and the upper and lower edges of the front are moving in regions of different optical density. The upper edge will consequently move faster than the lower, and the front will gradually wheel around, which means that the direction of propagation, or the direction of the ray, is constantly changing. We can treat the case by Huygens's construction by describing secondary wavelets of constantly decreasing radii around points on the wave-front, the enveloping plane representing the front in its next position (Fig. 68). By repeating the process we can show the gradual change of direction. The resulting curved rays are concave towards the direction of higher refractive index.

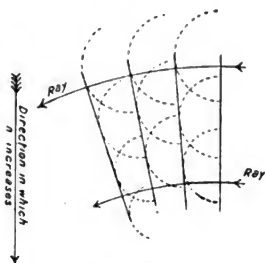


FIG. 68.

**Astronomical Refraction.**—The optical density or refractive index of the earth's atmosphere decreases as we ascend from the surface, consequently the rays of light, which reach our eyes from the stars, move in curved paths, except when the star is in the zenith. Since the direction in which the star appears to be, is the direction from which the ray comes when it enters the eye, the true position of the star can only be determined by taking the refraction of the atmosphere into account. The effect of refraction is to make the star appear higher up above the horizon, or nearer the zenith than it really is.

Now we have seen that the final direction of the ray is independent of the layers intervening between the medium in which the observation is made, and the region from which the light comes; it is therefore apparent that the change in direction can be determined by determining the refractive index of the air at the point where the instrument is situated, which can be done by observing its temperature, pressure, etc.

The curvature of light rays in the atmosphere also influences the apparent positions of objects on the earth's surface, the usual effect being an elevation of an object above its true position, a circumstance which must be taken into account in all geodetic observations.

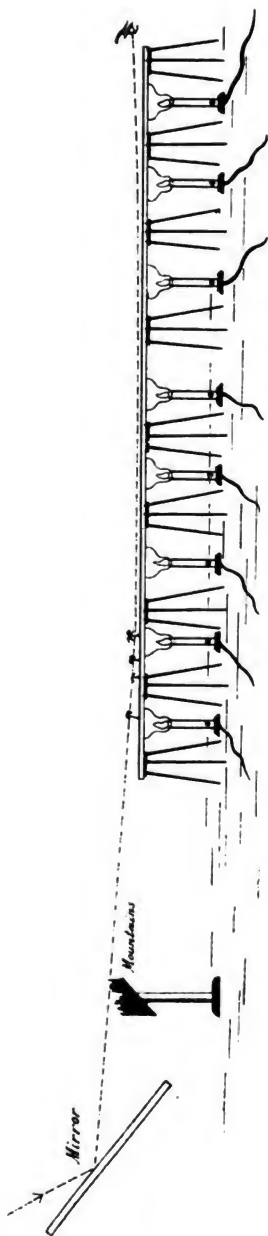
The normal variation of the refractive index in the atmosphere is often disturbed by temperature variations, as when the air near the surface is warmed by the heated ground, or when a layer of cold air flows over a layer of warm air, as may occasionally happen. These abnormal conditions in the atmosphere give rise to the phenomenon of mirage, the commonest type being that seen on the desert where the air is heated by the hot sand. In this case the refractive index is abnormally low along the ground, rises to a maximum as we ascend, and then decreases more slowly according to the usual law. Rays of light near the surface are therefore concave *upwards*, while those travelling at greater elevations are concave *downwards*. The result of this is, that rays which would ordinarily strike the ground are turned upwards and reach the eye, appearing as if reflected in the ground, while other rays starting originally from the same point may reach the eye by the usual path. The point thus appears double. The sky at the horizon may thus appear as if mirrored in the sand, and since the only reflecting body in nature capable of acting in this manner with which we are familiar is a smooth sheet of water, the natural inference is that a lake exists



FIG. 69.—DIAGRAM OF DESERT MIRAGE.

between us and the horizon. Where the sky is broken by mountains, we see their inverted images mirrored. The paths of the rays in mirage of this type are shown in Fig. 69.

It can be very beautifully reproduced by a method which was described by the author in the *Philosophical Magazine* in 1899, and which has since been somewhat improved. Three slabs of slate each a metre long, 20 cms. wide and 1 cm. thick (black-board slate), are mounted on iron tripods and carefully brought into the same plane, so that the upper surface is continuous and flat, which can be ascertained by "sighting" it from one end (Fig. 70). The surface is sprinkled with sand, to prevent reflection, which may occur at grazing incidence. A sheet of ground glass with an arc-light behind it represents the sky, or a mirror mounted so as to reflect the sky when viewed from the opposite end of the desert. The artificial sky must come down to the level of the sanded surface, and in front of it a chain of mountains cut out of pasteboard is mounted, with peaks varying from 1 to 2 cms. in height, and valleys which come quite down to the sand. The desert is heated by a long gas burner made by drilling numerous small holes in a piece of gas pipe. The flames should only be 4 or 5 cms. high, otherwise the slate is liable to crack. If we look along the sand, holding the eye an inch or two above the plane of the surface, we shall see, as the desert warms up, what appears to be a brilliant pool of water on the sand, in which the inverted images of the mountains and sky appear reflected. Photographs of this artificial mirage are shown in Fig. 71, Plate II.



ARTIFICIAL DESERT MIRAGE.

Fig 70.



1.



2.



3.

Fig 71.

## PLATE II

To face p 70.

Another type of mirage, sometimes seen at sea, can be referred to a stratum of hot air at a considerable distance above the earth's surface, which behaves in a manner precisely similar to the hot air on the ground, giving rise to inverted images of distant ships high above the horizon. Objects ordinarily below the horizon are frequently brought into view, by the curvature of the rays resulting from an abnormally rapid change in the refractive index of the air, a case being on record where ships moored off the French coast across the English channel 20 miles distant were seen from Dover.

Mirages are frequently seen on cold autumn mornings over large bodies of water, the air in the vicinity of the surface of the water being warmed. A frequent illusion, known as *Fata Morgana*, is the apparent elevation of objects on a distant shore into pinnacles and columns. It results from a distribution of density similar to that causing the desert mirage, the transition being less abrupt however. A medium stratified in horizontal layers, with a maximum refractive index along the central plane, will render divergent rays parallel and then convergent, the medium acting as a sort of continuous lens.

Fig. 72 is a somewhat exaggerated diagram of this effect. An eye at  $O$  receives rays from  $O'$  which have come over a number of different paths, and can be considered as situated at a focus towards which these rays converge.  $O'$  will therefore appear magnified in the vertical direction into a column  $AB$ . As the curvature of the rays is only in vertical planes there will be no corresponding horizontal magnification. Rocks and other objects lying along the shore are thus seen raised to the dignity of lofty cliffs, and blocks of ice floating in the water appear as white pinnacles.



FIG. 72.

If there was a similar variation in the refractive index in horizontal directions, magnification in all directions would occur. In a medium capable of acting in this way the equi-indical surfaces, or layers, of equal refractive index will be coaxial cylinders, the highest refractive index being along the axis. Exner has shown that the eyes of some insects are arranged in this way, the convergence of the rays to a focus resulting from the action of a non-homogeneous medium.

Cylinders of gelatine soaked in water were found by Exner and Matthiessen to behave in the same way.

These pseudo-lenses are not at all difficult to prepare and are extremely interesting. A quantity of photographic gelatine is soaked in clean water until quite soft, and the excess of water decanted. The gelatine is then heated until quite fluid and filtered through ordinary filter paper, warming the funnel from time to time. Boiling the gelatine with the beaten white of egg and skimming is said to clarify it. This operation is not necessary unless ordinary cooking gelatine is used. After filtering, the solution is evaporated on a water bath until almost as thick as syrup, and poured into a test tube. In an hour or so it will have solidified, and the cylinder can be removed by cracking a small hole in the bottom of the tube for the admission of

air, heating the tube quickly in a flame, and shaking or pushing out the contents. By means of a hot knife a section 1.5 cms. thick is cut from the cylinder, care being taken to get the ends as flat as possible. This section is placed on a small piece of thin plate glass, and a second plate laid on the top. The glass plates prevent evaporation from the ends, and as the water dries out, the refractive index rises, the highest value existing on the outside and the least along the axis. In from twelve to twenty four hours the action is strongly divergent, a lamp flame viewed through the cylinder being reduced anywhere from five to fifteen diameters. By combining the cylinder with a convex lens a Galilean telescope can be made. The rest of the cylindrical rod should be hung up to dry, which will take several days. A section cut from the dry cylinder and immersed in water becomes in time analogous in its action to a convex lens, and a telescope can be made of the two cylinders combined.

Schott has prepared similar cylinders of glass, by pouring the molten glass into iron tubes. The sudden chilling of the outer layer produced tension in the glass cylinder, and a corresponding variation in the refractive index, plane parallel plates cut from the cylinder acting as concave lenses.

Following the mathematical treatment which Exner gave for cylinders of this nature, we arrive at two interesting conclusions, namely that the ordinary lens formula  $\frac{1}{u} + \frac{1}{v} = \frac{1}{f}$  holds for them, and that to be *aplanatic*, the equation which shows the relation between the refractive index and the distance from the axis of the cylinder, is that of a parabola.

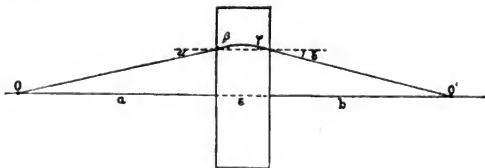


FIG. 73.

Let  $O$  (Fig. 73) be the source and  $O'$  the conjugate focus, at distances  $a$  and  $b$  from the surfaces of a cylinder of thickness  $\epsilon$ .

Consider the ray incident at angle  $\alpha$ , at a point at distance  $x$  from the axis, where the refractive index is  $n$ . The angle of refraction  $\beta = \frac{\alpha}{n}$ . From now on the ray moves in a path which is approximately circular for rays near the axis, turning through an angle  $\frac{\epsilon}{r}$  before reaching the second surface ( $r$  = radius of curvature of ray).

The ray meets the second surface at an incidence angle  $\gamma = \frac{\epsilon}{r} - \beta$ , for  $\beta$  and  $\gamma$  can be regarded as angles of a triangle whose exterior angle  $\frac{\epsilon}{r}$  is equal to their sum. The angle of refraction into air is

$$\delta = n\gamma = \frac{n\epsilon}{r} - n\beta = \frac{n\epsilon}{r} - \alpha.$$

Since we are considering rays near the axis  $x_1 = x_2 = x$ , where  $x$  is the distance of any point on the ray from the axis

$$\frac{x}{b} = \delta = \frac{n\epsilon}{r} - \alpha = \frac{n\epsilon}{r} - \frac{x}{a}, \text{ or } \frac{x}{a} + \frac{x}{b} = \frac{n\epsilon}{r}. \quad (1)$$

We now require an expression for  $r$  the radius of curvature of the ray.

Let  $GH$  be the element of wave front; the velocities and refractive indices at  $G$  and  $H$  are respectively  $v'$ ,  $n'$  and  $v$ ,  $n$  which are so related that  $v' > v$  and  $n > n'$ .

If the radius of the secondary wavelet around  $G$  is  $GL$  and the radius of the one around  $H$  is  $HK$ , the radius of curvature  $r$  will be found by producing  $LK$  until it meets  $GH$  produced, say at  $J$ . If  $GH$  equals  $dx$ ,

$$\frac{r}{r+dx} = \frac{HK}{GL} \quad (\text{Homol. sides of } \triangle s) = \frac{v'}{v} = \frac{n'}{n} = \frac{n+dn}{n},$$

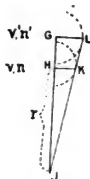


FIG. 74.

where  $dn$  is the increment of  $n$  corresponding to the increment  $dx$ .

$$rn = (r+dx)(n+dn), \quad rdn = -ndx, \quad \frac{1}{r} = -\frac{1}{n} \frac{dn}{dx}.$$

By equation (1) 
$$\frac{x}{a} + \frac{x}{b} = -\epsilon \frac{dn}{dx}. \quad (2)$$

Now  $n = f(x) = n_1 + c_2x + c_3x^2 + c_4x^3 \dots$  in which  $n_1$  equals the index at the axis, and on account of the symmetrical distribution of  $n$  around the axis,  $n$  has the same value for  $-x$  as for  $+x$ , and  $c_2 = c_4 = c_6 = 0$ ;

$$\therefore \text{ for nearly central rays } n = n_1 + c_3x^2, \quad (3)$$

$$\frac{dn}{dx} = 2c_3x, \quad \frac{1}{a} + \frac{1}{b} = -2c_3\epsilon = \text{Constant},$$

in which  $c_3$  is a constant depending on the law of change of the refractive index.

If we let  $-2c_3\epsilon = \frac{1}{p}$ , we have (4)  $\frac{1}{a} + \frac{1}{b} = \frac{1}{p}$ , the classical formula for lenses. This formula shows us that the focus is inversely proportional to the length of the cylinder.

By combining (3) and (4), we get  $n = n_1 - \frac{1}{2p\epsilon}x^2$ ,

$$n_1 - n = \frac{x^2}{2p\epsilon} \text{ and if } n_1 - n = \Delta n, \quad x^2 = 2p\epsilon\Delta n.$$

If we plot  $x$  and  $\Delta n$  on a system of rectangular coordinates, we shall get a parabola.

We have thus far confined our attention to nearly axial rays for which  $x$  is small, and obtained our final expression by neglecting powers higher than the square in the series. Let us now consider rays farther removed from the axis, and determine the law governing the change

of refractive index which must hold if the cylinder is to bring all rays to the same focus, or be aplanatic.

$$\text{By (2), we have} \quad -\frac{\epsilon}{x} \frac{dn}{dx} = \frac{1}{p},$$

$$-\epsilon p dn = x dx.$$

$$(\text{Integrating}) \quad -\epsilon p n = \frac{x^2}{2} + \text{Const.}$$

For  $x=0$ ,  $n=n_1$ , and since the above equation holds for all values of  $x$  it holds when  $x=0$ . From this relation we determine the constant to be  $-\epsilon p n_1$ ,

$$-\epsilon p n = \frac{x^2}{2} - \epsilon p n_1, \quad n = n_1 - \frac{x^2}{2p\epsilon}, \quad x^2 = 2p\epsilon\Delta n,$$

an expression identical with the one which we obtained for the axial rays.

The nature of the function is therefore parabolic, and if we plot ordinates equal to the refractive indices, and abscissae equal to the plus and minus values of  $x$ , we shall get a parabola, which is concave down or up according to whether the highest or lowest value of  $n$  is found along the axis, or according as we consider a cylinder (Fig. 75) which acts as a concave or convex lens.

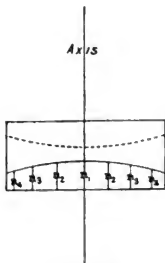


FIG. 75.

If we consider the cylinder to be of infinite length the rays will come to a focus on the axis, then diverge again, and moving in curved paths come to a focus again at a point further along on the axis.

The same thing will take place in vertical planes or in two dimensions if the equi-indical surfaces are parallel planes instead of coaxial cylinders. Such conditions sometimes exist in the atmosphere as Professor Everett has shown, and if the eye be situated at the focus where the rays come together, vertical magnification of the object will result as we have seen.

A ray of light entering a medium of this description will be bent towards, and cross the line of maximum optical density, where it changes its curvature and is again bent towards the line, which it may thus cross again and again, traversing a path which is approximately a sine curve. If we are dealing with a diverging pencil of rays, the rays will alternately converge to, and diverge from a focus, passing in this way through a number of successive foci. These effects can be well shown by the following device, which was described by the author in the *Philosophical Magazine* for April 1899.

A glass trough 50 cms. long by 10 cms. high and 2 cms. wide, with plate glass ends, is filled to the depth of 3 cms. with a strong solution of alum. On this is floated a layer of water containing 10% of alcohol, which is very much lighter than the alum solution, though



Fig 76.



Fig 77.

PLATE III

To face p 74.

having about the same refractive index. A mixture of glycerine and 85% alcohol has a much higher refractive index, but a specific gravity intermediate between these two liquids, consequently it is possible by means of a glass syphon, drawn down to a small aperture which is bent in a horizontal direction, to introduce a layer of it between the alum solution and the supernatant water. The necessary precautions and fuller directions will be found in the original paper. The three solutions were previously acidified with sulphuric acid, and rendered fluorescent with sulphate of quinine in order that the paths of the rays could be followed. By cautious stirring the diffusion of the layers into each other can be assisted, and we shall have as a result a medium in which the refractive index increases from the surface towards the median plane, and then decreases from this plane towards the bottom, the condition being similar to the atmospheric condition producing the Fata Morgana. If a very narrow beam of light from an arc-lamp, made parallel by means of a condensing lens, be thrown obliquely into one end of the trough, it will be seen to traverse the liquid in the form of a most beautiful blue wave, the curvature of which varies with the angle at which the ray enters. A ray of light travelling in a sine curve is shown in Fig. 76, Plate III., which was photographed directly from the trough.

The alternate convergence and divergence of rays, and the successive foci can be shown by allowing a parallel bundle of rays to enter one end of the trough in a horizontal direction. A photograph of this phenomenon is also shown in Fig. 77, Plate III.

**Scintillation.**—In addition to the more or less regular gradations in the refractive index of the atmosphere there exist striae, or small regions of sudden change due to the mixing of hot and cold currents, somewhat similar to the conditions existing in a mixture of glycerine and water. When a wave-front of light passes through a region where the refractive index is low it gains, and while travelling in a region of high refractive index it loses. The result of this is, that the striae deform the plane wave of light coming from the stars into a corrugated wave, portions of which are convex in the direction of propagation, while other portions are concave. The concave portions naturally converge, while the convex portions diverge; the result being that the energy concentrates itself in certain areas at the expense of the adjacent areas, as shown in Fig. 78.

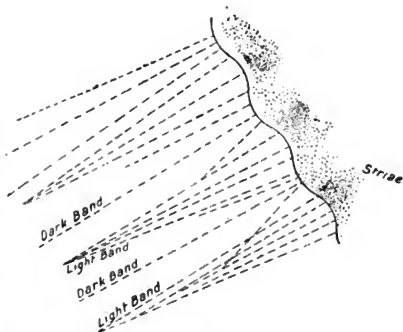


FIG. 78.

This uneven distribution of light produces the familiar phenomenon of scintillation or twinkling of the stars. If the intensity of the light from a star were sufficient, we should find that instead of illuminating a white surface uniformly, as does the sun or moon, it would illuminate it unevenly, dark and light patches alternating over the surface. This uneven illumination is actually observed during the few moments immediately preceding totality of a solar eclipse, the patches of light and shadow being arranged in more or less parallel bands. The bands are sometimes erroneously referred to as "diffraction fringes" bordering the moon's shadow. They move along over the ground with a velocity depending on the velocity of the upper currents of the atmosphere, usually from ten to twenty feet per second. The same phenomenon occurs also in the case of star light, except that the light is too feeble to produce shadow bands which are visible. Their presence, however, can be inferred from the well-known fact that the brilliancy of the star observed by the eye appears to suffer rapid periodic changes, the star appearing bright or feeble according to whether the eye is in a light or dark area of the moving system of shadows. The width of the bands is frequently not over 3 or 4 cms. This means that it may easily happen that one eye is in a dark, while the other is in a bright area at the same moment. If we look at a star with the eyes slightly converged, which we can easily do by focusing them on some object at a distance of five or six feet, and in a line with the star, the star will appear doubled and the two images will fluctuate in intensity, but the fluctuations will not be "in step," one eye seeing the star dark at the moment when the other eye sees it bright.

If a star is viewed through a telescope of large aperture, the resultant illumination at the focus is the integral of the bright and dark bands covering the object-glass at the moment, and this average illumination is practically constant, therefore scintillation is no longer observed. If the aperture of the instrument be contracted by a diaphragm of such size that only the light of a single bright or dark band can enter the instrument, the twinkling reappears. It is possible in this way to actually measure the radius of curvature of the corrugations of the wave-front in the case of star light. Suppose that at a given instant the wave entering the small aperture of the telescope is concave, it will come to a focus at a point slightly nearer the object-glass than the focus of the telescope for objects at infinity. At another instant when the aperture is in a dark band where the wave is convex, the focal point for this wave will be behind the principal focus. As the dark and light bands sweep across the aperture the image of the star will alternately appear sharp and blurred. If the eye-piece is at the focus for the concave wave it will be inside the focus for the convex wave. By pushing the eye-piece in up to a point where it is possible to occasionally catch a glimpse of a sharp image of the star, and then drawing it out to a point outside the focus, for which the same conditions prevail, it will be possible to determine the minimum radius of curvature of the convex and concave portions of the wave-front. Measurements made in this way show that the average radius of curvature is about 6000 metres,

although it may sometimes fall as low as 1800 metres, or rise as high as 20,000 metres. Obviously the conditions most favourable for work with astronomical instruments are to be found when the radius of curvature of the corrugations is very large. This means that the waves are approximately plane.

One other point is worthy of mention in this connection, namely, the difference between planets and stars in the matter of twinkling. In the case of planets the light comes from a luminous disc of an appreciable size, every point of which produces a system of shadow bands of its own. It is true that the inclination to each other of the rays coming from the different portions of the planet to the eye is very slight, but when we consider that they have traversed a distance of, say 6000 metres, in coming to a focus, that is, in forming a bright band, it is easy to see that the light from one side of the planet may easily produce a system of shadow-bands exactly out of step with those produced by the light of the other side of the disc. The supposition of a large number of shadow systems results in practically uniform illumination and absence of scintillation. This explanation of scintillation, while it accounts for alterations in the intensity of the light, does not account for the peculiar chromatic changes which were first observed by Respighi. If the spectrum of a scintillating star is observed it is seen to be traversed by broad dark bands, parallel to the Fraunhofer lines, which travel from the red to the violet, or from the violet to the red according as the star is in the west or the east. If the star is in the zenith, the motion of the bands is oscillatory. This indicates that the chromatic changes are due to the rotation of the earth. Owing to the dispersion of the air the violet rays will reach the eye over different paths from those traversed by the red rays. In Fig. 79 the dotted arc

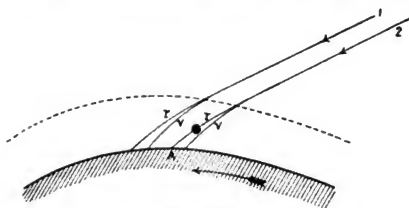


FIG. 79.

represents the upper limit of the atmosphere of the earth, which rotates in the direction of the arrow. Let the observer's eye be situated at *A*. Two parallel rays from the star are designated by 1 and 2. The dispersed rays are designated by *r* and *v*. It is clear that the violet rays from 1, and the red rays from 2, enter the eye or the spectroscope at *A*, if 1 and 2 are taken a little closer together than in the diagram. The violet rays thus traverse paths lying *above* those traversed by the red. Let us now imagine that one of the irregularities in the air previously mentioned, indicated by a black spot, lies in the path of the red rays, and suppose further that this irregularity is of such a nature as to cause the wave-front, originally plane, to become convex. The

intensity in the red region will consequently become less, for reasons already given. The rotation of the earth now comes into play, carrying this irregularity up and into the path of the violet rays, causing the minimum of intensity to shift from the red towards the violet end of the spectrum. This is the sequence observed by Respighi for a star in the west, which is the case represented in the diagram. These chromatic changes manifest themselves to the eye as an irregular change in the color of the star.

**The Method of Striae.**—A very ingenious and beautiful method was originated by Töpler (*Wied. Ann.*, cxxxi., p. 33) (which he named the "Schlieren-methode") for making visible regions in a transparent medium where the refractive index differed slightly from that of the surrounding regions. By employing as a source of light the instantaneous flash of an electric spark he was able to actually see the spherical sound waves sent off from another spark which occurred a moment before. Mach has used the method extensively for studying by photography the air waves given off by sparks, and accompanying rifle bullets in their flight, and an extensive series of photographs were made by Wood (*Phil. Mag.*, Aug. 1899, July 1900, May 1901) of sound waves undergoing reflection and refraction, to illustrate some of the fundamental principles of optics. The apparatus for showing these waves can be set up in a few minutes, with very little trouble, and as the experiment is a very beautiful and instructive one it will be described in detail.

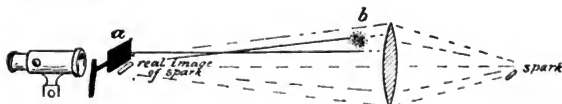


FIG. 80.

The general arrangement of the "Schlieren" apparatus is shown in Fig. 80. A good-sized achromatic lens of the finest quality obtainable, and of rather long focus, is the most important part of the device. The object-glass of a small telescope three or four inches in diameter is about right.

This lens is mounted in front of a suitable source of light (in the present case an electric spark), which should be at such a distance that its image on the other side of the lens is at a distance of about fifteen feet.

The image of the spark, which we will suppose to be straight, horizontal, and very narrow, is about two-thirds covered with a horizontal diaphragm (*a*), and immediately behind this is placed the viewing-telescope. On looking into the telescope we see the field of the lens uniformly illuminated by the light that passes under the diaphragm, since every part of the image of the spark receives light from the whole lens. If the diaphragm be lowered the field will darken, if it be raised the illumination will be increased. In general it is best to have the diaphragm so adjusted that the lens is quite feebly illuminated, though this is not true for photographic work. Let us now suppose that there is a globular mass of air (*b*) in front of the lens

of slightly greater optical density than the surrounding air. The rays of light going through the upper portion of this denser mass will be bent down, and will form an image of the spark below the diaphragm, allowing more light to enter the telescope from this particular part of the field; consequently, on looking into the instrument, we shall see the upper portion of the globular mass of air brighter than the rest of the field. The rays which traverse the under part of "*b*," however, will be bent up, forming an image of the spark higher up, and wholly covered by the diaphragm, consequently this part of the field will appear black. It will readily be understood, that with the long path between the lens and the image a very slight change in the optical density of any portion of the medium in front of the lens will be sufficient to raise or depress the image above or below the edge of the diaphragm, and will consequently make itself manifest in the telescope.

The importance of using a lens of first-class quality is quite apparent, since variations in the density of the glass of the lens will act in the same way as variations in the density of the medium before it, and produce unequal illumination of the field. It is impossible to find a lens which will give an absolutely even, feeble illumination, but a good achromatic telescope-objective is perfect enough for every purpose. A more complete discussion of the operation of the apparatus will be found in Toepler's original paper in the *Annalen*. The sound-waves, which are regions of condensation, and consequent greater optical density, make themselves apparent in the same way as the globular mass of air already referred to. They must be illuminated by a flash of exceedingly short duration, which must occur while the wave is in the field of view.

Toepler showed that this could be done by starting the sound-wave with an electric spark, and illuminating it with the flash of a second spark occurring a moment later, while the wave was still in the field. A diagram of the apparatus used is shown in Fig. 81. In front

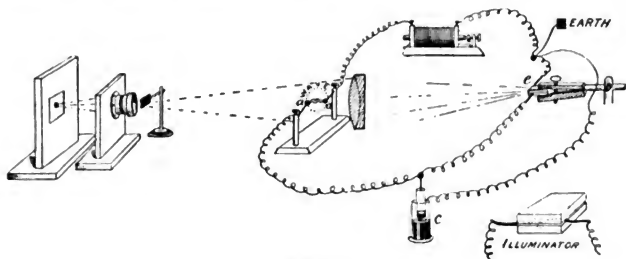


FIG. 81.

of the lens are two brass balls (*a, a*), between which the spark of an induction-coil passes, immediately charging the leyden-jar *c*, which discharges across the gap at *e* an instant later. The capacity of the jar is so regulated that the interval between the two sparks is about

one ten-thousandth of a second. The field of the lens is thus illuminated by the flash of the second spark before the sound-wave started by the first spark has gone beyond the edge of the lens.

To secure the proper time-interval between the two sparks it is necessary that the capacity of the jar be quite small. This limits the length and brilliancy of the illuminating spark, and with the device employed by Toepler it was impossible to get enough light to secure photographs of the waves. After some experimenting it was found that if the spark of the jar was passed between two thin pieces of magnesium ribbon pressed between two pieces of thick plate-glass, a very marked improvement resulted. With this form of illuminator five or six times as much light could be obtained as by the old method of passing the spark between two brass balls.

The spark is flattened out into a band, and is kept always in the same plane, the light issuing in a thin sheet from between the plates. By this arrangement we secure a light source of considerable length, great intensity, and bounded by straight edges, the three essentials for securing good results. The glass plates, with the ribbon terminals between them, must be clamped in some sort of a holder and directed so that the thin sheet of light strikes the lens: this can be accomplished by darkening the room, fastening a sheet of paper in front of the lens, and then adjusting the plates so that the paper is illuminated as much as possible. The image formed by the lens will be found to have very sharp straight edges, on one of which the edge of the diaphragm can be set in such a manner as to allow but very little light to pass when the intervening medium is homogeneous; a very slight change, however, in any portion may be sufficient to cause the entire amount of light passing through that portion to pass below the diaphragm and enter the telescope.

For photographing the waves the telescope is removed and a photographic objective put in its place. A vertical board is firmly clamped behind this in such a position that the image of the balls, between which the sound-spark passed, would be in focus on a plate held against it. This arrangement is used instead of a camera, because it is necessary to move the plate rapidly during the exposure, to prevent the image of more than one wave being formed on the same place. It was found that simply holding the plate in the hand against the vertical board and advancing it slowly from left to right, at the same time giving it a rapid up-and-down motion, answered every purpose.

The images obtained in this manner show the waves in different stages of development, for the time-interval between the two sparks varies between rather wide limits. This is really an advantage, for on a single plate it is possible to pick out a series showing the successive changes in the form of the wave-front produced by reflexion, refraction, etc. Each picture shows the circular field of the telescope-lens with the two brass rods crossing it and supporting in the center the two balls between which the sound spark passes. The hot air rising from the spark appears in most of the pictures like a puff of steam above the ball.

A few words regarding the apparatus may be helpful to those wishing to repeat the very beautiful experiments of Toepler. An induction-

coil capable of giving a three or four inch spark is about right, while a good-sized test-tube partly filled with mercury, and standing in a cylinder of mercury, will be found most convenient for a leyden jar. The balls between which the sound-spark passes should be adjusted so as to obtain almost the maximum spark possible, which will in general be rather less than half as long as the coil will give between its terminals. The best results are obtained when the sparks give off the same crackle found desirable in experiments with Hertz waves. Fresh plates of glass should be put in the illuminator every little while.

It is not at all difficult to get the apparatus to work properly, and doubtless it could be made to work on quite a small scale with a good photographic objective of rather long focus. The objective of a good-sized spy-glass would also give good results. Toepler was, I believe, of the opinion that he got more uniform results with an influence-machine than with the coil. He certainly found the time-interval between the two sparks to be more constant. This, however, is no object in photographic work, for the wide variation is the very thing that makes the pictures interesting.

The refraction of sound in a medium denser than air is shown in Fig. 82, where we have a rectangular tank with sides made of plane-parallel glass, and covered with a collodion film of soap-bubble thickness made by the method described by Toepler. Ordinary collodion is diluted with about ten parts of ether, poured on a small piece of plate-glass and immediately drained off. As soon as it is quite dry a rectangle is cut with a sharp knife on the film. Toepler's method of removing the film was to place a drop of water on one of the cuts, and allow it to run in by capillarity. The following method will, however, be found better. One end of the plate is lowered into a shallow dish of water, and the plate inclined until the water comes up to one of the cuts. By looking at the reflexion of a window in the water it is possible to see whether the film commences to detach itself from the glass. If all goes well it will float off on the surface of the water along the line of the knife-cut, and the plate should be slowly lowered (one end resting on the bottom of the dish) until the rectangular piece detaches itself and floats freely on the surface. The edges of the tank are well greased and then lowered carefully upon the film, to which they will adhere. The whole must then be lifted from the water in an oblique direction, when the film will be found covering the tank and exhibiting the most beautiful interference-colors. The tank was filled with carbonic acid and placed under the origin of the sound-wave. On striking the collodion-film the wave was partly reflected and partly transmitted, and it will be seen that the reflected component in the air

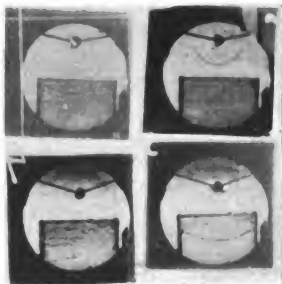


FIG. 82.

P.O.

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has moved farther than the transmitted component in the carbonic acid. The spherical wave-front is transformed into an hyperboloid on entering the denser medium. This is well shown in No. 3 of the series, where the wave in air, moving at higher velocity, has passed out of the field entirely, and there remains only the slower-moving disturbance in the denser gas. In No 4 the wave is seen returning up through the tank after having struck the bottom.

A prism can be made with its two refracting faces of this exceedingly thin collodion, which, when filled with carbonic acid, showed the bending of the wave front, exactly as we figure it in diagrams for light. It is necessary to have the collodion thinner than before, since if we are to photograph the wave after twice traversing the film, we must lose as

little energy as possible by reflexion. Fig. 83 shows the refraction in a carbonic-acid prism, the bending being particularly noticeable in No. 4, on which I have, with a pair of dividers, traced out the position which the wave-front would have occupied had it not traversed the prism.



FIG. 83.

Photographs were also obtained of the refraction of the wave, when the tank was filled with hydrogen, in which the sound travels faster than in air. The bulging down of the wave-front was very noticeable, though it is not as great as we should expect.

These explosive waves travel at a much higher velocity than ordinary sound-waves (nearly double the speed), and it is highly probable that the relative speeds in two different media is not the same as for ordinary sounds.

Various other cases of refraction can also be shown, for example, the transformation of a spherical into a plane wave by a carbonic-acid lens. The construction of the cylindrical lens, of exceedingly thin collodion, is a matter of considerable difficulty, the circular flat ends of very thin mica, free from striae, enabling the passage of the wave through the lens to be followed.



FIG. 84.

In examining liquids or solids for striae, or regions of variable refractive index, we can employ a flat gas flame as our source of light, covering the lower part of it with an opaque screen having a straight

edge. The lens will form an inverted image of this in front of the objective of the viewing telescope, and all but a strip half a millimeter or so in width is to be cut off from above by a second screen. The object to be examined is placed immediately in front of the lens. A piece of ordinary window glass makes a good object. The heated air rising from the hand can also be seen, and if a tank made of optical glass, filled with warm water, is placed before the lens, a drop lifted out and allowed to fall back can be seen descending through the liquid: the change in the refractive index is obviously due to the cooling by evaporation. Opaque objects placed before the lens appear with brilliantly illuminated margins, the light in this case being diffracted: with the arrangement of screens described only the upper and lower edges appear illuminated, since lateral deflection of the rays is without effect. The method is an extremely useful one, and can be applied to many lines of investigation, and the student should be thoroughly familiar with its possibilities.

**Invisibility of Objects.**—Opaque substances are seen by the light reflected from their surfaces; transparent substances in part by reflected light and in part by transmitted light. If we analyse carefully the appearance of a cut-glass decanter stopper we shall find it to be extremely complicated. Each facet reflects the image of some object in the room from its surface, and in addition to this shows some other object by refracted rays which have entered some other facet, these latter being in general more or less spread out into a spectrum by dispersion. If the stopper is wholly or in part made of colored glass, the refracted rays passing through the colored portions are modified by absorption, and affect the appearance. This remarkable complex, we say, looks like a stopper, and unless we try to paint a picture of it, or have our attention drawn to the details, we are apt to regard its appearance as quite simple.

We thus see that reflection, refraction, and absorption all play a part in making objects visible. It is interesting to examine into the conditions under which objects are invisible. If they are immersed in a medium of the same refractive index and dispersion, reflection and refraction disappear; and if they possess in addition the quality of perfect transparency, they will be absolutely invisible, the light rays passing through them without any modification either in intensity or direction. Could a transparent solid be found whose refractive index was the same as that of air, objects made of it would be invisible. The effect of immersing a transparent solid in a medium of similar optical properties is usually illustrated by dipping a glass rod into Canada balsam or oil of cedar, the immersed portion being practically invisible. A still better medium can be made by dissolving chloral hydrate in glycerine by the aid of heat. Only a little glycerine should be taken, as it is necessary to dissolve some eight or ten times its volume of the chloral before the solution acquires the right optical density. A glass rod, if free from bubbles or striae, becomes absolutely invisible when dipped in the liquid, and if withdrawn presents a curious appearance, the end appearing to melt and run freely in drops.

As a matter of fact, transparent objects are only visible by virtue of non-uniform illumination, as is pointed out by Lord Rayleigh in

his article on optics in the *Encyclopaedia Britannica*. If the illumination were the same on all sides they would be invisible, even if immersed in a medium of very different optical index. A condition approaching uniform illumination might, he says, be attained on the top of a monument in a dense fog. The author has devised a very simple method of showing this curious phenomenon, which, in brief, is to place the object within a hollow globe, the interior surface of which is painted with Balmain's luminous paint, and view the interior through a small hole.

The apparatus can be made in a few minutes in the following manner: A quantity of Canada balsam is boiled down, until a drop placed on cold glass solidifies. The Balmain paint, in the form of a dry powder, is stirred into the hot balsam until the whole has the consistency of thick paint. Two glass evaporating dishes of equal size are carefully cleaned and warmed, and coated on the outside with the hot mixture, which can be flowed over the glass, and by the dexterous manipulation of a small Bunsen flame made to cover the entire outer surface. Probably two perfectly plain hemispherical finger-bowls could be used instead of the evaporating dishes. As soon as the coating has become hard a small hole is cut through it through which the interior is to be viewed. If the lips of the dishes are placed together the interior can be seen through the small opening, but in this case the line of junction, which is always more or less dark, comes opposite the aperture, which is a disadvantageous arrangement.

If the inner surfaces be exposed to bright daylight, sun or electric light, and the apparatus taken into a dark room, a crystal ball or the cut-glass stopper of a decanter placed inside, will be found to be quite invisible when viewed through the small aperture. A uniform blue glow fills the interior of the ball, and only the most careful scrutiny reveals the presence of a solid object within it. One or two of the side facets of the stopper may appear if they happen to reflect or show by refraction any portion of the line of junction of the two hemispheres.

## CHAPTER V.

### DISPERSION.

IN our treatment of refraction we have assumed a constant retardation of the waves for a medium of given refractive index. We have seen that the velocity of light in the free ether of space is independent of the color or wave-length. Such, however, is not the case in refracting media, for here the waves not only travel slower than they do in space, but waves of different length travel with very different velocities. In all such media as air, water, and glass, the long waves travel faster than the short ones; consequently the deviation of the ray, or the angle through which the wave-front turns when encountering the boundary of the medium, depends on the color of the light as well as on the optical density of the medium.

When white light enters a transparent medium, the long red waves forge ahead of the green ones, which in their turn get ahead of the blue. If we imagine an instantaneous flash of white light traversing a refracting medium, we must conceive it as drawn out into a sort of linear spectrum in the medium, that is the red waves lead the train, the orange, yellow, green, blue and violet following in succession. The length of this train will increase with the length of the medium traversed. On emerging again into free ether the train will move on without any further alteration in its length.

We can form some idea of the actual magnitudes involved in the following way. Suppose we have a block of perfectly transparent glass (of ref. index 1.52) twelve miles in thickness. Red light will traverse it in  $1/10000$  of a second, and on emerging will be about 1.8 miles in advance of the blue light which entered at the same time. If white light were to traverse this mass of glass, the time elapsing between the arrival of the first red and the first blue light at the eye would be less than  $1/6000$  of a second. Michelson's determination of the velocity of light in carbon bisulphide showed that the red waves gained on the blue waves during their transit through the tube of liquid. The absence of any change of color in the variable star Algol furnishes direct evidence that the blue and red rays traverse space with the same velocity. In this case the distance is so vast, and the time of transit so long, that the white light coming from the star during one of its periodic increases in brilliancy, would arrive at the earth with its red component so far in advance of the blue that the eye would be able to take cognizance of the fact.

Inasmuch as the deviation of a ray of light depends on the change of velocity of a wave on going say, from a rare into a denser medium, we infer that those rays which are deviated the most, namely the violet, suffer the greatest change of velocity or move the slowest. Later on, when we come to the study of interference, we shall find other evidence that such is the fact.

Newton was the first to systematically study the phenomenon of dispersion. He discovered that ordinary white light was made up of different colors which could be separated from each other by passing the light through a prism. His most complete and convincing experiment may be briefly summed up as follows. The light of the sun was admitted to a darkened room through a small hole in the shutter, and the narrow beam passed through a prism. Instead of a round white image of the sun, there now appeared on the screen a colored band of light or spectrum, made up in reality of an infinite number of differently colored images of the sun superposed, but slightly displaced with reference to one another.

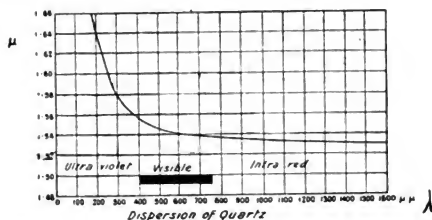


FIG. 85.

A small perforation was made in the screen which allowed light of approximately a single color to pass. This ray was transmitted through a second prism, and was found to form a fairly sharp image of the sun on a second screen, proving that monochromatic light suffers no decomposition or dispersion in the prism. By slightly turning the first prism, the spectrum could be moved so as to allow light of any color to pass through the perforation and be refracted by the second prism. Newton found that the colored image of the sun on the screen changed its position with every change of color, the deviation being greatest when the color was violet, and least when it was red.

The refractive index of a substance varies then with the wave-length of the light employed. To determine the relation between the two we can measure the index of a prism for light of known wave-lengths, *e.g.* the bright lines in the spark spectrum, or the dark lines in the solar spectrum, and plot the results on co-ordinate paper, taking the refractive indices as ordinates, and the wave-lengths as abscissae.

We shall find that the refractive index increases more rapidly than the wave-length decreases as we approach the violet, the curve having the general form shown in Fig. 85, which is the dispersion curve for quartz.

Having plotted such a curve for a given prism, we can determine the wave-length of any other line in the spectrum by determining the refractive index for the line, and finding the corresponding ordinate on the curve. If a prism spectroscope is to be used for wave-length determinations, it must be calibrated in this manner, for different samples of glass have very different dispersive powers. Were the deviations proportional to the wave-lengths, the curve would be a straight line, and we should have what is known as a normal spectrum. Such a spectrum can be formed by a diffraction grating, but never by a prism. The dispersion curve can be shown experimentally in the following way. Let a vertical normal spectrum, formed by a diffraction grating, be viewed or projected through a prism standing with its refraction edge vertical. The entire spectrum will be deviated by the prism, but the deviation will increase very rapidly as we near the blue, the spectrum being bent into a curve. This method of "crossed prisms," due to Newton, is of use in studying the remarkable phenomenon of anomalous dispersion, which we shall come to presently.

Newton came to the erroneous conclusion that the dispersion was proportional to the refraction, that is to say that substances of high refractive index had great dispersive powers, or gave wide spectra, while the reverse was true for substances of low refractive index. While this is apt to be the case, it is not always true, for we find that there are substances the mean refractive indices of which are small, while their dispersive powers are large, and *vice versa*.

**Achromatism.**—The independence of dispersion of refractive index makes it possible to arrange two prisms of different kinds of glass, with their refracting angles turned in opposite directions, which shall have the power of deviating a ray without spreading it out into a spectrum. One of the prisms almost entirely annuls the dispersion of the other, without entirely annulling the deviation, a thing which Newton considered impossible. Such a combination is known as an achromatic prism. Let us see just how such a system operates.

Flint glass has a much higher dispersive power in proportion to its mean refractive index than crown glass. The refractive indices of the two glasses for red, yellow and bluish-green light of wave-lengths corresponding to the *C*, *D*, and *F* lines in the solar spectrum are as follows:

			<i>C</i>	<i>D</i>	<i>F</i>
Flint glass,	-	-	1.630	1.635	1.648
Crown glass,	-	-	1.527	1.530	1.536

If prisms of small angle are employed we can write the deviations for these colors as proportional to the refractive indices less 1, that is for flint glass the distance from a point on a screen where the direct ray falls, to the points where the red-green and blue rays fall when the prism is put in the path of the light, will be 630, 635, and 648.

The length of the spectrum, or rather the distance between the *C* and *F* lines, is obviously 648 - 630 or 18.

For crown glass the distances will be 527, 530, and 536, and the distance between the *C* and *F* lines will be 536 - 527 or 9. The dispersion of the flint glass is therefore double that of the crown glass. If now we make a crown-glass prism of twice the angle of the flint-

glass prism, the distance between the *C* and *F* lines will be the same as with the flint prism, while the distances of the lines from the spot where the direct ray falls will be twice as great as before, or 1054, 1060, and 1072.

Suppose now we place the two prisms together with their refracting angles turned in opposite directions. The crown prism alone would shift the *F* line to a distance of 1072, but the flint prism shifts it back a distance of 648, and its resulting position is  $1072 - 648$  or 424 from the spot where the direct ray would fall. The *C* line would be deviated by the crown prism to a distance 1054, but the flint one moves it back 630, and its position is  $1054 - 630$  or 424. The *C* and *F* lines are thus deviated the same amount, and the dispersion is annulled so far as these two colors are concerned. The combination is achromatic for red and greenish-blue light, deviating both to a distance of 424, a trifle less than the deviation produced by the flint prism acting alone. Let us now see if the yellow light falls in the same place. The position of the *D* line will be given by  $1060 - 635$  or 425, that is, it will be deviated a very little more than the *C* and *F* lines, consequently the combination is not perfectly achromatic. By means of two prisms it is possible to bring any two parts of the spectrum together, the other colors lying a little to the right or left of the superposed portions, forming what is known as the secondary spectrum.

The general rule to follow in the construction of an achromatic prism is as follows. To bring any two lines of the spectrum together, the angles of the two prisms must be so proportioned that the distance between the lines in question is the same for each prism. Were the distances between the other lines the same for both prisms, the combination would be truly achromatic, but such is not the case, owing to the irrationality of dispersion.

Achromatic prisms are of very little practical use, but the principle is of great importance in connection with achromatic lenses.

**Direct Vision Prisms.**—By referring to the table of refractive indices for crown and flint glass it is easy to see how a combination of two prisms can give dispersion without deviation, that is, yield an undeviated spectrum. If, instead of giving the crown prism an angle double that of the flint, we make it 1.2 times as great, and make the same calculation as before, we shall find that we have a spectrum the length of which is 7, and the center of which falls on the spot where the undeviated ray would fall. Such a combination is known as a direct vision prism, and is employed in cases where any considerable deviation would be detrimental, as when compactness of the instrument is desirable.

**Achromatic Lenses.**—We are now in a position to consider the principle on which the achromatic lens is made. Any lens can be considered as a prism of varying angle, or rather as a solid formed by the rotation of a thin section of a curved prism around its base. Since the distance of the focus of a lens from its center depends on the deviation of the rays, it follows that the focus will be different for the different colors, the blue rays which are bent the most meeting nearest the lens, and the red, which are bent to a less degree, coming together farther away, an effect known as chromatic aberration. What we

require is a combination which will produce an equal deviation, and consequently a common meeting point for rays of all colors. If we can arrange two prisms of crown and flint glass which will give deviation without dispersion, we can in the same way, by employing a double convex lens of crown and a plane concave of flint glass, give exactly the same deviation to two colors widely separated in the spectrum, and very nearly the same deviation to the other colors, with the result that rays of different refrangibility come together at very nearly the same point.

Reference to Fig. 86 will make the analogy between the achromatic lens and prism clear. The blackened parts indicate how each portion of the lens combination can be considered as two opposed prisms. We found that in the case of the prism the ratio between the angles was 1 : 2, and applying this to the lens it is easy to see that if the surfaces *A*, *B*, and *C* have the same curvature, the surface *D* of the flint lens must be plane, since the angle of the elementary prismatic portion of the flint lens must be everywhere 1/2 that of the opposed elementary crown prism. Just as by employing two prisms we could unite two lines of the spectrum, so by the use of two lenses we can bring rays of any two different colors to the same focus.



FIG. 86.

**Calculation of Achromatic and Direct Vision Prisms.**—For prisms of small angle the deviation  $\delta = \rho(n - 1)$ , in which  $\rho$  is the prism angle and  $n$  the refractive index. The deviation for two definite colors (say lines *F* and *C* of the solar spectrum) are given by

$$\delta_F = \rho(n_F - 1),$$

$$\delta_C = \rho(n_C - 1).$$

Subtracting,  $\delta_F - \delta_C = \rho(n_F - n_C) \dots \dots \dots (1)$

The difference  $\delta_F - \delta_C$  we can designate as the dispersion angle for the colors in question, and for brevity write it  $\zeta_{FC}$ . For a second prism of a different angle and composed of a different glass we have similar equations.

We will take into account three colors corresponding to the lines *F*, *D*, and *C*, for which we have equations:

$$\text{Prism 1. } \delta_D = \rho(n_D - 1), \quad \zeta_{FC} = \rho(n_F - n_C).$$

$$\text{Prism 2. } \delta'_D = \rho'(n'_D - 1), \quad \zeta'_{FC} = \rho'(n'_F - n'_C).$$

Suppose now that these two prisms are opposed. We shall then have the total deviation  $g$  of the color *D* represented by  $\delta_D - \delta'_D$ , and the dispersion angle  $\omega$  between the rays *F* and *C* by  $\zeta_{FC} - \zeta'_{FC}$ .

$$g_D = \delta_D - \delta'_D = \rho(n_D - 1) - \rho'(n'_D - 1), \dots \dots \dots (2)$$

$$\omega_{FC} = \zeta_{FC} - \zeta'_{FC} = \rho(n_F - n_C) - \rho'(n'_F - n'_C). \dots \dots \dots (3)$$

For a *direct vision prism* in which the ray of color *D* is undeviated, we set  $g_D = 0$ , and obtain at once from equation (3) the relation

$$\frac{\rho}{\rho'} = \frac{n'_D - 1}{n_D - 1},$$

or the angles must stand in inverse ratio to the refractive indices less 1.

For an achromatic prism which is to deviate the rays of colors  $F$  and  $C$  the same amount, we write  $\omega_{FC} = 0$  and get from equation (4) the relation

$$\frac{\rho}{\rho'} = \frac{n'_F - n'_C}{n_F - n_C}, \quad \frac{\rho}{\rho'} = \frac{n'_F - n'_C}{n_F - n_C}.$$

We will now investigate the spectrum produced by a direct vision prism which transmits the ray  $D$  without deviation. For this we have

$$g_D = 0 = \rho(n_D - 1) - \rho'(n'_D - 1),$$

or 
$$\rho' = \rho \frac{(n_D - 1)}{n'_D - 1}.$$

Substituting this value of  $\rho'$  in equation (3) and transposing gives us for the angular separation of the rays  $F$  and  $C$

$$\omega_{FC} = \rho(n_D - 1) \left[ \frac{n_F - n_C}{n_D - 1} - \frac{n'_F - n'_C}{n'_D - 1} \right].$$

The quantity  $\frac{n_D - 1}{n_F - n_C} = \nu$  is called the relative dispersion of a substance, and it is usually given in the tables which represent the optical properties of different kinds of glass.

Our equation now becomes

$$\omega_{FC} = \rho(n_D - 1) \left[ \frac{1}{\nu} - \frac{1}{\nu'} \right].$$

In a similar manner we can derive an expression for the deviation of the ray  $D$  by an achromatic prism which deviates rays  $F$  and  $C$  by the same amount. In this case

$$\omega_{FC} = 0,$$

$$g_D = \rho(n_F - n_C)(\nu - \nu').$$

If in addition to deviating the rays  $F$  and  $C$  by the same amount, rays corresponding to the line  $A$  and  $D$  are to be united, we have

$$\frac{\rho}{\rho'} = \frac{n'_F - n'_C}{n_F - n_C},$$

$$\frac{\rho}{\rho'} = \frac{n'_D - n'_A}{n_D - n_A},$$

$$\frac{n'_D - n'_A}{n'_F - n'_C} = \frac{n_D - n_A}{n_F - n_C}.$$

The quantity  $n_F - n_C$  is known as the mean dispersion, and quantities such as  $n_D - n_A$  or  $n_F - n_D$  partial dispersions.

**Resolving Power of Prisms.**—Lord Rayleigh has investigated the subject of the resolving power of prisms, and his treatment follows. It will hardly be understood until after the chapter on Diffraction has been read, and is introduced at this point only for future reference.

“Let  $A_0B_0$  (Fig. 87) be a plane wave-surface of the light before it falls upon the prisms,  $AB$  the corresponding wave-surface for a particular part of the spectrum after the light has passed the prism, or after it has passed the eye-piece of the observing-telescope. The

path of the ray from the wave-surface  $A_0B_0$  to  $A$  or  $B$  is determined by the condition that the optical distance, represented by  $\int \mu ds$ , is a minimum; and as  $AB$  is by supposition a wave-surface, this optical distance is the same for both points. Thus

$$\int \mu ds \text{ (for } A) = \int \mu ds \text{ (for } B). \dots\dots\dots(2)$$

We have now to consider the behavior of light belonging to a neighboring part of the spectrum. The path of a ray from the wave-surface  $A_0B_0$  to  $A$  is changed; but in virtue of the minimum property the change may be neglected in calculating the optical distance, as it influences the result by quantities of the second order only in the change of refrangibility. Accordingly the optical distance from  $A_0B_0$

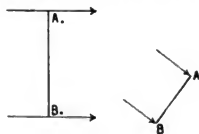


FIG. 87.

to  $A$  is represented by  $\int (\mu + \delta\mu) ds$ , the integration being along the path  $A_0 \dots A$ ; and, similarly, the optical distance between  $A_0B_0$  and  $B$  is represented by  $\int (\mu + \delta\mu) ds$ , where the integration is along the path  $B_0 \dots B$ . In virtue of (2) the difference of the optical distance is

$$\int \delta\mu ds \text{ (along } B_0 \dots B) - \int \delta\mu ds \text{ (along } A_0 \dots A). \dots\dots\dots(3)$$

The new wave-surface is formed in such a position that the optical distance is constant; and therefore the *dispersion*, or the angle through which the wave surface is turned by the change in refrangibility, is found simply by dividing (3) by the distance  $AB$ . If, as in common flint-glass spectroscopes, there is only one dispersing substance,  $\int \delta\mu ds = \delta\mu s$ , where  $s$  is simply the thickness traversed by the ray.

If we call the width of the emergent beam  $a$ , the dispersion is represented by  $\delta\mu(s_2 - s_1)a$ ,  $s_1$  and  $s_2$  being the thicknesses traversed by the extreme rays. In a properly constructed instrument  $s_1$  is negligible, and  $s_2$  is the aggregate thickness of the prisms at their thick ends, which we will call  $t$ ; so that the dispersion ( $\theta$ ) is given by

$$\theta = \frac{t\delta\mu}{a}. \dots\dots\dots(4)$$

By § 2 the condition of resolution of a double line whose components subtend an angle  $\theta$  is that  $\theta$  must exceed  $\frac{\lambda}{a}$ . Hence from (4), in order that a double line may be resolved whose components have indices  $\mu$  and  $\mu + \delta\mu$ , it is necessary that  $t$  should exceed the value given by the following equation:

$$t = \frac{\lambda}{\delta\mu}, \dots\dots\dots(5)$$

which expresses that the relative retardation of the extreme rays due to the change of refrangibility is the same ( $\lambda$ ) as that incurred without

a change of refrangibility when we pass from the principal direction to that corresponding to the first minimum of illumination.

That the resolving power of a prismatic spectroscope of given dispersive material is proportional to the total thickness used, without regard to the number of angles, or setting of the prisms, is a most important, perhaps the most important, proposition in connection with this subject. Hitherto in descriptions of spectroscopes far too much stress has been laid upon the amount of dispersion produced by the prisms; but this element by itself tells nothing as to the power of an instrument. It is well known that by a sufficiently close approach to a grazing emergence, the dispersion of a prism of given thickness may be increased without limit; but there is no corresponding gain in resolving-power. So far as resolving-power is concerned, it is a matter of indifference whether dispersion be effected by the prisms or by the telescope."

The expression for the resolving-power of a prism is usually written in the form  $\frac{\lambda}{\delta\lambda} = t \frac{\delta\mu}{\delta\lambda}$ , which follows at once from (5).

This equation states that two lines of wave-lengths  $\lambda$  and  $\lambda + \delta\lambda$  will be just barely separated when the thickness of the prism's base  $t$ , multiplied by  $\frac{\delta\mu}{\delta\lambda}$ , is equal to  $\frac{\lambda}{\delta\lambda}$ .

As an example we may calculate the thickness of a prism which will just separate the sodium lines. We must first get a value for  $\frac{\delta\mu}{\delta\lambda}$ .

This we can do by differentiating the dispersion formula

$$\mu = A + \frac{B}{\lambda^2},$$

$$\frac{\delta\mu}{\delta\lambda} = -\frac{2B}{\lambda^3}.$$

The value of  $B$  varies with the material of the prism. Let

$$B = .984 \times 10^{-10}, \text{ which is for extra dense flint,}$$

$$\lambda = 5.890 \times 10^{-5} \text{ cms.,}$$

$$\delta\lambda = .006 \times 10^{-5} \text{ cms. (difference between } D_1 \text{ and } D_2).$$

$$\text{Therefore } t = \frac{\lambda^4}{2B\delta\lambda} = \frac{10^{10}\lambda^4}{1.968\delta\lambda} = 1.02 \text{ cms.}$$

The base of the prism must thus be at least a centimeter thick if the sodium lines are to appear separated.

Lord Rayleigh found as a result of a number of experiments that from 1.2 to 1.4 cms. were actually required, depending on the observer.

**Christiansen's Experiment** (*Wied. Ann.*, Nov. 1884).—While engaged upon some determinations of the refractive indices of white powders, by the method of immersing them in liquid mixtures of the same refractive index, Christiansen observed some very remarkable and interesting effects. Owing to the different dispersive powers of the liquid and powder, complete transparency could only be obtained for

monochromatic light. If white light was employed the transmitted light was highly colored, the transmitted color corresponding to the particular wave-length for which the two substances happened to have the same refractive index. Finely-powdered glass immersed in a mixture of benzol, and bisulphide of carbon was found to exhibit the colors well. The powder must be quite free from dirt, the elimination of which is sometimes very difficult. The author has obtained the best results with the powdered quartz, which can be procured from the large chemical houses. The powder is boiled in nitro-muriatic acid to free it from impurities and thoroughly washed in clean water. It is then dried and placed in a small flask with enough bisulphide of carbon to wet it thoroughly. Benzol is then added a little at a time until the mixture begins to get transparent. It will be found that red light is transmitted first, then yellow, green, and blue in succession as more benzol is added. It is best to stop when the transmitted light is yellow. In the general illumination of a brightly-lighted room the colors are not very pronounced, and it is best to employ a distant lamp in a fairly dark room as the source of light. If a permanent preparation is desired, the following method gives good results. A quantity of the quartz powder is introduced into a 100 c.c. flask (not more than  $\frac{1}{10}$  of the volume of the flask), the neck of which is then drawn down in a blast until it has a diameter of only a few mms. The liquid previously adjusted in the manner described is then introduced in sufficient quantity to form a rather thick pasty mass, which will stick in a thick layer to the walls of the flask if it is shaken. The flask is then packed in powdered ice and salt and the neck closed by fusion in the flame. The freezing mixture is necessary on account of the inflammability of the vapor and its tension, and it is a good plan to wrap a towel around the beaker containing the flask and cooling mixture in case of explosion. On removing the flask from the ice it will be found to be quite opaque, owing to the change in the refractive index of the liquid. As the temperature rises red light is transmitted first, and by slightly warming the flask in spots by momentary contact with a flame or even with the fingers all colors of the rainbow may be made to appear simultaneously, the whole appearing like a great opal. The reason of these beautiful temperature changes will be readily understood by reference to Fig. 88.

Suppose *A* to be a linear source of light which is deviated to the right and spread out into two spectra by prisms of the same angle, the one composed of quartz the other of the liquid mixture. The refractive indices having the same value for yellow light, the yellow of one spectrum will fall immediately above the yellow of the other. Since, however, the liquid has a much higher dispersion it will yield a longer spectrum and the other colors will not be in coincidence, or in other words the refractive indices are different for all the other colors. The color transmitted will obviously be the one for which we have

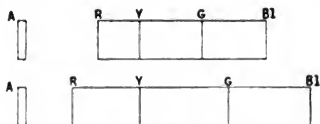


Fig. 88.

coincidence in the above schematic diagram. The other colors will be more or less scattered by irregular refractions and reflections. Suppose now we heat the mixture, the increase of temperature will cause the liquid to expand and its refractive index to decrease, while the effect upon the quartz is comparatively slight. This will mean a shift of the lower spectrum in the diagram towards the left, the green regions of the two spectra coming into coincidence, while a further increase of temperature will bring the blue regions together. The effect of the warming is thus to shift the region of transmission down the spectrum towards the blue.

In general, unless the thickness of the heterogeneous medium is considerable, the light which is not directly transmitted emerges to some extent as diffused light. The color of this diffused light is complementary to the transmitted, and the green image of a lamp flame seen through a thin layer of the paste is surrounded by a purple halo. The colors seen when equal volumes of glycerine and turpentine are shaken together into an emulsion are of similar nature, though erroneously attributed to interference in some text books. The opalescent precipitate obtained by the addition of hydrofluosilic acid to a solution of potassium chloride has been found by the author to be another case, the color of the transmitted light changing in a most beautiful manner upon the addition of water, which diminishes the refractive index of the liquid, precisely as the rise of temperature did in the case of the benzol mixture. Fuller particulars regarding these curious mixtures will be found in the original papers of Christiansen, and in an interesting paper by Lord Rayleigh (*Phil. Mag.*, xx. 358, 1885).

**Determination of the Dispersion of a Substance in the Form of a Powder.**—If a transparent substance in the form of a powder be mixed with a liquid of the same refractive index, the whole becomes optically homogeneous, and the opacity resulting from the irregular reflection and refraction of the particles disappears. Owing to the irrationality of dispersion it is not possible to obtain a liquid of exactly the same refractive index and dispersion, the mixture being optically homogeneous for a single color only: this color is transmitted as we have seen, while the other colors are scattered, and to a greater or less extent refused transmission. Suppose we wish to determine the dispersion of precipitated potassium fluo-silicate, which in the solution of KCl, in which it is formed, shows brilliant opalescent colors by transmitted light. Introduce the mixture into a hollow prism and allow it to stand until the precipitate has settled. Place the prism on the table of a spectrometer, cover the upper part of the prism with a card to cut off the light which passes through the clear liquid and examine the transmitted light with a telescope. It will be found to consist of some definite portion of the spectrum, which can be considerably narrowed by shielding all of the prism except the base. Set the cross hair of the eye-piece on the center of this band, uncover the upper portion of the prism and note the wave-length of the Fraunhofer line which comes nearest to the cross hair. Determine the refractive index of the liquid for this line in the usual manner, which will be also the refractive index of the powder for the same color. By adding KCl or water we can vary the refractive index

of the liquid, making it coincide with that of the powder for the other colors of the spectrum, and in this way the dispersion of the powder can be determined. Compare this with the dispersion of the liquid at such a density, say, that it is optically the equivalent of the powder for green light.

**Anomalous Dispersion.**—In the case of transparent substances the dispersion is said to be normal, that is, the refractive index increases as the wave-length decreases, though the rate of change varies according to the nature of the substance.

In the case of substances which show selective absorption this is not generally the case, the refractive index for the short waves on the blue side of the absorption band being less than the index for the red light on the other side of the band.

This phenomenon has been named anomalous dispersion, but, as we shall see presently, there is nothing anomalous about it, the so-called normal dispersion being nothing more than a special case of the anomalous. Fox Talbot appears to have been the first to notice the peculiar effect, but his discovery was not followed up. In 1860 Le Roux (*Ann. de Chimie et de Physique*, 3rd series, vol. xli, p. 285, 1861) discovered that a prism containing iodine vapor deviated the red rays more than the blue, the indices at a temperature of 700° C. for the red and violet being 1.0205 and 1.019. Christiansen in 1870 (*Pogg. Ann.* 1870) detected anomalous dispersion in the case of an alcoholic solution of fuchsine, which is one of the aniline dyes having a strong absorption band in the green. Of the remaining colors, the red, orange and yellow occur in the same order as in the case of a glass prism. The violet, however, is less refracted than the red, and separated from it by a dark interval. Christiansen's prism was made of two glass plates inclined at an angle of 1 degree, the solution being held between them by capillarity. The subject was next investigated by Kundt, whose papers will be found in *Pogg. Ann.* 1871, 1872. His observations

showed that the phenomenon is to be observed in the case of all bodies which possess what is known as surface color, that is, bodies which selectively reflect certain wave-lengths. Ordinary pigments do not belong to this class, the color being produced by absorption, as we shall see in a succeeding chapter.

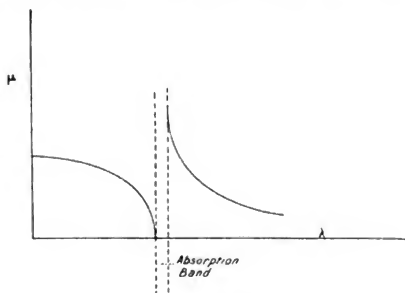


FIG. 89.

Kundt applied the method of crossed prisms, due originally to Newton, to the investigation of anomalous dispersion. If a spectrum is formed by a glass prism with its refracting edge vertical, and this spectrum is further deviated by a prism formed of an alcoholic solution of some aniline dye with its

refracting edge horizontal, the appearance seen will be similar to that shown in Fig. 89. Kundt established the law that on approaching an absorption band from the red side the refractive index is abnormally increased by the presence of the band, while if the approach is from the blue side the index is abnormally decreased. So great is the difficulty of seeing the effect with the small dispersion obtainable by alcoholic solutions, that the earlier results of Kundt were not at first accepted by some physicists of repute, the effect being attributed to a want of achromatism of the eye. The demonstration by means of crossed prisms, however, removed all doubts regarding the reality of the phenomenon.

1. Considerable trouble is usually found in repeating Kundt's experiment with fluid prisms.

The phenomenon can be studied to much better advantage by means of prisms formed by squeezing fused cyanine between plates of glass. A certain amount of dexterity is required to make good prisms, which can only be acquired by practice. Small rectangular pieces of thin German plate glass are prepared

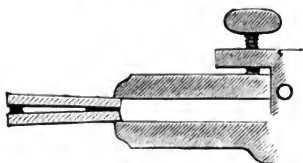


FIG. 90.

(measuring about  $2 \times 3$  cm.), and a thin strip cut from a visiting-card glued along the short side of one. A piece of cyanine about the size of a coarse shot is placed near the opposite side, and the edge of the plate heated over a small flame until the dye fuses, holding another cover-strip in the flame at the same time, in order to have both at about the same temperature. The hot edge

of the cover is now to be brought down into the cyanine, and the plate gently lowered until the edge rests on the strip of card. The plates must be at once placed under pressure in a small clamp, where they are to remain until cold. The pressure is to be applied close to the refracting edge of the prism only, as shown in figure. This is very important. Experience is the only guide to the degree of pressure required.

It will be found that there is a very narrow strip of clear glass at the refracting edge, where the glass plates have come into optical contact. This produces a diffraction-band superposed on the anomalous spectrum, but it is so faint that it is not troublesome. One has only to view a gas flame turned edge wise through the prism, the anomalous spectrum showing colors in the order orange, red, blue, green, the latter being the least deviated.

It is usually necessary to turn the prism slightly to get the green part of the spectrum; that is, the incidence should not be normal.

If a prism of this nature is covered with a small diffraction grating, the lines of which are perpendicular to the edge of the prism, the oppositely curved branches of the diffraction spectra appear most beautifully when an arc light is viewed through the combination.<sup>1</sup> If a grating is not available, the cyanine prism can be mounted over a

<sup>1</sup> Wood, *Phil. Mag.*, June 1901.

small aperture in a card and combined with a glass prism of low dispersion, or better a water prism, both being mounted on the table of a spectrometer illuminated with sun or arc light.

Other remarkable cases will be described in the following chapter.

**Anomalous Dispersion in its bearing on Solar Phenomena.**—In a communication published in the *Proceedings of the Royal Academy of Sciences*, Amsterdam,<sup>1</sup> W. H. Julius makes the very brilliant suggestion that the "flash spectrum" seen immediately at totality may be due to photosphere light abnormally refracted in the atmosphere of metallic vapors surrounding the sun: in other words, the light of the flash spectrum does not come from the reversing layer at all, but from the photosphere. He shows that the light which will be thus abnormally refracted will be of wave-lengths almost identical with the wave-lengths which the metallic vapors are themselves capable of radiating, that is, it will be light of wave-lengths nearly identical with those of the absorption bands of the vapors. This beautiful theory not only explains the apparent shallowness of the reversing layer, a thing that has always puzzled astrophysicists, but it accounts for the extraordinary brilliancy of the lines.

The theory of Julius supposes the sun to be surrounded by an atmosphere of metallic vapors, the density and refractive index of which decrease with increasing distance from the surface. In this atmosphere the rays of light coming from the photosphere will move in curved paths similar to those of rays in our own atmosphere.

The refractive index is, however, very small except for wave-lengths very near those which are absorbed by the vapor, consequently the light most strongly refracted, if it could be sorted out and examined with the spectroscope, would resemble very closely the light emitted by the vapors. Julius shows that this sorting out of the more refrangible rays may account for the bright line spectrum usually attributed to the reversing layer, these rays moving in curved paths in the solar atmosphere, thus reaching us after the photosphere has been hidden by the moon.

This phenomenon, namely the production of a bright line spectrum by the anomalous refraction of light from a white hot source, was reproduced in the laboratory by Wood, and independently by Ebert at about the same time. The conditions supposed by Julius to exist at the surface of the sun were imitated as closely as possible, and a spectrum of bright lines was obtained with light from a source showing a continuous spectrum, by means of anomalous dispersion in an incandescent metallic vapor.

For the reproduction of the phenomenon in the laboratory it is necessary to form an atmosphere of metallic vapor in which the refractive index changes rapidly from layer to layer. This was accomplished by allowing the flame of a Bunsen burner fed with metallic sodium to play against the under side of a white plaster plate. On looking along the surface of the plate it was seen that a dark space existed between the flame and the cold surface, resembling somewhat the dark space surrounding the cathode of a Crookes's tube.

<sup>1</sup> See also *Astrophysical Journal*, xii. 185.

It seemed highly probable that, inasmuch as the temperature of the flame was lowered by contact with the plate, the density of the sodium vapor would increase very rapidly from the surface of the plate downward. The under surface of the plaster plate having been thus covered with a non-homogeneous layer of sodium vapor, a spot at the edge of the flame was illuminated with sunlight concentrated by a large mirror. This spot radiated white light in every direction and corresponded to the incandescent photosphere of the sun (Fig. 91). A telescope provided with an objective direct vision prism

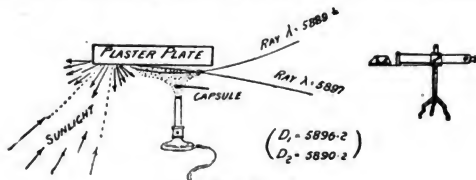


FIG. 91.

was directed toward the white spot and moved into such a position that, owing to the reduction in the width of the source of light by foreshortening, the Fraunhofer lines appeared in the spectrum. This represented the stage of an eclipse when only the thin crescent of the sun is visible. The sodium flame appeared superposed on the spectrum, of course. On moving the spectroscope until it was well inside of the plane of the illuminated surface and feeding the flame with fresh sodium, the solar spectrum vanished and there suddenly blazed out two narrow bright yellow lines, almost exactly in the place of the



FIG. 92.—FLASH SPECTRUM OF SODIUM PRODUCED BY ANOMALOUS DISPERSION.

sodium lines, as is shown in Fig. 92, in which the inverted sodium flame appears on the continuous spectrum. Cutting off the sunlight with a screen caused the instant disappearance of the lines.

Repeating the experiment it was found that the bright lines came into view on the sides of the sodium lines towards the blue, that is to say, it is light for which the medium has an abnormally low refractive index that is bent around the edge of the plate and enters the instrument. This is precisely what

we should expect, for sodium vapor has a refractive index of less than 1 for waves slightly shorter than  $D_1$  and  $D_2$ , as was shown by

Julius in his paper. The rays then will be concave upward in a medium in which the refractive index varies, as it has been supposed to vary in the present case. If the sodium vapor is very dense we see only a single bright line bordering  $D_2$ , owing to the complete absorption of the light between the lines.

A search was next instituted for the light of a wave-length slightly greater than that of the sodium lines. For these waves the vapor has a refractive index greater than 1, consequently the rays will be concave downward in the layer of vapor. If we move our prismatic telescope down in a search for these rays, the solar spectrum will appear and drown out everything, but if we set up a screen (shown in Fig. 93) in

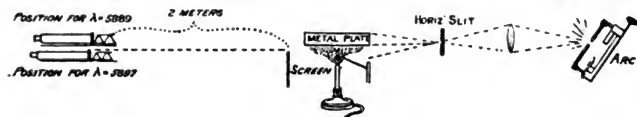


FIG. 93.

such a position as to just cut off the light from the illuminated spot, and feed the flame with sodium, we shall presently see bright lines appear on the side of the sodium lines towards the red. In this case when the vapor is dense we get only a single line bordering  $D_1$ . The path of these rays is indicated (on an exaggerated scale) in Fig. 91. The arrangement described is inconvenient in many ways to work with, and was accordingly modified in the following way.

The light of an arc lamp is focused on a horizontal slit, and a flat metal plate supported so that the plane in which its under surface lies coincides with the plane of the slit. The plate should be an inch or so thick, with a fairly level surface. At a distance of about two meters a telescope provided with a prism (direct vision if possible), arranged so as to give a vertical spectrum, is placed at such a height that the prism barely catches the rays coming from the slit and grazing the surface of the plate (Fig. 93). On looking into the telescope we see a bright continuous spectrum, and the telescope is to be raised until this becomes quite faint. The Bunsen burner beneath the plate is now to be lighted and a bit of sodium, in a small iron capsule, introduced into the center of the flame. The results obtained are practically identical with those which have been described. The flash spectrum of potassium has been obtained in a similar manner, consisting of lines in the extreme red, from one to three in number according to the density of the vapor and position of the telescope. Fair results have also been obtained with thallium.

Julius applies the anomalous dispersion theory to the prominences as well as to the reversing layer. This phenomenon can also be reproduced in the laboratory. Referring to Fig. 93, we see that its principle is identical with that of the "schlieren" apparatus of Töpler, described in the previous chapter. By arranging a similar apparatus illuminated with arc light, and setting the screens so that the field is dark, most interesting results can be obtained by heating a small capsule containing a bit of metallic sodium in front of the large lens, and placing a large direct vision prism in front of the telescope.

## CHAPTER VI.

### INTERFERENCE OF LIGHT.

THUS far we have treated single disturbances only, and have not considered the effect at a point when two or more trains of waves act on it simultaneously. We know from observation that two rays of light will cross each other without in anyway interfering with one another. The feeble rays from a faintly illuminated object will cross a region traversed by rays of great intensity without being influenced in any way so far as we can see. In this respect then light does not interfere with light. When two light waves strike the same particle of ether at the same time, its displacement is the algebraic sum of the displacements that would be produced by the waves acting separately. This is known as the principle of superposition. It was stated by Huygens in 1678 as follows. "The displacement, due to a source of small vibrations, is the same whether it acts alone or in conjunction with other sources, provided the displacements are small. This is the fundamental principle which underlies the whole subject of interference. When we consider the effect at a point which is simultaneously acted upon by two separate waves, we have then merely to sum the separate effects.

Thus, if either of the two waves acting alone would cause the ether particle to execute a vibration of unit amplitude, both together will cause it to vibrate over double the path, if the waves are in the same phase, that is, if they both reach the point at the same moment. If they reach the point in opposite phase, that is half a wave-length apart, the displacements are equal but in opposite directions, the resultant displacement being zero, or in other words the particle does not move.

We must note carefully, however, that the interference is only at this point. The waves have not destroyed each other, for each runs along beyond the point in question precisely the same as if it had not encountered the other. Indeed this must be so, for waves represent energy, and energy cannot be destroyed.

Interference then does not destroy any of the energy, and we shall see later on that whenever we produce a decrease in the illumination at any point by means of interference, we shall produce a corresponding increase at some other point, or the total illumination remains the same. That this is strictly true we shall prove presently.

The intensity of the illumination obviously depends on the amplitude of the vibration, but the relation between them is not at once obvious. We say in general that two candles produce double the illumination that one candle does, that three candles produce triple the illumination, etc. What is it that we have doubled at the point by lighting the second candle? At first sight it might appear as if we had doubled the amplitude, but we shall show presently that this is not the case. One thing, however, we can be pretty sure of, namely that we have doubled the amount of energy at the point. Now the energy in wave-motion exists partly as kinetic and partly as potential, that is, we have displaced particles at rest but possessing potential energy in virtue of their displacement from their position of equilibrium, and particles moving across the line of equilibrium which possess kinetic energy only. Other particles on the wave possess both potential and kinetic energy, and it can be shown that the total energy of the wave is equally divided between potential and kinetic. Let us now determine the relation existing between the energy and the amplitude.

**Average Kinetic Energy of a Vibrating Particle.**—The displacement of a particle at any time  $t$  is given by the equation

$$y = a \sin(\omega t - a).$$

Its velocity at any moment then will be  $v = \frac{dy}{dt} = a\omega \cos(\omega t - a)$  and its kinetic energy  $\frac{1}{2}mv^2$ , where  $m$  represents the mass of the particle.

The velocity varies from 0 to  $a\omega$ , as is clear from the above formula, and the mean energy during a complete vibration, of periodic time  $T$  is

$$\frac{1}{T} \int_0^T \frac{1}{2}mv^2 dt = \frac{ma^2\omega^2}{4T} \int_0^T 2 \cos^2(\omega t - a) dt = \frac{ma^2\omega^2}{4T} \int_0^T \left\{ 1 + \cos 2(\omega t - a) \right\} dt$$

$$= \frac{ma^2\omega^2}{4T} \left\{ t + \frac{1}{2\omega} \sin 2(\omega t - a) \right\} = \frac{1}{4}ma^2\omega^2, \text{ in which } \omega = \frac{2\pi}{T}.$$

The average energy is therefore  $\frac{ma^2\omega^2}{4} = \frac{m\pi^2a^2}{T^2},$

which can be taken as the measure of the intensity, if we define intensity as the energy in unit volume of the vibrating medium. It can be proven that the total energy is evenly divided between kinetic and potential, and since we have only considered the kinetic energy in the above treatment the total energy will be double the amount calculated. We can also define intensity as the quantity of energy transmitted in unit time across unit cross section of a plane perpendicular to the direction in which the energy is travelling. In this case the velocity of propagation enters as a factor, and we must multiply the quantity calculated above by  $v = \frac{\lambda}{T}$ , which gives us  $\frac{m\pi^2a^2\lambda}{T^3}.$

The important thing to notice is that the intensity varies directly as the square of the amplitude, and inversely as the square of the periodic time. The first is of importance in the study of interference, the second in considering the laws of radiation which will form the subject of a subsequent chapter.

If we are dealing with two sources of light which emit monochromatic radiations of the same periodic time or wave-length, their intensities are in the ratio of the squares of their amplitudes or

$$\frac{I}{I'} = \frac{a^2}{a'^2}$$

In comparing the intensities when the periodic times are different, we cannot use the eye, for it is impossible to judge accurately of the equality between two different colors. Moreover the eye cannot directly determine the true intensity, for, as we know, the true intensity or energy of the extreme red end of the spectrum is far greater than that of the yellow, while the eye is more strongly impressed by the latter. In comparing the intensities of two sources which do not emit similar radiations, we must resort to some measuring instrument which reduces them to energy of the same type, for example the thermopile or bolometer, which measures their heating power. Since the intensity of radiation varies as the inverse square of the distance from the source, as can be proven by the most elementary methods, it follows that the amplitude varies inversely as the distance.

**Composition of Vibrations.**—If we have a point moving in a circular orbit with a uniform velocity, the projection of this point on any diameter of the circle moves with harmonic motion, just as does a particle vibrating under the influence of a force directly proportional to its distance from its position of equilibrium. The point moving in a circle has an acceleration  $V^2/r$  (directed towards the center), where  $V$  = the orbital velocity, and  $r$  = the radius of the circle. This acceleration can be resolved into two components parallel and at right angles to the given diameter  $AA'$ . The one parallel to the diameter is  $V^2/r \times x/r$ , where  $x$  is the distance of  $P$ , the projection of the point on the diameter, from the center of the circle.

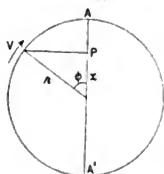


FIG. 94.

The acceleration of  $P$  is then  $\frac{V^2}{r^2} \cdot x$ , directed always towards the center, and proportional to its distance from the center. This acceleration is similar to that which the particles of an elastic body receive when moved out of their position of equilibrium, and we assume the ether particles acted on by a force of a kindred nature. The velocity with which the point  $P$  moves on the diameter is  $v = V \sin \phi$ , where  $\phi$  represents the phase.

Suppose now we require the effect on a point of two harmonic motions of equal periods and different amplitudes and phases. We can represent their motions by constructing two concentric circles with radii proportional to the amplitudes (Fig. 95).

The two harmonic motions will be represented by the projections on a diameter of two points  $G$  and  $G'$ , which move around these circles with equal angular velocity.  $P$  will then represent the position of the particle at a given time as due to the motion represented by  $G$  alone, while  $P'$  will represent its position at the same time as due to the motion represented by  $G'$  alone. If both these motions are

impressed simultaneously, the position of the particle will be represented by  $R$ , so situated that  $RC = PC + P'C$  (by the principle of superposition). The phase difference between the two vibrations is the angle  $GCG'$ , which of course remains constant. If we complete the parallelogram  $GCG'S$ ,  $R$  will represent the projection of  $S$  on the diameter, and as the parallelogram turns with  $G$  and  $G'$ , the motion of  $R$ , the projection of  $S$ , will represent the resultant motion. The diagonal of the parallelogram is evidently the amplitude of the resultant vibration, and its square measures the intensity. Now the square of the diameter of a parallelogram is by Geometry equal to the sum of the squares on two adjacent sides, plus twice the product of the sides into the cosine of the included angle. Consequently if  $a$  and  $a'$  are the amplitudes of the component vibrations, and  $e$  the phase difference between them, the resultant intensity will be

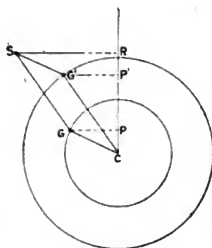


FIG. 95.

$$I = a^2 + a'^2 + 2aa' \cos e.$$

Suppose now that we have two waves of equal length and amplitude, arriving at a point in the same phase. In the above formula  $a$  will equal  $a'$ , and  $\cos e$  will equal one, therefore the resultant intensity will be  $4a^2$ , or quadruple the illumination produced by one wave alone. If the two waves reach the point a quarter of a wave-length apart the phase difference will be  $90^\circ$ , and the illumination  $2a^2$ , or twice that due to a single wave. If the phase difference is  $180^\circ$ , then  $\cos e = -1$  and the illumination becomes zero.

**Distribution of Illumination.**—If we have two similar sources of light, which are vibrating in unison, the value of  $e$  in the expression which we have just deduced will vary from point to point. Let us consider the distribution of illumination along a line, perpendicular to the direction in which the two sources lie. In this case we will consider that  $a = a'$  since the sources are similar, and we will consider the sources as lying on each side of the axis of ordinates. Taking distances along one line as abscissae, and representing the illuminations as ordinates, we have the illumination due to one source represented by a

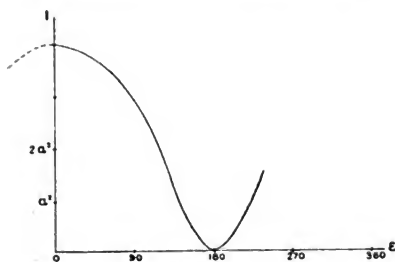


FIG. 96.

straight line parallel to the axis of abscissae, the ordinate of which is  $a^2$ . With both sources acting together the amplitude will vary from point to point; on the axis of ordinates, where the disturbances arrive

in the same phase, we have the amplitude  $2a$  and the intensity  $4a^2$ . We can express our abscissae in terms of the phase difference. If this is  $90^\circ$ , or the waves arrive a quarter of a wave-length apart,  $\cos \epsilon = 0$  and the illumination is  $2a^2$ , or double that due to one source acting alone. For  $\epsilon = 180$ ,  $\cos \epsilon = -1$  and the illumination is zero. Intermediate points can be determined by assigning different values to  $\epsilon$ , a curve similar to that shown in Fig. 96 being the result.

If now there be no loss of energy the total illumination must remain the same; we can represent this by the area comprised between the curved line and the axis of abscissae. The total intensity due to the two sources acting without interference, which would be the case if they did not vibrate in unison, would be  $2a^2$ . This is true of course only when we consider the average illumination for a time which is long in comparison to the time between certain assumed abrupt changes in the phases of the vibrating sources.

If no energy is lost the area between a line parallel to the axis of abscissae of ordinate  $2a^2$ , and the two ordinates erected at  $\epsilon = 0$  and  $\epsilon = 360$  should be equal to the area of the curve within the same limits.

The total illumination along a distance  $\Delta x$  on the axis of abscissae is

$$I_1 = 2a^2\Delta x,$$

if we assume no interference.

With interference taking place the total illumination is

$$I_2 = \int_x^{x+\Delta x} (2a^2 + 2a^2 \cos \epsilon) dx \text{—in which } x \text{ is the value for which } \epsilon = 0.$$

Since  $\epsilon$  is a linear function of  $x$  we can write  $\epsilon = Kx$ , in which  $K$  is a constant, and if  $\Delta x$  represents the distance from  $\epsilon = 0$  to  $\epsilon = 360$

$$K\Delta x = 2\pi.$$

Integrating we have

$$I_2 = 2a^2\Delta x + \int_x^{x+\Delta x} 2a^2 \cos Kx dx,$$

$$I_2 = 2a^2\Delta x + \frac{2a^2}{K} [\sin K(x + \Delta x) - \sin Kx],$$

$$I_2 = 2a^2\Delta x = I_1.$$

It must be understood clearly at the outset that to have *permanent* interference, the phase relation between the two sources must remain constant, or they must be similar; their periodic times of vibration must be the same, and any changes of phase which occur in one must occur also in the other. The only way in which this condition can be attained experimentally is by making one source the image of the other, or by dividing the bundle of rays which issue from a single source into two portions, either by reflection or refraction, and then reuniting them.

**Resultant of a Large Number of Disturbances of Arbitrary Phase.**—

We have seen that when two waves in the same phase act on a point, double the amplitude, and consequently four times the illumination, results. The question now perhaps occurs, why do not two candles produce twice the amplitude of one candle, and consequently four times the illumination? The answer to this will be readily seen

if we consider carefully the manner in which any given point is illuminated by a candle. In the flame of a candle there are countless radiating particles, in all possible phases of vibration. The illumination is due to the joint action of them all, and to determine it we must find the *resultant* of a *large* number of vibrations of arbitrary phase. In other words, if we have a great number of particles, each one of which alone would give an amplitude 1 and unit illumination at a given point, what will be the amplitude produced by all of them acting together? If there are  $n$  particles, and it so happened that all of them were vibrating in such a manner as to send vibrations in similar phase to the point, the resultant amplitude would be  $n$  and the illumination  $n^2$ . Another candle with  $n$  particles vibrating in the same manner and "in time" with the first, acting with the first, would give us an amplitude  $2n$  and an illumination  $(2n)^2$  or  $4n^2$ , that is four times the illumination of a single candle. It is obviously impossible, however, for all the particles to send their waves to the point in the same phase, for they are all vibrating independently of each other, and they are, moreover, at different distances from the illuminated point. The amplitude produced cannot then be equal to  $n$ . To determine the effect we will follow the method adopted by Lord Rayleigh. We will first simplify the problem by supposing the possible phases restricted to two opposite phases. That is, the phase difference at the point of any two waves is either 0 or 180. We thus have to consider the effect of a large number ( $n$ ) of + and - effects of equal value. If all are plus the intensity will be  $n^2$ ; if there are as many plus as minus it will be zero. If rather less than half happen to be minus, their effects will be neutralized by a corresponding number of plus quantities, and the resultant intensity will be represented by the square of the number of outstanding plus effects. It is clear that, no matter how many vibrating particles we have, the intensity will fluctuate rapidly, for it can remain constant only on the condition that the same number of (say) + effects are in excess, a circumstance that is about as unlikely as that all should be alike. But if we suppose that countless rearrangements of phase take place in the shortest time that the mind is capable of taking cognizance of, we can show that the average illumination will have a certain definite value.

If there are  $n$  particles the chance that all will be positive in any single arrangement is  $\frac{1}{2^n}$ , as can be seen by considering the analogous case of pitching coins. If three are pitched there are 8 possible arrangements of heads and tails, of which only one gives us all heads: the chance of this event happening is  $1/8$ , or  $1/2^3$ , since in this case  $n = 3$ .

The expectation of intensity corresponding to the contingency that all the effects are + is the product of the chance and the intensity, that is

$$1/2^n \cdot n^2.$$

It will be remembered that by expectation of an event we mean the actual value of an event, one chance in ten of getting \$50 is mathematically worth one chance in 50 of getting \$250.

In considering the total effect of a large number of arbitrary re-arrangements,  $1/2^n \cdot n^2$ , represents the actual value of the contingency that all effects are positive. The expectation when one effect is minus, and all the rest plus, or that  $n-1$  are +, is  $1/2^n \cdot n(n-2)^2$ .

The whole expectation of intensity is thus

$$1/2^n \left\{ 1 \cdot n^2 + n(n-2)^2 + \frac{n(n-1)}{1 \cdot 2} (n-4)^2 + \dots \right\},$$

and the sum of  $n+1$  terms of this series can be shown to be simply  $n$ . The expectation of intensity is therefore  $n$ , or the average illumination is  $n$  times the illumination produced by a single particle.

Lord Rayleigh shows that the same is true when the phases are arbitrary instead of being restricted to + and -. (*"Wave Theory": Encyclo. Brit.*)

If now we take two candles of equal power we simply increase the number of particles to  $2n$ , without in any other way affecting the conditions of the problem and the illumination or intensity is seen to be  $2n$  or twice that produced by a single candle.

**Interference of Light.**—Grimaldi, who was the first to accurately observe and describe diffraction, or the bending of light around the edges of obstacles, described as early as 1665 an experiment which he believed proved that darkness could be produced by the addition of light to light. He admitted sunlight into a darkened room through two neighboring pinholes, and received the light on a white screen. Each pinhole cast on the screen a circular image of the sun surrounded by a feebly illuminated ring. By placing the screen at such a distance from the pinholes that the outer rings overlapped, the outer edge of the ring formed by one of the holes being tangent to the outer edge of the sun's image formed by the other, he observed that the edge of the ring was less brilliant in the overlapping portion than at other places. We shall see presently that interference could not have occurred under these conditions, for two sources of light, in order to produce permanent destructive interference at a given point, must be similar—that is, must be vibrating in unison with similar amplitude and period—and two pinholes illuminated by sunlight would

not fulfil these conditions unless they were less than 0.05 mm. apart, as will be proven later on.

A century later this experiment was modified by Young, and true destructive interference of light observed. Young passed the sunlight through a pinhole, and then received the diverging cone upon two other pinholes (Fig. 97). From each one of these a divergent cone of light spread out, and where these

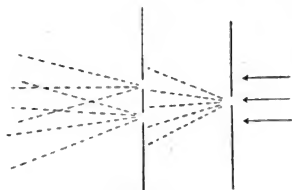


FIG. 97.

two cones of light overlapped on a screen, he observed dark and light bands. In this experiment, the two pinholes lie on the wave-front of the disturbances coming from the first hole, consequently they are always in the same phase. The dark bands are the loci of points

situated at distances from the two pinholes, differing by an odd number of half waves. The fringes in this experiment, being produced by diffracted light, did not prove that two streams of ordinary light could destroy one another at a point. Diffracted light was not well understood at the time; some modification was supposed to have taken place, and the fringes might be due in some way to this modification.

Fresnel realized the importance of producing two streams of light, capable of interfering and containing no diffracted light. The streams must come from two similar sources, and not pass the edges of any obstacles. This was accomplished by Fresnel by reflecting the rays from a point source of light from two mirrors inclined very slightly towards one another. Two virtual images of the point were thus formed behind the mirrors, separated by a very small distance, depending on the angle between the mirrors. Two mirrors of silvered or black glass,  $AB$  and  $BC$ , receive light from a point source at  $S$  (Fig. 98).

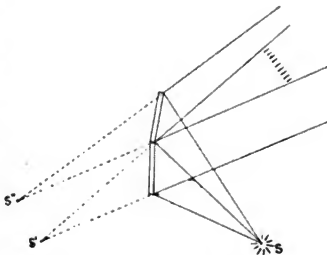


FIG. 98.

The light reflected from the two mirrors comes then from two virtual images  $S'$  and  $S''$ , which lie very close together if the angle between the mirrors is small. We thus have rays coming from the two similar sources,  $S'$  and  $S''$ , and within the region where they overlap interference takes place. The light, instead of being uniformly distributed, is collected, as it were, into bright lines with dark spaces between them. The dark bands are the places where the waves from the two sources arrive half a wave-length apart and destroy each other: at the bright bands the waves arrive together, and we have reinforcement. It is evident that as the angle between the mirrors increases, the two virtual sources  $S'$  and  $S''$  approach, coalescing when the angle equals  $180^\circ$ .

Let us now examine the form and position of the fringes.

Inasmuch as we can consider the virtual sources  $S'$  and  $S''$  as if they were real points of light, we will suppose the mirrors removed, and consider the illumination on a screen placed at a distance " $a$ " from the sources. Let  $AB$  be a section of the screen. At  $P$ , which is on a line perpendicular to the line joining the sources at its middle point, we shall have a maximum illumination, since  $P$  is equidistant from the sources, and the waves starting together reach it in similar phase and reinforce.

Going away from  $P$  we shall find a point  $M$  half a wave-length nearer  $S_2$  than  $S_1$ , and here the waves will arrive half a wave-length apart, and destroy one another. If we advance

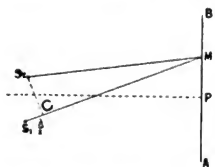


FIG. 99.

a little further along the line  $AB$  we shall reach a point where the path difference is a whole wave-length, and we shall have another maximum. Let us determine the distance of any bright or dark band from  $P$  in terms of the distance  $a$ , the distance between the sources and the wave-length of light. Around  $M$  as a center with a radius  $MS_2$ , describe an arc cutting  $MS_1$ , at  $c$ . Since  $S_1S_2$  is small in comparison to  $a$ , this arc is approximately a straight line perpendicular to the line  $MO$  ( $O$  being the point midway between the two sources).

$S_1S_2$  is perpendicular to  $OP$ , and therefore the angle  $S_1S_2C = \text{angle } MOP$ .

If the angles are equal, so also are their circular measures, or  $\frac{MP}{OP} = \frac{S_1c}{S_1S_2}$ , or calling  $x$  the distance of the dark band from the center of the fringe system, and  $s$  the distance between the sources, we have

$$\frac{x}{a} = \frac{\lambda/2}{s}.$$

The general expression then for the position of any bright or dark band will be  $x = \frac{a}{s} n \frac{\lambda}{2}$ , odd values of  $n$  corresponding to dark bands, even values to light.

It is clear from the diagram that the point  $P$  will be a maximum for light of any color or wave-length. If the source of light is white this central band will also be white. The positions of other maxima being a function of the wave-length, it follows that the spacing between the bands will be different for the different colors, consequently there will be an overlapping, and instead of white fringes with dark spaces between we shall have colored fringes, the dark minima being absent except in the immediate vicinity of the central white band.

*at* We will now take up a more complete investigation of the distribution of the maxima and minima in space. The locus of all points equidistant from two points is a plane perpendicular to the middle point of a line joining the points. The first maximum is then a plane lying between the two sources. The second maximum is the locus of all points in space so situated that the differences between their distances from the sources is one wave-length. Points fulfilling this condition lie on a hyperboloid of revolution, the sources being the foci, for by definition an hyperboloid is a surface generated by the movement of a point in such a way, that the difference between its distances from two fixed points is a constant. The locus of the second maximum will be another hyperboloid with a constant difference of 2. The loci of the maxima and minima in space form a system of confocal hyperboloids, and the fringes formed on a screen intercepting them will be hyperbolae.

A very good notion of the arrangement of these hyperboloid surfaces may be obtained in the following way. Stick two tacks into the wall about half a meter apart, and fasten to them the ends of a string about 1.5 meters long. The strings represent the paths of the rays and the tacks the sources. Hold the strings between the thumb and finger in such a way that the lengths going to the two tacks are equal, and the hand can only move up and down in a circle if the two strings are kept

tight. If the paths be increased by constantly feeding equal amounts of string through the fingers, moving the hand up and down all the while, the locus of the first maximum will be traced. Now suppose the wavelength to be represented by 5 cms. and arrange the strings so that one is 5 cms. longer than the other, and move the hand up and down; it will move in a circle as before, the circle being a section of the hyperboloid. It is possible by a little experimenting to feed equal increments of length to the two strings through the fingers, the difference between them being always 5 cms., and the hand in moving up and down will trace out the surface of the first hyperboloid, or rather one branch of it. The best way to insure feeding equal increments is to hold both strings tight behind the moving hand and allow the thumb and finger to slowly slide back, taking care that neither cord behind it becomes slack. Now increase the path difference to 10 cms., then 15, and then 20, tracing the surface for each. In this way we form a rough picture in our mind of the way the surfaces lie in space. In the case of our interference experiments the luminous points are so near together, and the screen so far removed, that its intersections with the hyperboloids are approximately straight lines.

Very satisfactory Fresnel mirrors can be made of modern mirror glass, or even of thin plate glass, unsilvered. Silvered glass is preferable owing to its greater reflecting power. The varnish can be dissolved from the silvered surface with alcohol, and the metal film polished. If glass of this description cannot be procured, a piece of thin plate glass can be chemically silvered. Two pieces measuring about 2 cms. along each edge are laid side by side on a second piece of plate glass, the outer edge of one being raised slightly by means of a narrow strip of thin paper. The edges of the plates should be in contact and both should be pressed against the supporting plate. They are then fastened in this position with a little sealing-wax. The angle between the plates should be such as to make the reflected images of an illuminated slit (situated at a distance of forty or fifty cms.) appear about 3 mms. apart. A suitable slit can be made by ruling a line on a piece of the mirror glass with the point of a knife. It should be backed with a bright sodium flame and the mirrors mounted about 30 cms. from it. The dividing line between the mirrors should be adjusted accurately parallel to the reflected images, which lie on either side of it, and the field examined at a distance of 20 or 30 cms. from the mirrors with an eye-piece or pocket magnifying-glass. The eye-piece should be held at the point at which both of the reflected images are seen. If the fringes do not appear at once they can usually be brought into view by re-adjusting the mirror for parallelism with the slit, the field being watched with the eye-piece. The distance of the  $n$ th fringe from the center of the system is given by

$$x = \frac{a+b}{2a\omega} n \frac{\lambda}{2},$$

$a$  and  $b$  being the distances of the slit and the plane in which the fringes are seen from the mirrors, and  $\omega$  the very acute exterior angle between the mirrors. If we measure this angle, which we can do with a

spectrometer, and the distance between the fringes, we can determine roughly the wave-length of the sodium light.

**Lloyd's Single Mirror.**—Even simpler than the Fresnel mirrors is the device employed by Dr. Lloyd. Here the light streams from the source and its reflected image are made to interfere. The experiment is easily repeated with a strip of plate-glass thirty or forty centimeters long and three or four wide, mounted in a clamp-stand with its surface vertical. The illuminated slit is placed a little beyond the further end, and one or two millimeters in front of the plane of the surface. If the eye is now brought up to the opposite end, the slit and its reflected image are both seen, and the fringes are easily found at this point with an eye-piece. Dr. Lloyd found that the center of the system did not lie on the plane of the surface, as might be expected, but was displaced by the width of half a fringe. This is due to the phase change which the light experiences on reflection. As the mirror is turned slowly about a vertical axis, the distance between the fringes changes. With the images close together they are broad and very easily seen; with the images farther apart they are very fine, and only seen with difficulty. This piece of apparatus is the easiest of all to work with, it being almost impossible to miss finding the fringes at the first attempt.

**Fresnel's Bi-Prism.**—In this experiment the beam of light is divided by refraction by means of a prism of very obtuse angle, as shown in Fig. 99a.

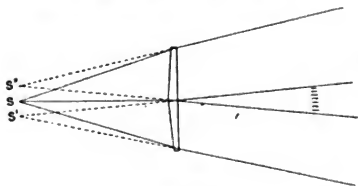


FIG. 99a.

The rays, originally emanating from a source at  $s$ , after refraction have directions as if they came from the two sources  $s'$  and  $s''$ . The illuminated slit should be parallel to the edge joining the two opposed prisms.

The wave-length of the light can be approximately determined with the bi-prism.

If  $a$  is the distance of the source from the prism,  $b$  is the distance of the plane in which the fringes are observed, and  $c$  the distance between  $s_1$  and  $s_2$ , we have, if we call  $\delta$  the angle of deviation produced by each half of the prism,

$$c = 2a \sin \delta = 2a(\mu - 1)\epsilon,$$

in which  $\mu$  is the refractive index of the glass and  $\epsilon$  the prism angle. The distance of the  $n$ th fringe from the center of the system is given by

$$x = \frac{a+b}{c} n \frac{\lambda}{2} = \frac{a+b}{2a(\mu-1)\epsilon} n \frac{\lambda}{2},$$

which shows us that the bi-prism is equivalent to a pair of Fresnel mirrors inclined at an angle  $(\mu - 1)\epsilon$ . A bi-prism can be easily made in the following manner. Heat a little Canada balsam in a watch glass over a small flame until a drop becomes nearly solid in cooling. Cut two pieces of thin plate glass measuring  $1 \times 2$  cms., and cement them, with the long edges in contact, to a second piece of plate glass with a little of the balsam, pressing the outer edges into contact with the supporting

plate; and allowing the inner edges to be slightly raised (0.5 mm. is about right) by the balsam layer, as shown in Fig. 100. A prism made in this way works quite as well as those supplied by opticians, which are made of a single piece of glass.

In using the prism, it should be mounted at a distance of about 40 cms. from the illuminated slit, and the dividing line between the plates made parallel to it. The fringes can be found with the eye piece in the same way as in the case of the mirrors, a little re-adjusting of the prism being perhaps necessary. The fringes obtained by all of the devices thus mentioned are modified by diffraction effects, due to the fact that the waves which diverge from the two virtual sources are not complete, but are abruptly cut off at the point of union of the mirrors, or at the obtuse angle of the prism.



FIG. 100.

**The Corresponding Points of the Sources.**—It is clear that continuous interference can only result between streams of light which come from corresponding parts of the two sources. Our slit is backed by a sodium flame, and even if we make it extremely narrow, the phase of the vibration will by no means be constant across its width. We must remember that the sodium flame contains countless vibrating sources of light, and continuous interference can only result between the rays emitted by one of these and its image, or between the two images of the same vibrator. A large number of these sources will be comprised

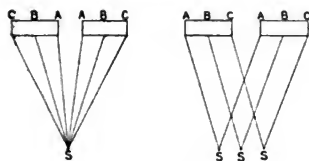


FIG. 101.

by the width of the slit, consequently interference takes place between streams of light which come from corresponding parts of the images. In the case of the Fresnel mirrors and the bi-prism the corresponding parts lie on the same side of the images, while in the case of Lloyd's single mirror they are on opposite sides. In the latter case the axis of symmetry, or the position of the central fringe, for which the path difference is zero, is the same for all the corresponding points; in the former case it is different for each pair of points. The fringes obtained with Lloyd's mirror are therefore more sharply defined, and a wider slit can be used. This will be clear by reference to Fig. 101, in which *A, B, C* are corresponding points, and *s* the axis of symmetry, which in the case of the bi-prism sources is seen to have a different position for each pair of corresponding points.

**Limit to the Number of Fringes.**—Very interesting conclusions regarding the vibrations of the molecules in the flame can be drawn from the number of fringes which can be counted. At the first dark fringe it is clear that we have destructive interference between vibrations which left the corresponding points at the same instant. At the 100th dark fringe we have interference between a vibration from one source with a vibration from the other which left the source 100 *T* earlier, if *T* is the time of the vibration. At the 1000th dark fringe we

have interference between vibrations which left the sources 1000  $\lambda$  apart. If now we consider that our molecules execute only about 1000 vibrations without any abrupt phase change, it is clear that we cannot have more than 1000 dark fringes; for, for a difference of path greater than 1000 wave-lengths, we shall have a train of waves meeting another train which left the source under different phase conditions, and which may therefore reinforce, instead of destroy, one another. The number of fringes which can be observed gives us therefore information regarding the number of regular vibrations performed by each molecule before an abrupt change occurs. Fizeau counted as many as 50,000 fringes in the case of sodium light, while improved apparatus and methods of modern times have raised the number to a million in the case of certain kinds of monochromatic light, from which we conclude that under favourable conditions as many as a million vibrations can be performed before any abrupt change takes place. We can liken the molecule to a tuning-fork struck at regular intervals with a hammer. At every blow there is an abrupt change of phase. If our fork vibrates 3000 times per second, and we strike it every two seconds, we could obtain interference with a path difference of something less than 6000 wave-lengths, while if we struck it but once in 10 seconds the path difference could be increased to nearly 30,000 wave-lengths. In this case the waves would be four inches long and the maximum path difference about 2 miles. It is obviously impossible to perform such an experiment, but the analogy is useful.

**Shift of the Fringes by Introduction of Thin Transparent Plate.**—If a thin plate of some transparent substance is put in the path of one of the interfering streams of light, the optical path will be increased owing to the retardation of the waves in the glass. If the refractive index is  $\mu$  and the thickness  $\epsilon$ , the increment of path is  $(\mu - 1)\epsilon$ , in which there are  $(\mu - 1)\frac{\epsilon}{\lambda}$  waves. Increasing the path by one whole wave-length will cause a bright or dark fringe to shift into the position of its neighbor, therefore in the above case the shift will be  $n$  fringe widths;  $n = (\mu - 1)\frac{\epsilon}{\lambda}$  if we call a fringe width the distance between two bright fringes.

When the fringes are formed with white light the introduction of the plate produces a somewhat more complicated effect. This case will be discussed presently. It is clear that we can determine the refractive index of a thin plate if we know its thickness, and measure the shift of the fringes. The above formula only holds for monochromatic light, and with light of this description the fringes are similar in appearance, and the shift cannot be determined if it exceeds one fringe width, unless it can be produced gradually as by introducing a gas slowly into a tube, the ends of which are closed with glass plates, and watching the drift of the fringes. More will be said on this subject when we come to the subject of the interferometer. If we require the actual distance through which the central fringe is shifted we can easily deduce the expression  $x = (\mu - 1)\epsilon\frac{a}{s}$ , in which  $s$  is the distance between the sources and  $a$  the distance of the screen.

**Interference Fringes with White Light.**—We have thus far considered our sources as sending out light of a single wave-length only. If we illuminate the slit with white light and examine the fringe system, we find that only a few rain-bow colored bands are visible. The cause of this is at once apparent. The formulæ which we have deduced for the distances between the fringes show us that this distance is a function of  $\lambda$  the wave-length, which occurs in the numerator of our expression for  $\chi$ , the distance of a given fringe from the center. It is thus clear that the shorter the wave, the closer together will the fringes lie. If we assume white light to be made up of waves of various lengths, the fringes will be out of-step at every point save on the axis of symmetry. The central bright fringe will coincide for all the colors, but since the red bands are about twice as broad as the violet, the bands soon get completely out-of-step, and we have practically uniform illumination.

The condition of things is shown in Fig. 102, in which the red fringes are represented by the unbroken line, the violet by the dotted line. The first dark fringe on each side of the central bright one will therefore be tinged deeply with violet. The other colors will produce other systems of bands of intermediate spacing, and it is clear that at



FIG. 102.

points a short distance from the center, we shall have maximum illumination for a large number of wave-lengths regularly distributed throughout the spectrum. The resultant illumination cannot be distinguished from white light by the unaided eye, and the field therefore appears uniformly illuminated. Interference is taking place, however, in this region just as before, as we can readily prove by substituting a small spectroscope for the eye-piece, when the spectrum will be found to be crossed by dark bands corresponding to the wave-lengths, for which the position of the slit of the instrument is a position of zero illumination. We can get a better idea of the state of affairs if we consider what happens if we place the slit of the spectroscope on the central bright band and then move it slowly out into the fringe system. At the central bright band we have all colors present, and consequently see a continuous spectrum. On moving the instrument the slit enters presently into the first dark fringe for violet, and the violet of the spectrum disappears. As we move the slit along the other colors disappear in turn, a dark band moving up the spectrum. By the time that we reach the first dark band for red, we are again in a region of maximum illumination for violet, which, therefore, appears again as the dark band in the spectrum nears the red end. It is clear that owing to the difference of spacing of the fringes, the dark bands will enter the spectrum at the violet end more rapidly than they leave it at the red end: they will consequently accumulate in the spectrum, the number increasing as we move the spectroscope further and further away from the central band. The experiment can be easily performed

by means of Lloyd's single mirror, illuminating the slit with sun or lamp-light, and substituting a small pocket spectroscope for the eye-piece. The instrument should be mounted in a clamp-stand and pointed towards the double source, the slit being close to the edge of the plate.

We will now deduce an expression for the number of bands in the spectrum. Take a point in the fringe system corresponding to the  $n$ th maximum for  $\lambda_1$ . The path difference will be

$$\delta = 2n \frac{\lambda_1}{2}.$$

Now let the wave-length decrease to  $\lambda_2$ , such that the same path difference is represented by

$$\delta = (2n + 1) \frac{\lambda_2}{2}.$$

This new value  $\lambda_2$  represents the wave-length for which the point is a minimum.

Writing 
$$2n \frac{\lambda_1}{2} = (2n + 1) \frac{\lambda_2}{2},$$

$$2n\lambda_1 = 2n\lambda_2 + \lambda_2, \text{ or adding } \lambda_1 \text{ to each side,}$$

$$2n\lambda_1 - 2n\lambda_2 - \lambda_2 + \lambda_1 = \lambda_1,$$

$$(\lambda_1 - \lambda_2)(2n + 1) = \lambda_1,$$

$$\lambda_1 - \lambda_2 = \frac{\lambda_1}{2n + 1}.$$

This expression shows us that the change in wave-length  $\lambda_1 - \lambda_2$  which is necessary to change the point from a maximum to a minimum is equal to the wave-length divided by  $2n + 1$ . If  $n$  is large, *i.e.* if we are far out in the system, the necessary change will be very small. For example, let  $\lambda_1 = .0005$ , and suppose that we are at the 50th fringe, then

$$.0005 - \lambda_2 = \frac{.0005}{101}, \quad \lambda_2 = .000495.$$

The point will therefore be a maximum or minimum for a large number of wave-lengths, within the range of the visible spectrum. Suppose now that we are at unknown point in the fringe system and wish to determine the path difference.

Let  $\lambda_1, \lambda_2, \lambda_3$ , etc., correspond to the wave-lengths (going from red to violet) of the dark bands in the spectrum.

$$\delta = (2n + 1) \frac{\lambda_1}{2} = (2n + 3) \frac{\lambda_2}{2} = (2n + 2p - 1) \frac{\lambda_p}{2}.$$

We count the number of bands between two widely separated Fraunhofer lines (if we are using sunlight). This number is  $p$  in the above equation.  $\lambda_1$  and  $\lambda_p$  are the wave-lengths of the lines between which we have  $p$  dark bands; then

$$\delta = (2n + 1) \frac{\lambda_1}{2} = (2n + 2p - 1) \frac{\lambda_p}{2}.$$

Suppose  $p = 50$ , and  $\lambda_1 = 6399$  (*C* line) and  $\lambda_p = 3967$  (*H* line).

Substituting these values, we find that  $n = 80$ , which shows us that we are dealing with the 80th minimum for each color. From this we can

calculate the path difference, which we find to be 259 half wave-lengths for the violet and 161 for the red.

Fizeau and Foucault were enabled in this way to detect interference with a path difference of 7000 waves. This result has been interpreted by many writers as indicating that the elementary components of white light must consist of periodic wave-trains, several thousand regular vibrations being executed without abrupt change of phase. Lord Rayleigh has shown, however, that we can infer nothing whatsoever about the regularity of the vibrations of the source in this case, the limit of the number of bands seen in the spectroscope depending solely on its resolving power. We shall study the case more in detail when we come to the subject of white light.

**Interference of Waves of Different Lengths. Light-Beats.**—When two tuning forks of slightly different pitch are sounded simultaneously we hear a fluttering sound, the intensity rising and falling. The interference in this case is not continuous in time at a given point in space. If we draw two wave-trains of slightly different wave-length we shall see that they are “in step” and “out-of-step” at periodic intervals. Where they are in step we shall have maximum amplitude, where they are out-of-step we shall have minimum or zero amplitude. As the double wave-trains sweep by a given point it will be in alternation the seat of large and small disturbances.

If we seek for the optical analogy it is easy to see that two sources of monochromatic light, of slightly different period, should give us a moving system of interference fringes, any given point in space being alternately the seat of maximum and minimum illumination. The frequency of the beats being equal to the difference between the two interfering trains, the wave-length of the beat is  $\frac{\lambda'\lambda}{\lambda' - \lambda}$ .

Light-beats have never been obtained by uniting two streams of light from sources of different color. Righi has, however, performed an experiment in which the frequency of vibration of one of the two streams of light which form a system of fringes can be increased any desired number of times per second by passing the light through a revolving Nicol prism. This experiment will be described in detail in the chapter on Elliptical Polarization.

It is also worthy of remark that the moving fringes observed in a Michelson interferometer, as the back mirror is advanced, can be regarded as a manifestation of light-beats, the wave-lengths of one of the interfering trains being lessened by reflection from the moving mirror by Doppler's principle. This manner of regarding the phenomenon appears to be due to Mr. A. B. Porter. The analogy has perhaps occurred to others, but my attention was first directed to it by Mr. Porter's note in *Science* 1905.

The moving system of interference fringes which constitute beats can be most beautifully shown by means of capillary waves on a mercury surface, the disturbances being originated by two tuning forks of slightly different pitch.

Two forks of the same pitch are thrown slightly out of tune by fastening small lumps of soft wax to the prongs of one of them. A light bit of wire is fastened to a prong of each fork, and after setting

the forks in vibration, the tips of the wires are dipped into the surface of clean mercury. The hyperboloid fringes will be seen to be in motion, sweeping around in a most beautiful manner. Between the wire points they will be observed to travel from one vibrating point towards the other. If the wax lumps are removed, the fringes immediately become motionless. The phenomenon can be projected on a screen to advantage, by reflecting the light down upon the mercury surface, and thence to the screen through a projecting lens, by means of a pair of mirrors or large reflecting prisms.

**Achromatic Interference Fringes.**—As we have seen, the fringes obtained with Lloyd's mirror and a source illuminated with white light, soon blend into a uniformly illuminated field, owing to the fact that the distance between the maxima and minima varies with the color. If by some artifice we can make the widths of the fringes the same, the system will become achromatic, and we can count a large number of fringes even with white light. This can be accomplished by using, as our source, a short spectrum with its blue end towards the reflecting plate. The blue sources will thus be closer together than the red, and if the adjustments are right the blue sources will give fringes of the same width as the red, which are farther apart.

The condition is best realized by employing a diffraction grating and a lens to form the spectrum. A vertical slit is illuminated with sun or arc light, and a glass grating with two or three thousand lines

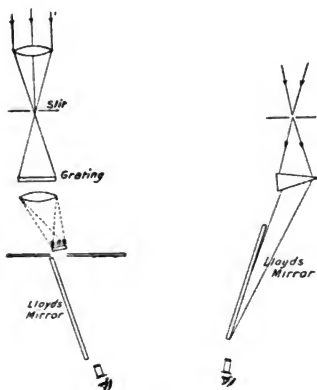


FIG. 103.

to the inch, combined with a lens of four or five inches focus, is so arranged as to form a series of diffraction spectra on a card mounted in the focal plane of the lens. The card should be perforated with a small hole through which the light of one of the first order spectra is allowed to pass. The Lloyd plate is placed in such a position as to furnish a reflected image of the spectrum, the blue end of which is turned towards the original spectrum, as shown in Fig. 103. The reflecting surface should be pointed exactly at the central image formed by the grating and lens, if perfect achromatization is desired. The fringes are viewed as before with an eye-piece, a little adjusting

of the plate being all that is necessary to completely fill the field of view with fine black and white lines.

If the spectrum is formed by a prism of about  $20^\circ$ , which can be made of plate glass, and filled with water, less perfect achromatization is obtained, still a large number of fringes can be seen.

The spectrum in this case can be virtual, *i.e.* no lens need be used, the prism being mounted between the slit and the mirror, as shown in the second diagram (Fig. 103). If the prism is placed between the mirror and the eye-piece no achromatization results, for in this case the two spectra are not opposed.

**Introduction of Thin Transparent Plate.**—As we have just seen, a plate of thickness  $e$  and refractive index  $\mu$  shifts the central fringe for monochromatic light through a distance

$$x = (\mu - 1)\epsilon \frac{a}{s} \quad S$$

This distance will be different for the different colored systems, since  $\mu$  varies with the wave-length, and there will therefore be no point at which the waves of all lengths will arrive in the same phase, or, in other words, there will be no strictly achromatic fringe.

There will, however, be a system of colored fringes with a central band which appears nearly achromatic, the determining condition of which is not that the path difference be equal to zero, but that the change in phase with change of  $\lambda$  be a minimum.

Let the distance of the approximately achromatic fringe from the original centre of the system be  $x$ .

The geometrical path difference at this point, disregarding the plate, is  $\frac{xs}{a}$ , using the same notation as before. The actual optical difference

of path is  $\frac{xs}{a} - (\mu - 1)\epsilon$ , since the shift is towards the side on which the plate is introduced, and the original short path is lengthened by the introduction of the plate. Now  $(\mu - 1)\epsilon$  is a function of  $\lambda$ , and we will write it  $f(\lambda)$ . The difference of phase at the point in question for any value of  $\lambda$  will be, writing  $D = (\mu - 1)\epsilon = f(\lambda)$  and  $D' = \frac{xs}{a}$ ,

$$2\pi \frac{D' - f(\lambda)}{\lambda}.$$

The required condition of minimum phase variation with  $\lambda$  is given by differentiating this expression with respect to  $\lambda$  and equating to zero; performing the operation we get, putting  $f'(\lambda) = \frac{df(\lambda)}{d\lambda}$

$$- \frac{2\pi f'(\lambda)\lambda}{\lambda^2} - \frac{2\pi D'}{\lambda^2} - \frac{2\pi f(\lambda)}{\lambda^2} = 0, \quad \text{or } D' = f(\lambda) - \lambda f'(\lambda).$$

The central fringe corresponding to wave-length  $\lambda$  is shifted by the plate to the position of the  $n$ th fringe given by

$$n = \frac{(\mu - 1)\epsilon}{\lambda} = \frac{f(\lambda)}{\lambda}.$$

By our original supposition regarding the position of the shifted achromatic (approximately) fringe it occupies the position of a fringe of order  $n'$  given by

$$n' = \frac{xs}{a\lambda} = \frac{D'}{\lambda} = n - f'(\lambda),$$

and is therefore shifted relatively to central fringe for monochromatic light of wave-length  $\lambda$  by a number of fringes given by  $n' - n = -f'(\lambda)$ .

The variation of  $\mu$  with  $\lambda$  is well expressed in the present case by

the equation 
$$\mu = A + \frac{B}{\lambda^2},$$

which we shall discuss more in detail when we come to the subject of dispersion.

$$\begin{aligned} f(\lambda) &= (\mu - 1)t, \\ \frac{df(\lambda)}{d\lambda} &= f'(\lambda) = -\frac{2Bt}{\lambda^3}, \\ n' - n &= \frac{2Bt}{\lambda^3}, \end{aligned}$$

which shows us that the shift of the approximately achromatic fringe obtained with white light, with respect to the central fringe obtained with monochromatic light of wave-length  $\lambda$ , varies inversely as the cube of the wave-length, and directly as the thickness of the plate. We shall have occasion to make use of this formula when we come to the subject of the interferometer.

A remarkable instance of the shifting of the region of fringe visibility far out into the system was observed by Wood<sup>1</sup> in studying the dispersion of sodium vapor with the interferometer. The path difference under which it is possible to obtain interference fringes with the  $D_3$  light of a helium tube can be nearly trebled by the introduction of a small amount of sodium vapor into the path of one of the interfering beams. This development of fringes far out in the system by the dispersive action of the vapor is accompanied by their complete disappearance at the center of the system, where the difference of path is zero.

The introduction of a medium into the path of one of the interfering beams causes a shift of the fringe system as a whole, and if the medium is dispersing, the shifts will be different for the different colors. The red, green, and blue fringes, which are out-of-step at a given point, may thus be brought into coincidence by the inequality of their respective displacements. In this case, however, since the systems are shifted as a whole, the fringes will be thrown out-of-step at the center of the system, consequently we have obtained an increased visibility far out in the system at the expense of visibility at the center. Now the helium light is very near the  $D$  lines of sodium, and sodium vapor, in this region of the spectrum has a dispersive power so great that a prism of it giving the same deviation as a  $60^\circ$  glass prism (if it could be formed) would separate two lines only  $\frac{1}{26}$  as far apart as the  $D$  lines, by an amount as great as the distance between the red and the greenish-blue of the spectrum yielded by the glass prism. This enormous dispersive power may well be expected to modify profoundly the appearance of the fringe system produced even with light as monochromatic as that of the  $D_3$  line. That a change is produced depends

<sup>1</sup> "Achromatization of approximately monochromatic interference fringes by a highly dispersive medium," *Phil. Mag.*, September 1904.

on the fact that no light is absolutely monochromatic, the finest spectrum lines having an appreciable width. We can thus consider the  $D_3$  light as an extremely short spectrum, and apply the same reasoning as in the case of a thin transparent plate introduced into the path of one of the interfering streams of white light.

The treatment will be better understood after a study of the interferometer and the resolution of spectral lines, but it is given here, on account of the identity of the phenomenon with the displacement of the white center. The helium fringes under ordinary circumstances disappear when the path difference is between 1.5 and 2 cms., there being no recurrence of visibility by further increment of path difference as in the case of sodium light. We must therefore regard the helium ( $D_3$ ) line as a single line of finite breadth or a close group of lines. In Fig. 104 let  $BC$  represent the intensity curve of the helium light,  $C$

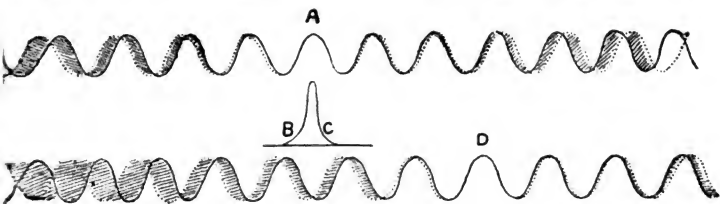


FIG. 104.

being the edge of shorter wave-length. Immediately above we have a schematic representation of the fringe system, with its center at  $A$ . Light from the side  $B$  of the  $D_3$  line will produce the fringes indicated by the dotted line, which are farther apart than the fringes formed by the light of shorter wave-length coming from the side  $C$  of the line. There will, in addition, be an infinite number of other systems formed by light of wave-lengths intermediate between  $B$  and  $C$  which are indicated by light shading.

Now suppose sodium vapor to be introduced into one path of the instrument, and the whole system shifted slightly to the left in consequence. Owing to the enormous dispersive power of the vapor, the dotted system (longer  $\lambda$ 's) will be shifted more than the other, since the  $D_3$  line lies on the blue side of the sodium absorption-band, and the change in the velocity of the light is greatest for the longest waves, namely, those on the  $B$  side of the line. The result of this dispersive action is that the fringes are brought into step at a point  $D$ , to the right of the center, thrown out-of-step at the center and still more out-of-step to the left of the center.

The achromatizing action of the sodium vapor is most beautifully shown if we illuminate the interferometer with white light.

Under ordinary conditions only two or three black and white fringes are seen, bordered on each side by perhaps a dozen rainbow-colored bands, which fade rapidly into a uniform illumination. If sodium vapor is formed in one of the interferometer paths, the colored fringes rapidly achromatize, and increase in number, breaking up, however,

into groups as shown in Fig. 105. As the density of the vapor increases the number of groups increases, each group, however, containing fewer fringes. The position of the center of the grouped system drifts in the same direction as the point of maximum visibility in the previous experiments.

The explanation of the altered appearance of the fringes in this case

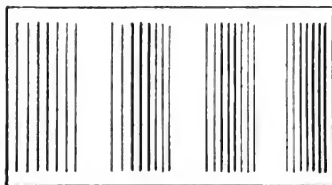


FIG. 105.

is not as simple as in those previously considered. We are dealing with two wide ranges of wave-lengths on opposite sides of the absorption-band. The fringe shifts of the two spectral regions will be in opposite directions, while the drifts of the points of maximum visibility will be in the same direction. Each set will be more or less

perfectly achromatized, and in the region in which they overlap we have a periodic visibility, owing to the difference in the widths of the fringes of the two systems.

The following treatment, which is rigorous, has been given by Lord Rayleigh.

"The remarkable shift of the bands of helium light when a layer of sodium vapour is interposed in the path of one of the interfering pencils, is of the same nature as the displacement of the white center found by Airy and Stokes to follow the insertion of a thin plate of glass. If  $D$  denote the thickness of the plate and  $\mu$  its refractive index,  $(\mu - 1)D$  is the retardation due to the insertion of the plate, and if  $R$  be the relative retardation due to other causes, the whole relative retardation is

$$R + (\mu - 1)D, \dots\dots\dots(1)$$

in which  $R$  and  $D$  are supposed to be independent of the wave-length  $\lambda$ , while  $\mu$  does depend upon it. The order of the band ( $n$ ) is given by

$$n = \frac{R + (\mu - 1)D}{\lambda} \dots\dots\dots(2)$$

For the achromatic band in the case of white light, or for the place of greatest distinctness when the bands are formed with light approximately homogeneous,  $n$  must be stationary as  $\lambda$  varies, *i.e.*

$$\frac{dn}{d\lambda} = 0. \dots\dots\dots(3)$$

For a small range of wave-length we may write

$$\lambda = \lambda_0 + \delta\lambda,$$

so that 
$$n = \frac{R + \left(\mu_0 + \frac{d\mu}{d\lambda_0} \delta\lambda - 1\right)D}{\lambda_0 + \delta\lambda}$$

$$= \frac{R + (\mu_0 - 1)D}{\lambda_0} + \frac{\delta\lambda}{\lambda_0} \left( D \frac{d\mu}{d\lambda_0} - \frac{R + (\mu_0 - 1)D}{\lambda_0} \right). \dots\dots\dots(4)$$

The achromatic band occurs, not when the whole relative retardation (1) vanishes, but when

$$R + (\mu_0 - 1)D = D\lambda_0 \frac{d\mu}{d\lambda_0}. \quad (5)$$

If  $D$  be great enough, there is no limit to the shift that may be caused by the introduction of the dispersive plate.

As Schuster has especially emphasized, the question here is really one of the *group-velocity*. Approximately homogeneous light consists of a train of waves in which the amplitude and wave-length slowly vary. A *local* peculiarity of amplitude or wave-length travels in a dispersive medium with the *group* and not with the *wave-velocity*; and the relative retardation with which we are concerned is the relative retardation of the groups. From this point of view it is obvious that, what is to be made to vanish is not (1), in which  $\mu$  is the ratio of wave-velocities  $V_0/V$ , but that derived from it by replacing  $\mu$  by  $U_0/U$ , or by  $V_0/U$ , where  $U$  is the group-velocity in the dispersive medium. In vacuum the distinction between  $U_0$  and  $V_0$  disappears, but in the dispersive medium

$$U = \frac{d(kV)}{dk}, \quad (6)^1$$

$k$  being the reciprocal of the wave-length in the *medium*. If we denote as usual the wave-length in *vacuo* by  $\lambda$ ,

$$k = \frac{2\pi\mu}{\lambda} = \frac{2\pi V_0}{\lambda V}. \quad (7)$$

Accordingly 
$$\frac{V_0}{U} = \frac{V_0 dk}{d(kV)} = \frac{d(\mu/\lambda)}{d(1/\lambda)} = \mu - \lambda \frac{d\mu}{d\lambda}. \quad (8)$$

Substituting this for  $\mu$  in (1), we see that the position of the most distinct band is given by

$$R + \left( \mu - 1 - \lambda \frac{d\mu}{d\lambda} \right) D = 0, \quad (9)$$

in agreement with (5)."

**Distribution of Phase over Small Area illuminated by Source of Finite Size.**—In Young's celebrated experiment, where interference was observed between the diffracted rays coming through two pin-holes illuminated by sunlight coming from another small hole, it is clear that the phase of the vibration must be the same at the two pin-holes. If the source of light were infinitely small, the phase relation between the vibrations passing through the two holes would be permanent, even if the holes were widely separated, but if the source has finite size, as is always the case, this will not be true. The reason for this is clear. The vibration at each hole is the resultant of the disturbances coming from the various points of the source, and this will vary with the position of the holes. We will now derive an expression for the maximum distance allowable between the holes, in terms of the size of the source and its distance, or what amounts to the same thing, the area over which we can regard the phase as constant.

<sup>1</sup> *Theory of Sound*, § 191, 1877.

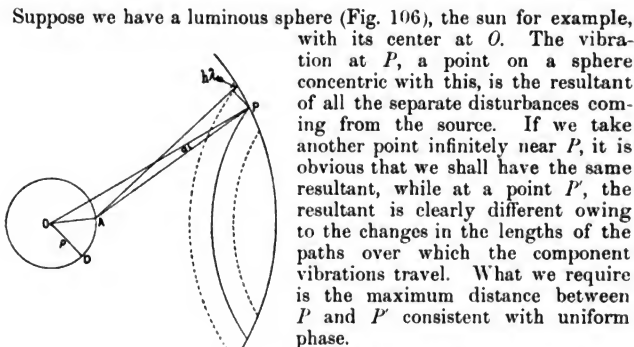


FIG. 106.

$P$ , the plane of which is perpendicular to  $OA$ , since all points on it are equidistant from  $A$ .

We now construct two other circles on the outer sphere, one just within, the other without the first circle, such that their distances from  $A$  differ by only a small fraction of a wave-length from the distance of the first circle. This will give us a narrow circular zone, over which the phase, due to vibrations coming from  $A$ , is constant.

The width of this zone will be represented by

$$x = \frac{h\lambda}{\sin \alpha},$$

in which  $\alpha$  is the angle subtended from  $P$  by  $AO$ , and  $x$  is the half width of the zone:  $h$  is a small fraction which should not be over  $\frac{1}{4}$ .

$P$ , however, receives vibrations from all other points on the luminous hemisphere, and for each one of these we can construct a zone passing through  $P$  in exactly the same manner. There will be a small area around  $P$  common to all the zones, over which the resultant phase due to all the vibrations will be the same. The largest value which  $\sin \alpha$  can have is attained when the luminous point considered is at  $D$ ,

when  $\sin \alpha = \frac{\rho}{R}$ , in which  $\rho$  is the radius of the source and  $R$  the radius of the large sphere. Inserting this value in the above equation gives us the semi-diameter of the small area which we are after. The diameter of the area is given by

$$2x = \frac{R\lambda}{2\rho} \text{ if we take } h = \frac{1}{4}.$$

Since  $\frac{2\rho}{R}$  represents the apparent diameter of the luminous source when viewed from  $P$ , we can say in general that the phase can be considered constant over a circular area not greater in diameter than

the wave-length of light divided by the apparent diameter of the source. In the case of sunlight  $\frac{\rho}{R} = \tan 16' = \cdot 005$ , and  $\lambda = \cdot 0005$  mm.

$$2x = \frac{\cdot 0005}{\cdot 01} = \cdot 05 \text{ mm.}$$

In ordinary sunlight, therefore, the phase is the same over an area measuring only  $\cdot 05$  mm. in diameter, or in a square millimeter there are 400 different states of vibration. We can easily apply our formula to Young's experiment. Suppose we form an image of the sun with a lens of 5 mm. focus. Its diameter will be  $\cdot 05$  mm. and from a distance of 1 meter  $\frac{\rho}{R}$  will be  $\cdot 00005$ . ✓

The diameter of our circle of similar phase will be 5 mm., i.e. the two pin-holes should not be over 5 mm. apart if we are to regard them as similar sources, which is the condition which we must fulfil if we wish to obtain interference fringes.

**Interference Fringes when the Sources are in Line.**—If we consider the hyperboloid fringes formed in space when waves radiate in all directions from two similar sources, to be cut by a plane perpendicular

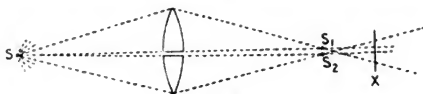


FIG. 107.

to the line joining the sources, the maxima and minima form circles which have a common center on the prolongation of the line joining the sources. We can easily see how this happens if we repeat the experiment with the string exploring the region around the line which joins the two tacks.

Fringes of this description were obtained by Meslin (*Compt. Rendus*,

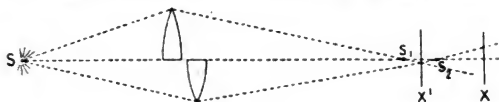


FIG. 108.

1893) by an ingenious modification of the arrangement of the Billet split lens. This instrument consists of a double convex lens, cut in halves, the two portions being slightly separated, so as to form two images  $s_1, s_2$  of the source, the fringes being observed in the plane  $X$  (Fig. 107). By displacing one of the halves as shown in Fig. 108 the sources are brought into the line of sight. The fringes will be circles in this case, but they will not be found as before in the plane  $X$ , since the beams from the two sources do not overlap, but in the plane  $X'$  between the sources, where overlapping occurs. This is in reality interference between waves radiating from a source, with waves converging to a similar source.

There are other methods by which two similar sources in line can be obtained.

It would seem at first sight as if the conditions could be easily fulfilled by putting a very minute source of monochromatic light, in front of and close to a silvered reflecting surface. A minute electric flame, obtained by passing the discharge of a small induction coil between two metal points charged with sodium, and mounted very close together, would apparently fulfil the conditions. If the experiment is tried, no fringes are seen however. This is possibly due to the impossibility of getting the source small enough, but more probably a rather peculiar factor comes into play. The light which forms the reflected image leaves the flame in a direction opposite to the stream with which it is to interfere. It is doubtful if we can consider the sources similar in this case. If we were dealing with a single sodium molecule the case would be different, but we must remember that the sodium flame has the power of absorbing precisely the radiations which it emits, consequently we should have more light from the back of the flame in the case of the reflected image, while the direct beam would consist chiefly of light from the front of the flame. Interference between these two portions is obviously impossible. There are other objections to the arrangement, as a little consideration will show.

If, however, we reflect a point source of light at two parallel surfaces we have beams capable of interfering and producing circular fringes. The reflecting surfaces must be equidistant, *i.e.* parallel, and the incidence normal. The axis of the circular fringe system will then coincide with the ray normal to the surfaces, and an eye-piece cannot be brought into such a position as to show the circular maxima and minima.

If the reflecting surfaces are half-silvered, that is coated with a film of silver of such thickness as to reflect and transmit equal quantities of light, the circular fringes can be seen when an eye-piece is held behind the plates. The distance between the source and its virtual image formed by a double reflection between the plates, is twice the distance between the reflecting surfaces. This device is essentially the form of interferometer designed by Fabry and Perot, and will be discussed more in detail presently.

**Interference Fringes along Caustics.**—In the chapter on Reflection we have seen that the caustic surfaces are traced by the cusped wave-fronts. Just within the caustic we therefore have two wave-fronts, which travel obliquely with respect to each other, and which came originally from the same source. Let  $c, c'$  be the caustic traced by the cusped waves, the crests of which are represented by solid lines, the troughs by dotted lines (Fig. 109). Along the line  $A$  we have crests meeting troughs, and a consequent destructive interference, while along the line  $B$  we have similar phases, and maximum illumination. It is clear from the diagram that the interference is between a portion of the wave which has passed through a focus with one which is converging to a focus, the condition being



FIG. 109.

similar in some respects to experiment with the Billet split lens, described in the previous section. These fringes are easily seen by concentrating sunlight upon a pin hole and reflecting the light from an oblique concave mirror. The fringes will be found in the region between the primary and secondary focal lines, *i.e.* along the caustic.

**Interference of Polarized Light.**—The study of the interference of polarized light was taken up by Fresnel and Arago in 1816. Young's explanation of the colors of thin plates of doubly refracting substances in polarized light was not wholly satisfactory to Fresnel. The non-appearance of color in the absence of the polarizer and analyzer had not been accounted for, Young's explanation being simply that the color was produced by interference between the ordinary and extraordinary rays emerging from the thin plate.

Fresnel made as a preliminary experiment the following. Employing a thin crystal of Iceland spar in the same manner as a bi-prism, he looked for interference fringes in the overlapping portion of the two bundles of rays into which the doubly refracting crystal divided the incident light. He had, what amounted to two similar sources of light, radiating beams polarized at right angles to each other. No fringes were observed. To compensate for the path difference resulting from the difference of retardation between the two rays, Fresnel placed a glass plate of calculated thickness in the path of the least retarded beam. This seemed necessary to Fresnel, because at the time he was unaware of the fact that interference was possible under conditions involving considerable difference of path. The introduction of the retarding plate gave rise however to diffraction fringes, which made it difficult to draw conclusions.

Fresnel accordingly modified the experiment, reflecting the light after its passage through the crystal, from a glass plate of such thickness that the path difference between rays reflected from the front and back surface was the same as the path difference between the ordinary and extraordinary rays emerging from the crystal plate. This arrangement might be expected to show interference between the ordinary ray reflected from the front surface of the glass plate and the extraordinary ray reflected from the back surface, but there was no trace of a fringe system. A still better arrangement was then tried by Fresnel, consisting of two crystals of equal thickness with their principal sections at right angles to one another. The ordinary ray from the first crystal is refracted as an extraordinary ray in the second and *vice versa*, the result being two beams polarized at right angles to each other with no path difference between them: in this case also no fringes were observed. These experiments established the fact that the two beams of light polarized at right angles to each other, into which a doubly refracting crystal divides ordinary light, are incapable of interfering. Arago then devised an experiment in which the two polarized rays were obtained independently of double refraction. Two parallel slits in close proximity were illuminated as in Young's experiment, and behind each was placed a pile of mica plates at the polarizing angle. By rotating either of the piles the transmitted polarized ray could be set at any angle. It was found that

when the planes of polarization were parallel, fringes were produced, but when the planes were at right angles the illumination was uniform.<sup>1</sup>

Fresnel then devised a very beautiful modification of the experiment. A selenite plate was placed in front of the two slits, and a set of fringes produced similar to those produced by the slits alone. Because of the selenite plate we must regard each slit as sending two beams polarized at right angles to each other. The two ordinary beams being polarized parallel to each other interfere and form a fringe system, and the two extraordinary rays, being also polarized parallel, give rise to a second system superposed on the first. If now two beams polarized at right angles were capable of interfering, we should have a set of fringes due to the interference of an ordinary beam from one slit, with an extraordinary beam from the other slit, and since the retardation of the two in the selenite plate is different, there would be a considerable path difference, and the fringe system would be displaced with reference to the first. As a matter of fact two systems would be found, one to the left, the other to the right of the original system. These extra fringe systems were not found however.

Fresnel then cut the selenite plate in two between the slits, and turned one half through a right angle. By this device the ordinary ray from one slit was brought into the same plane of polarization as the extraordinary ray from the other: the two sets interfered and produced two systems of fringes displaced to the right and left of the original system, owing to the retardation of the ordinary ray on the extraordinary. These experiments establish the first of the Fresnel-Arago laws that "Two rays polarized parallel will interfere, while two rays polarized at a right angle will not."

Fresnel and Arago then showed by a modification of the last experiment that two rays polarized at right angles, obtained from ordinary light, can be made parallel without thereby acquiring the property of interfering. The two piles of mica plates were placed behind the slits in such positions that the rays were polarized at a right angle, and a doubly refracting crystal was mounted behind them, with its principal section at an angle of 45 degrees to the planes of polarization. This crystal resolved each of the two plane polarized beams into two rectangularly polarized components of equal intensity. At first sight

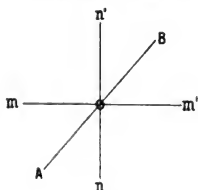


FIG. 110.

it might appear as if we had here exactly the same conditions that we had in the last experiment; that is, each slit furnishes two equally intense rectangularly polarized beams which would interfere in pairs and produce fringes. Fresnel found, however, that no fringes were formed. Let us see wherein the difference lies.

We will begin by assuming ordinary light to consist of plane polarized light, the plane of polarization changing constantly with great rapidity. Suppose at a given moment the plane of polarization of the light incident on the slits to be represented

<sup>1</sup> Two plates of tourmaline furnish an easier means of repeating this experiment.

by the line  $AB$  (Fig. 110): furthermore, let  $mm'$  and  $nn'$  be the planes of polarization of the beams transmitted by the mica plates. The vibration  $OB$  is resolved into the vibrations  $on'$  and  $om'$ ; one pile of plates transmits the former and the other pile the latter component. We thus have beams from the slits polarized at a right angle to each other. We will now bring them into the same plane by means of the doubly refracting crystal, which we will suppose to be tourmaline, since this crystal has the property of absorbing one of the rectangularly polarized components into which it divides a ray. Suppose the crystal so placed that the plane of the transmitted vibration is parallel to the plane of the vibration of the incident light (which we consider plane polarized) for an infinitesimal of time. Call this plane  $AB$  as before (Fig. 111).  $on'$  will be resolved into components, one of which,  $od$ , is transmitted, and  $om'$  is also resolved into two components, one of which,  $oe$ , is transmitted. These two beams are of equal intensity and polarized in the same plane, and will accordingly produce a set of fringes. And now comes the important point.

The plane of polarization of the incident light is changing with inconceivable rapidity all the time. In the next infinitesimal of time suppose the plane turned through a right angle to the position  $A'B'$ . The piles of mica plates resolve this into  $on'$  and  $om$  at right angles to each other. The tourmaline plate only transmits the components which are parallel to  $AB$ , namely,  $od$  and  $od'$ , which do not coincide, but have a phase difference of  $180^\circ$ . The two equally intense beams give rise to a set of fringes as before, but a phase difference of  $180^\circ$  exists between them, the maxima fall in the places occupied by the minima of the previous case. The non-interference of two beams polarized at a right angle, obtained from ordinary light and brought to the same plane of polarization, is then only apparent. What we really have are maxima and minima, which change place as rapidly as the plane of polarization of ordinary light changes, and the result is uniform illumination. Were it possible to take an absolutely instantaneous photograph of the illuminated field we should probably find the fringes.

If two beams of light polarized at right angles were derived originally from a polarized beam, they will interfere when brought into the same plane of polarization.

This is simply the permanent condition of what in the previous case existed only for an infinitesimal of time, and was experimentally investigated by Fresnel, who found that the position of the maxima in the fringe system depended on whether the plane, into which the rays polarized at right angles were brought, was parallel or at right

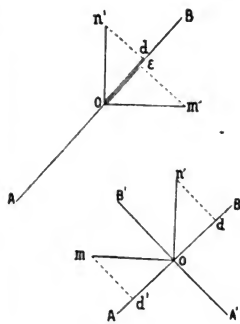


FIG. 111.

angles to the original plane, a phase difference of 180 being introduced in the latter case, as we have seen.

**The Fresnel-Arago Laws.**—These results can be summed up into what are known as the Fresnel-Arago laws.

1. Two rays polarized in the same plane interfere in the same manner as ordinary light.

2. Two rays polarized at right angles do not interfere.

3. Two rays polarized at right angles (obtained from ordinary light), and brought into the same plane of polarization, do not interfere in the ordinary sense.

4. Two rays polarized at right angles (obtained from plane polarized light) interfere when brought into the same plane of polarization.

5. In the latter case, under certain conditions, half a wave-length, corresponding to the phase difference of 180, must be added to the path difference.

**Mach's Experiment.**—The Fresnel-Arago experiments were repeated with certain modifications, including prismatic separation, by E. Mach, who devised a single piece of apparatus for demonstrating experimentally all five of the laws. The arrangement was as follows: The slit of a spectrometer is replaced by a pin-hole, on which white light is concentrated, and the telescope directed so as to view this point source. The eye-piece of the telescope is furnished with a direct vision prism, which draws out the point source into an exceedingly narrow spectrum, with the red above and the blue below. Between the telescope and collimator a vertical slit is mounted, which produces a set of diffraction fringes parallel to the edges of the slit. The distance between these fringes depends on the width of the slit and the color of the light. If we had no prismatic dispersion we should have innumerable fringe systems superposed, but the prism separates them, and gives us curved bands (*bbb*, Fig. 112), since the violet bands are closer together than the red. If now the width of the slit be doubled, the distance between the fringes is halved, and in the same space we shall have double the number, or *bbb* and *aaa*. The slit is now covered with two 1 mm. thick quartz plates (cut parallel to the axis), placed side by side, one with the axis vertical, the

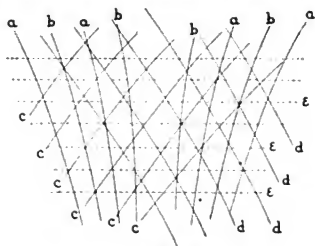


FIG. 112.

other with the axis horizontal, which divide the slit lengthwise into two equal portions. Four plane-polarized beams pass therefore through the slit. The directions of the vibrations of these four rays are represented by the arrows *aβγδ* in figure 113. Of these *γ* and *δ* will be accelerated on *a* and *β*, owing to the difference of velocity between the ordinary and the extraordinary rays in the crystal.

With this change of phase, due to retardation, the parallel vibrations *a* and *δ* would interfere, and produce the fringe system *a* and *b*. The left hand side of the slit

alone will give minima along the lines  $aa$ , and the right half alone will do the same.

The elementary waves from both sides acting together will produce minima along the lines  $bb$ . Any phase difference introduced between the waves coming from the two halves will not affect the minima  $aa$ , but will affect the minima  $bb$ , since these are produced by disturbances coming from both halves of the slit. The fringes  $bb$  will be displaced towards the side of the retarded ray. The phase difference due to the difference of velocity of the two components in the crystal is a function of the wave-length or color, being greatest for the violet, consequently the violet fringes will be shifted more than the red. The result of this is that the minima  $bb$ , instead of being parallel to  $aa$ , appear in an oblique direction, as indicated by the lines  $dd$ . Just as the vibrations  $a$  and  $d$  produce the minima  $a$  and  $d$ , the components  $b$  and  $c$  produce a set of minima coincident with the minima  $aa$ , and another oblique set  $cc$ , the displacement of which is opposite to that of the system  $dd$ . A Nicol prism is now added to the eye-piece of the viewing telescope, by means of which all the light is brought into the same plane of polarization. If the Nicol has such a position that only the components  $a$  and  $\delta$  are transmitted, the minima  $a$  and  $d$  are seen. If, however, it be turned through  $90^\circ$ , so that  $a$  and  $\delta$  are cut off, and  $\beta$  and  $\gamma$  are transmitted, the minima  $a$  and  $c$  are alone visible. If the Nicol be given a position midway between these two, all four components  $a, \beta, \gamma, \delta$  are broken up, each into two at right angles to each other, one only of each pair being transmitted. We might expect under these conditions a new set of fringes, due, for example, to the interference between the parallel components of  $a$  and  $\beta$ , but such is not the case, for we have here two rays polarized at right angles derived from ordinary light, and brought back to the same plane of polarization, and interference does not result under these conditions.

If we start with polarized light, as we can do by placing a Nicol prism before the slit, we get very interesting results. We will first set the plane of polarization of both Nicols at an angle of  $45^\circ$  degrees with the axes of the quartz plates. The plates resolve the incident vibration (vibrating in the plane  $A$ , Fig. 113) into rectangular components  $a, \beta, \gamma, \delta$ . The rectangular components  $a$  and  $\beta$ , with no phase difference between them, are brought into the same plane by the second Nicol, and consequently can interfere, since they were derived from light which was originally polarized. These produce minima  $a$  and  $b$ , as do also the components  $\gamma$  and  $\delta$ . The resultant of the parallel components of  $a$  and  $\beta$  can, moreover, interfere with the resultant of the parallel components of  $\gamma$  and  $\delta$ , and these two resultants have a phase difference depending only on the wave-length of the light, consequently a new set of horizontal minima,  $e$ , are formed, certain colors being wholly absent in the spectrum. It is important to see just how these new minima are produced: they are not diffraction minima, as are the others, but correspond to the dark bands seen in the spectrum of light which has passed through a

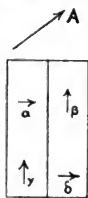


FIG. 113.

quartz plate placed between parallel Nicols. The absent colors are of course thrown out to one side by the reflecting film in the Nicol.

We have then the minima  $a$ ,  $b$ , and  $c$  when the Nicols are parallel to each other, and at angle of  $45^\circ$  to the axis of the quartz. Now suppose the eye-piece Nicol turned through  $90^\circ$ ; in its new position it will bring the components of  $a$ ,  $\beta$ ,  $\gamma$ , and  $\delta$  into a plane perpendicular to the original plane of the vibration. With parallel Nicols we have  $a$ ,  $\beta$ ,  $\gamma$ , and  $\delta$  brought into a plane parallel to the original plane, as  $a'$ ,  $\beta'$ ,  $\gamma'$ ,  $\delta'$  (Fig. 114): with Nicols perpendicular they are brought into a plane perpendicular to the original plane, as  $a''$ ,  $\beta''$ ,  $\gamma''$ , and  $\delta''$ . In the

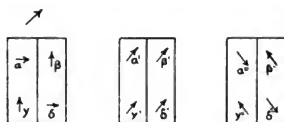


FIG. 114.

latter case we have a phase difference of  $180$  between  $a'$  and  $\beta'$ ,  $\gamma'$ , and  $\delta'$ ,  $a''$  and  $\gamma''$ , and  $\beta''$  and  $\delta''$ , as we can readily see if we construct a diagram illustrating the resolution. The result of this is that the  $b$  and  $c$  systems will show minima where the maxima were before. The  $c$  and  $d$  minima are absent in this case, as well as in the

former, only appearing when the first Nicol is parallel to one or the other of the axes of the quartz. If now the front Nicol is rotated, we obtain in succession the systems  $abe$ ,  $ad$ ,  $abe$ , and  $ac$ ; and if the Nicol is rotated rapidly the systems  $acd$  alone remain, as is the case when unpolarized light is used.

**The Colors of Thin Plates.**—The iridescent colors which are displayed by thin films of transparent substances were first investigated by Boyle. In 1665 Hooke devised the method of producing the colors by means of an air film between two lenses of large radius of curvature. He found that the colors were distributed in concentric rings, showing that they depended on the thickness of the film, and that equal thickness gave always the same color. Hooke explained the production of color as follows. A portion of the light is reflected from the upper surface, and a portion penetrates the film and is reflected from the lower surface. This portion has suffered two refractions and a reflection, and is weakened in consequence. This weaker impulse will reach the eye a moment later than the one coming from the upper surface, and Hooke supposed the sensation of yellow to be due to a weak impulse following a stronger one. If the thickness of the film increases, the weaker impulse will lag behind the stronger until it will unite with the next following stronger, finally lagging behind this sufficiently to once more produce the sensation of yellow. Thus he explained the recurrence of the color with increasing thickness. Hooke's notion was that sensation of color depended on successive impacts on the retina of strong and weak impulses. If the stronger preceded the weaker one color was produced. If *vice versa*, then another color resulted. He was right in explaining the color as produced by the union of the light streams reflected from the two surfaces, and being ignorant of the nature of white light and of wavelength as we speak of it, gave what seemed the simplest and most probable explanation of the regular sequence of the colors.

The subject was more carefully investigated by Newton, who made careful measurements of the colored rings (since known by his name) produced by the air film between a lens and a plate of glass.

It remained for Young, however, to give the true explanation that the rings were due to the interference between the wave-trains reflected from the upper and lower surface of the film.

**Newton's Rings.**—The thickness of a film of air at any point between a spherical and a plane surface in contact, is easily expressed in terms of the distance of the point from the point of tangency, and the radius of curvature of the surface.

This gave Newton the means of accurately determining the color produced by an air film of any thickness. A lens, the radius of curvature of which is known, is placed on a piece of plate glass and viewed by reflected light. Circular colored rings are seen surrounding the point of contact, the colors being most brilliant where the air film is very thin. We wish to determine its thickness  $\epsilon$ , for example, where the first yellow ring appears. Let the radius of curvature of the lens be  $R$ , and the radius of the yellow ring  $r$  (Fig. 115). We have

$$r^2 = R^2 - (R - \epsilon)^2 = 2R\epsilon - \epsilon^2 \text{ or } \epsilon = \frac{r^2}{2R}$$

since  $\epsilon^2$  is small in comparison to  $2R\epsilon$ , an expression which shows us that the thickness of the air film is proportional to the square of the radius of the ring. Newton found that with monochromatic light he got alternately bright and dark rings, and that the rings when produced by red light were larger than when produced by blue. With white light, then, we have an infinite number of ring systems superposed, just as we did fringe systems in the case of Fresnel's mirrors, and to the blending of these systems is due the complicated succession of colors observed by Newton.

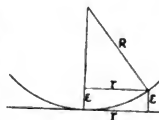


FIG. 115.

Newton found that the center of the ring system was black, and that the order of the colors was as follows: [Black, blue, white, yellow, red.] [Violet, blue, green, yellow, red.] [Purple, blue, green, yellow, red.] (Green red), (greenish blue red), [greenish blue], [reddish white]. He called the reds, the red of the first order, the second order, etc. The above list is grouped into orders by brackets. The explanation of these colors we will take up a little later. Let us first consider the manner in which a thin plate produces interference, first when the incidence is normal, then when it is oblique. It may be remarked here that in viewing the interference phenomena produced by thin plates, the eye must be focused upon the film.

Let  $AB$  be a ray incident nearly normally on the upper surface of the air film (Fig. 116). A portion is refracted to  $C$ , where it undergoes a second reflection, emerging from the upper surface after a second refraction at  $D$ . It then pursues a direction identical with the first reflected portion of some other ray  $A'D$ , very close to  $AB$ . (If the incidence is absolutely normal, it will coincide, of course, with the first reflected portion of  $AB$ .) Suppose the path  $BCD$  to be one-half

wave-length. Then the waves which have twice traversed the film will be half a wave-length behind those which are reflected from the upper surface, and if the amplitudes are the same there will be destructive interference, and no light will be reflected by the film. In other words, every ray which would ordinarily be reflected from the upper surface, will be destroyed by one coming from the under surface. If the film were infinitely thin the path  $BCD$  would be zero, and we should expect the waves to agree in phase and reinforce one another; but, as a matter of fact, we find that when the film is very thin exactly the reverse is true, *no light is reflected*: and when the path  $BCD$ , which is practically twice the film's thickness, is exactly one half wave-length, we have the two streams reinforcing each other instead of destroying each other. The explanation of this

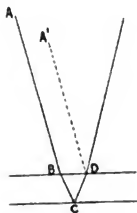


FIG. 116.

is that the two reflections take place under different conditions. At the upper surface the reflection is from a dense medium to a rare; at the lower surface, from a rare to a dense. The waves reflected at the rarer medium are reflected without change of phase, those reflected at the denser medium suffer a phase change of  $180^\circ$ . This is, of course, equivalent to a path difference of half a wave-length.

When the thickness of the plate is small in comparison with the wave-length, the waves reflected at the lower surface destroy those reflected from the upper surface, by virtue of this sudden change of phase, and no light is reflected.

This explanation was given by Young, who devised a very beautiful experiment in support of it. By using a lens of crown and a plate of flint glass with a film of cassia oil between them, he secured a system in which reflection from the upper and lower surface of the film took place under the same condition, the oil having a refractive index intermediate between the crown and flint glass. The ring system formed under these conditions had a white center, exactly in accordance with his theory.

Under normal incidence we have then the following equations for the thickness  $e$  of the film:

$$\left. \begin{aligned} e &= (2n - 1) \frac{\lambda}{4} \text{ for a maximum} \\ e &= 2(n - 1) \frac{\lambda}{4} \text{ for a minimum} \end{aligned} \right\} \text{ by reflection,}$$

where the reflection takes place under opposite conditions, and

$$\left. \begin{aligned} e &= 2n \frac{\lambda}{4} \text{ for a maximum} \\ e &= (2n + 1) \frac{\lambda}{4} \text{ for a minimum} \end{aligned} \right\} \text{ by reflection,}$$

where the conditions are the same, as in the case of the cassia oil experiment.

To determine the successive thicknesses of the film which will reflect light of wave-length  $\lambda$ , we give to  $n$  values 1, 2, 3, 4, etc., and find for  $e$  corresponding values  $\frac{\lambda}{4}$ ,  $3\frac{\lambda}{4}$ ,  $5\frac{\lambda}{4}$ , etc. That is, thicknesses corresponding to odd values of the quarter wave-length give maxima, and thicknesses corresponding to  $0$ ,  $2\frac{\lambda}{4}$ ,  $4\frac{\lambda}{4}$ , etc., give minima.

With films of such thickness that no light is reflected, the energy is not lost, but is transmitted; therefore such films have an increased transmitting power for monochromatic light, none being lost by reflection from the first surface.

The above equations show us that as we increase the thickness of the film it alternately reflects and refuses to reflect; therefore in the case of a film enclosed between a lens and a plate, the thickness of which increases as we go out from the point of contact, the locus of points in the film which reflect and which refuse to reflect, are concentric circles.

**Influence of Multiple Reflections.**—The theory of thin plates as it came from the hands of Young had an imperfection. The portions of the light reflected from the two surfaces are not equal, since the light which suffers reflection at the second surface has already been weakened by reflection at the first. The two portions should therefore never wholly destroy each other as they do when we employ monochromatic light. Poisson showed that we must take into account the multiple reflections which occur within the film. If the retardation of the ray  $A'B'CB$  on the ray  $AB$  is  $\delta = 2e \cos r$ , the retardation of consecutive rays incident at  $B''B'''$ , etc., are  $2\delta$ ,  $3\delta$ , etc. (Fig. 117). We thus know the phases of the components as they arrive at  $B$ , and to calculate their joint effects we must know their amplitudes. A certain percentage of the incident light will be reflected at the glass air surface, and a certain percentage at the air-glass surface, but we have no right to assume that the fractional part reflected is the same in each case. The following method of ascertaining the relations between the amplitudes of the reflected rays was used by Stokes.

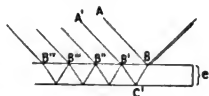


FIG. 117.

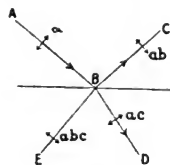


FIG. 118.

Let the amplitude of the incident ray be  $a$ , then the amplitude of the reflected ray will be  $ab$ , in which  $b$  is a fraction, and the amplitude of the refracted ray will be  $ac$ ,  $c$  being in general a fraction larger than  $b$ . By the principle of reversibility, if we send these two rays back along their paths, they should give rise to the original ray, reversed in direction of course, with the original amplitude  $a$ . If we reverse  $BC$ , however, it will give rise to two rays, one along  $BA$  of amplitude  $ab^2$ , and one along  $BE$  of amplitude  $abc$ . In reversing  $DB$  we cannot obtain the amplitudes of the reflected and refracted components by multiplying its amplitude by  $b$  and  $c$  respectively, since the reflection takes place

under different conditions. We will therefore designate the amplitudes of the reversed components of  $BD$  by  $acf$  (along  $BA$ ) and  $ace$  (along  $BE$ ). If the sum of these components is to represent a ray along  $BA$  of amplitude  $a$ , and we are to have no ray  $BE$ , as must be the case if the reversed rays give rise to the original ray only, the following relations between  $b$ ,  $c$ ,  $e$ , and  $f$  must hold:

$$acf + ab^2 = a \text{ and } ace + abc = 0.$$

These equations give us

$$cf = 1 - b^2 = 1 - \tilde{e} \text{ and } b = -e.$$

The latter equation shows us that the amplitude of the ray arising from reflection in passing from the upper to the lower medium is equal to the amplitude of a ray of equal intensity which has suffered reflection in passing from the lower to the upper medium. The fact that the sign of  $b$  is opposite to the sign of  $e$  indicates moreover that there is a relative phase change of half a period between the ray reflected under opposite conditions. This explains the absence of a ray along  $BE$  when we reverse the reflected and refracted components, the components along  $BE$  having a phase difference of 180 and destroying one another.

The perfect blackness of the interference fringes when monochromatic light is used follows at once from the above equations. The amplitude of the stream reflected from the first surface is  $ab$ . The transmitted amplitude is  $ac$ , of which  $abc$  is reflected from the lower surface, and  $abcf$  emerges into the upper medium. The amplitudes emerging into the upper medium owing to the multiple reflections form a series  $abcf + ab^3cf + ab^5cf + \dots$ . Complete interference will occur if the sum is equal to  $ab$ . This is seen to be the case, for

$$abcf(1 + b^2 + b^4, \text{ etc.}) = abcf = \frac{1 - b^{2n}}{1 - b^2} = ab \cdot \frac{cf}{1 - b^2} = ab,$$

since  $cf = 1 - b^2$ , as we have seen above.

**Colors of Iridescent Crystals and Opals.**—Some very remarkable phenomena connected with the colors of thin films are frequently exhibited in crystals of chlorate of potash. The cause of these colors was investigated by Stokes, and found to be due to the existence of planes within the crystal at which a periodic "twinning" had occurred. The colors are extremely brilliant and pure, much more so than any exhibited by soap films. An interesting paper by Lord Rayleigh will be found in the *Phil. Mag.*, xxvi., pp. 256-265, 1888. One of the most remarkable facts connected with the phenomenon is that the spectrum of the reflected light is frequently found to consist almost entirely of a comparatively narrow band. The same phenomenon is also exhibited by the fiery opal. One in possession of the author at a certain angle of incidence reflects yellow light which, when examined in the spectro-scope, is found to consist of a narrow band not much wider than the distance between the yellow mercury lines. In the case of a single thin film, of such thickness that but a single region of the spectrum is reflected, this region is always of considerable breadth. To account for the reflection of light of such a high degree of purity Lord Rayleigh

assumes that the reflection takes place at a number of thin laminae sensibly equidistant, the distance between any two being of the order of magnitude of the light wave. Quoting from his paper, "In order to explain the vigor and purity of the color reflected in certain crystals it is necessary to suppose that there are a considerable number of thin surfaces disposed at approximate equal intervals. At each angle of incidence there would be a particular wave-length for which the phases of the several reflections are in agreement. The selection of light of a particular wave-length would thus take place upon the same principle as in diffraction spectra, and might reach a high degree of perfection." Lord Rayleigh describes an interesting acoustical analogue. the sound of a bird call, giving a pure tone of high pitch, being most copiously reflected from a number of flat equidistant screens made of thin muslin, stretched upon brass rings at a certain distance apart (*Nature*, xl., p. 227, 1889).

This remarkable limitation of the reflected light to a narrow region of the spectrum will be better understood after a study of the diffraction grating. It will be interesting to compare this action of multiple films with the action of the Fabry and Perot Interferometer, which will be described in a subsequent chapter. The colored crystals of chloride of potash are not difficult to prepare. A hot saturated solution, containing a pound or more of the chlorate, is allowed to evaporate in a large flat dish. The colored crystals can easily be picked out by examining the mass in reflected light after the liquid has been poured off. The intensity of the light reflected from the thin planes will be found to vary in a remarkable manner as the crystal is turned about. The following laws were discovered by Stokes:

(1) If one of the crystalline plates be turned round in its own plane, without alteration of the angle of incidence, the peculiar reflection vanishes twice in a revolution, viz. when the plane of incidence coincides with the plane of symmetry of the crystal.

(2) As the angle of incidence is increased the reflected light becomes brighter and rises in refrangibility.

(3) The colors are not due to absorption, the transmitted light being strictly complementary to the reflected.

(4) The colored light is not polarized. It is produced indifferently whether the incident light be common light or polarized in any plane, and is seen whether the reflected light be viewed directly or through a Nicol's prism turned in any way.

(5) The spectrum of the reflected light is frequently found to consist almost entirely of a comparatively narrow band. When the angle of incidence is increased, the band moves in the direction of increasing refrangibility, and at the same time increases rapidly in width. In many cases the reflection appears to be almost total.

**Newton's Rings in White Light.**—The formula for the thickness of the plate required to produce a maximum or minimum, shows us that if we decrease  $\lambda$  we must decrease the thickness of the film; that to produce the first red maximum the thickness of the film must be  $\frac{1}{4}$  of the wave-length of the red wave, while to produce the first blue ring, it must be only  $\frac{1}{4}$  of the blue wave. The first blue maximum, therefore, lies nearer the center than the red, and the maximum for the other

colors occupy intermediate positions. An inspection of the formula for the maxima  $e = (2n - 1) \frac{\lambda}{4}$  shows us that with a given thickness, large in comparison to the wave-length, the formula will hold for a number of different values of  $\lambda$ , taking different values as we change  $\lambda$ . Thus a given thickness may fulfill the conditions of the equation for maxima for a large number of different colors. The analogy between this case and that of the fringes produced by the Fresnel mirrors with large path-difference is obvious. Suppose the thickness of the film to be .01 m., we then have  $.01 = n \frac{\lambda}{4}$  or  $.04 = n\lambda$  for a maximum when the value of  $n$  is any odd number. If we give  $\lambda$  its value for red, .0007, we find  $n$  to be 57, or we have  $n = 57$  for red, corresponding to the 28th maximum.

With violet light of wave-length .0003 we find  $n = 133$ , corresponding to the 66th maximum. Between these two values we shall have 66-28 other maxima for intermediate wave-lengths. Consequently a film measuring .01 mm. in thickness will reflect 38 different parts of spectrum and refuse to reflect 38 intermediate parts, or if we examine the light reflected from the film with the spectroscope we shall find the spectrum crossed by 38 dark bands.

It is plain that we have here a means of determining the thickness of a thin film. By examining the light reflected from it with a spectroscope and counting the number of dark bands between any two known points (Fraunhofer lines) in the spectrum, we can, by making the substitutions in the formula, calculate the thickness. In the formula which we have given we have supposed the incidence normal, and  $\lambda$  to be the value of the wave-length in the material of the film. If we are dealing with films of glass we must, of course, reduce the wave-length values to their values in glass.

The complete formula for determining the thickness of a plate of any substance with light reflected at any incidence is

$$e = \frac{n \lambda_1 \lambda_2}{2\mu \cos r (\lambda_1 - \lambda_2)},$$

in which  $n$  is the number of dark bands between wave-lengths  $\lambda_1, \lambda_2$ ,  $\mu$  the refractive index of the film, and  $r$  the angle of refraction.

Going back now to the rings themselves, we will look into the subject of the order of the colors as we go out from the center.

We know that the center is black and that violet will be the first color reflected and red the last as the film thickens. If we calculate the thickness of an air film, giving the first maximum for violet, we shall find it to be .000075 mm., while for extreme red it is .000175. But the second minimum for violet comes at .000150; therefore, before the red is reflected the violet fades away and the blue diminishes in brilliancy. If we analyze the light with a spectroscope and gradually increase the thickness, we shall see the violet appear first, followed by the other colors in order. Before the red appears a broad dark band sweeps in from the violet end (the second minimum), crossing the entire spectrum and leaving it at the red end.

These bands enter the spectrum more rapidly than they leave it, since the minima for the violet are closer together than for the red; consequently as the film increases in thickness the dark bands increase in numbers until they become too narrow to be visible.

It is clear that the color at any given distance from the center is represented by the residual spectrum. Elaborate tables have been prepared giving the sequence of the colors as they appear to the eye, but they are of little value, as they merely represent the compound tints due to the mixture of the residual parts of the spectrum. For example, suppose that a dark band cuts off the yellow and the blue, leaving the red and green. As we have seen a mixture of red light and green light produce a sensation of yellow, and we might record that at a certain thickness yellow was reflected, if we trusted to the eye, whereas the spectroscope would show that no yellow was reflected at all, but only red and green. This is an extreme case, but is illustrative of the way in which some of the colors of the rings are produced. The state of things can be best shown graphically. We will take as ordinates the thickness of the film represented in  $\frac{\lambda}{4}$  units and as abscissae the wave-lengths (Fig. 119).

Lay off on a line perpendicular to the  $x$  axis erected at a point where  $\lambda_r = .0007$ , ordinates equal to  $\frac{\lambda_r}{4}, \frac{2\lambda_r}{4}, \frac{3\lambda_r}{4} \dots \frac{n\lambda_r}{4}$ . These points will alternately represent the maxima and minima for red. Do the same when  $\lambda_v = .0003$ , using the new value of  $\lambda$  in forming the ordinates. Now join the successive pairs of points. The equation of the lowest line is evidently  $y = \frac{x}{4}$ , and the line therefore passes

through the origin. It gives us the thickness necessary to give the first maximum for any intermediate wave-length. The next line also passes through the center and gives us the thickness for the second minimum for any  $\lambda$ . The lines then represent the position in the spectrum of the alternate maxima and minima for various thicknesses. Cover up the diagram with a sheet of paper and move the paper slowly upward, thus representing a continually increasing thickness. The maxima and minima will enter the spectrum at the violet end and leave it at the red end, and the spectrum will gradually fill up with lines, just

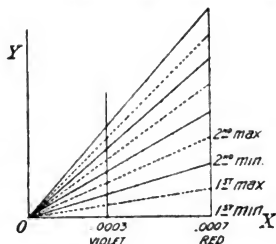


FIG. 119.

as it does in the spectroscope. The rings are visible by transmitted light as well as by reflected light, but are far feebler, for the reason that in no place is the illumination zero. The reflected system is the result of two streams of nearly the same intensity reflected from the bounding surfaces of the film, consequently the illumination of the minima is very nearly zero. The transmitted system results from the interference of two streams of very different

intensity, one directly transmitted, which is much more intense than the portion reflected, and the other a twice reflected ray. The minima are therefore not so very much darker than the maxima and the rings are not very distinct. If the reflecting surfaces are lightly silvered, the rings may be as distinct by transmitted light as they are by reflected. Two plates of glass half silvered, exhibit beautiful interference fringes when the silvered surfaces are pressed together and a sodium flame is viewed through the combination.

**Effect of a Prism upon Newton's Rings.**—It was observed by Newton that, when the colored ring system formed by a lens and a flat plate was viewed through a prism, the number of rings visible was greatly increased on one side of the system, the increase being about twelvefold. If the fringes were equidistant, as is the case with the Fresnel mirrors or the bi-prism, a prismatic shift would bring the fringes of different colors into step at a point far out in the system, but would throw them out of step at the center, so that their appearance would not be much altered. The rings, however, become narrower as we advance out into the system, and if we simplify the problem by supposing that we have only red and blue light, which are shifted through different distances by the prism, it is not difficult to see how the achromatization results, for the blue arcs, from a portion near the center, can be made to fit approximately over the red arcs in a more remote region. Now the blue rings are shifted more than the red, consequently the achromatization will occur on the side of center towards which the shifts have taken place. A full treatment of the subject will be found in Lord Rayleigh's paper on achromatic interference bands (*Phil. Mag.*, 1889). Fringes can often be found by this means on thin glass bulbs, easily made by blowing out a glass tube; sodium light will give fringes without the prism, but nothing can be seen with white light owing to the thickness of the glass.

**Achromatization of the Fringes formed by a Thin Reflecting Lamina.**—An arrangement was devised by Talbot which yielded achromatic fringes of equal widths. The achromatization which we have just considered depends upon the different widths in the different parts of the system, and cannot be applied to the equidistant fringes obtained with a wedge-shaped film.

To obtain achromatization in this case it is necessary to arrange matters so that the scale of the system is the same for the different colors. Now the scale depends on angle of the wedge (which is obviously fixed) and the angle of refraction. Under ordinary circumstances the angle of refraction is very nearly the same for the different colors, but if we employ an air film between glass plates, with the light incident in such a direction that the angle of refraction is nearly  $90^\circ$ , owing to the powerful dispersion the angle will vary with  $\lambda$ , and since the angle is greater for the blue than for the red, the blue fringes may be formed on the same scale as the red, and nearly perfect achromatization result.

To obtain a sufficiently large angle of refraction it is necessary to employ a prism of the form shown in Fig. 120. A right-angle prism answers nearly as well. It should be placed with its hypotenuse on a silvered glass plate or a plate of polished speculum metal. A horizontal

slit illuminated with white light is placed in such a position that the light is incident on the second surface at nearly the critical angle. This position can be found by lifting one end of the prism a little and watching the slit image, varying the height of the slit until a number of images appear side by side. On lowering the prism these images will run together. They are virtual images formed by multiple reflections between the prism face and the metal plate. This condition has been overlooked by the majority of previous writers, so far



FIG. 120.

as I have been able to find, and it modifies the appearance of the fringes, as we shall see presently. The colored fringes can now be found with an eye-piece, and by tilting up the edge of the prism which is nearest to the slit it is usually possible to achromatize them at once, 50 or 60 black and white bands appearing in the field. When the prism rests on the plate, the fringes are broad and highly colored. An eye accustomed to the appearance of fringe systems near the central

fringe will recognize that there is something peculiar in this case. The appearance is due to the fact that we have a number of virtual sources in line one behind the other. If the slit is illuminated with sodium light, the fringes will present a remarkable appearance. The maxima will appear bright and narrow, with broad minima between

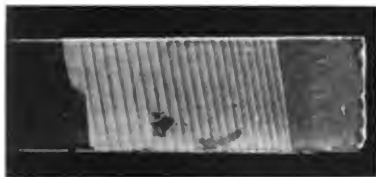


FIG. 121.

them. On one side of each maximum a number of fainter maxima will be seen, which gives a corrugated or shaded appearance to the system. A photograph of the system can be obtained by laying an orthochromatic plate on the face of the prism, and exposing it for a couple of hours, taking care to shield it from all light except that which comes through the slit. A picture obtained in this way is reproduced in Fig. 121.

The influence of the multiple reflection is to increase the steepness of the intensity curve for the maxima. A similar effect occurs in the case of the Fabry and Perot interferometer. The explanation of the phenomenon will be postponed until we come to the study of diffraction, and discuss the fringes produced by more than two similar sources.

**On the Polarized Fringes produced by the Interference of Two Streams of Light polarized at Right Angles.<sup>1</sup>** In the case of ordinary thin-film interferences the planes of vibration of the disturbances reflected from the two surfaces of the film are parallel.

<sup>1</sup> Wood, *Phil. Mag.*, April, 1904.

It is possible, however, to prepare a film which shall fulfil the requirement that the vibrations reflected from its upper surface make any desired angle with those coming from the lower surface. The path-difference between the two streams will vary with the thickness of the film; and if the amplitudes be equal we shall have the vibrations compounding into circular, elliptic, or plane ones, according to their phase-difference.

A thin glass or gelatine film, backed by a metallic reflecting surface, is all that is necessary. The incident sodium light should be polarized at an angle of  $45^\circ$  with the plane of incidence by passage through a Nicol, and the reflected light examined with an analyzing Nicol. The fringes obtained in this way present a most curious appearance, reminding one forcibly of a spectrum line with a fainter component seen in the Fabry and Perot interferometer. Their general appearance is shown in Fig. 122, which represents the fringes obtained by flowing a plate of speculum metal with a rather dilute solution of gelatine, and allowing it to dry in a slightly inclined position.

The easiest way to get them, however, is to blow out the end of a rather large glass tube into a large thin balloon of tissue glass, picking

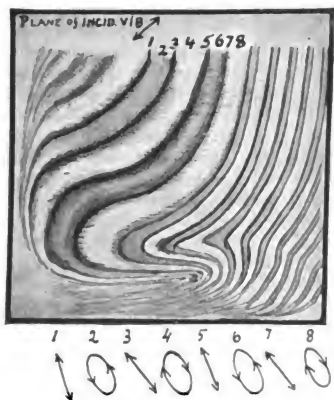


FIG. 122.

out a portion, by the light of a sodium flame, which shows fairly straight interference-fringes one or two millimeters apart. A small piece of the thin glass is laid, with its slightly convex side down, upon a clean mercury surface, and sodium light, polarized in azimuth  $45^\circ$ , reflected from the surface at an angle of about  $60^\circ$ . On viewing the reflected light through a Nicol, the curious double fringes can be easily found by slowly turning the Nicol. The light will be found to be plane-polarized along the lines 1, 3, 5, 7, etc., of Fig. 122, though in general the planes of polarization along one set of lines is inclined to the plane

of polarization along the alternate lines, as is indicated below the figure, the arrows representing the direction of the vibration (electric vector). Between the lines of plane-polarized light, which appear as dark fringes when the Nicol is so oriented as to extinguish the light, we have either elliptically or circularly-polarized light, as can at once be shown by the introduction of a quarter-wave plate, which enables us to extinguish the light along the lines 2, 4, 6, and 8, by suitable adjustment of the mica plate and analyzer. The direction of revolution of the vibration along lines 2 and 4 is opposite to that along lines 6 and 8.

To account for these fringes we must investigate the planes of polarization of the rays reflected from the two surfaces, and then compound them with various phase-differences.

The incident light vibrates in a plane indicated by the arrow at the top of Fig. 122. The light reflected from the glass surface is of course plane-polarized, vibrating parallel to the surface when the angle of incidence is equal to the polarizing angle. For larger angles of incidence, the plane of the reflected vibration makes an angle with the surface, depending on the magnitude of the reflected component, which lies in the plane of incidence. To determine the nature of the vibration coming from the glass-metal surface, it is necessary to get rid of the light reflected from the upper surface of the glass. This can be done by laying a small piece of rather thick plate-glass on a plate of speculum metal with a film of benzole between, and allowing a narrow beam of light to fall upon the surface of the glass. The images reflected from the two surfaces appear separated, and can be independently examined with a Nicol. The benzole film practically brings the metal surface into optical contact with the glass. The reflected beams will be found to be plane-polarized, the vibrations being in the directions shown in Fig. 122 for an angle of incidence near the polarizing angle. As the angle of incidence increases, the planes of the two vibrations both turn towards the vertical, and finally make an angle of  $90^\circ$  with each other, i.e.  $60^\circ$  and  $30^\circ$  with the reflecting surface.

If we compound the two components shown in Fig. 123*a* with varying phase-differences, we can account easily for the polarized fringes. In

Fig. 123*b* let  $BC$  be the vibration from the glass and  $BA$  that from the metal. When the path is zero or a whole number of waves, we have the plane-polarized resultant  $BE$ . If the path-difference is an odd number of half-waves, we have the plane-polarized resultant  $BF$ . These two states occur along the lines 1, 5 and

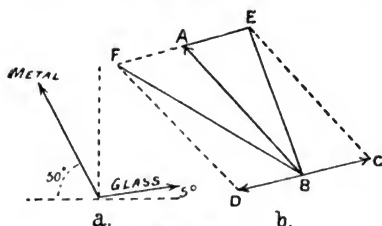


FIG. 123.

3, 7 respectively. The angles which the planes of vibration make with the reflecting surface should be respectively greater and less than the angle made by the component coming from the metal, as was found to be the case. As we increase the angle of incidence the component  $BC$  (from glass) increases in magnitude and turns up towards the vertical, the inclinations of the planes of polarization of the two sets of fringes becoming greater.

Between the lines along which the light reflected from the film is plane-polarized, we have lines of elliptically (or in some cases circularly) polarized light. The directions of revolution were determined with the quarter wave-plate, and are as shown in Fig. 122. Geometrical computations of the elliptic vibrations, resulting from two components such as are shown in Fig. 123*b*, agreed perfectly with the

experimentally determined orbits, both in respect to the directions of the major axes and the directions of rotation. The method used was similar to the one given in Müller-Powillet's text-book (vol. ii. 1, p. 1135), the only modification necessary being the rotation of one of the sets of parallel lines, which represent displacements, through a certain angle, since, in the case with which we are dealing, the two components are not at right angles. By sufficiently increasing the incidence angle, the components  $AB$  and  $BC$  can, however, be brought to very nearly a right angle. At the same time the intensity of the component from the glass surface has increased to such a degree as to be about equal to the one from the metal, and we have practically circular polarization along certain lines.

**Preparation of Films for the Exhibition of Newton's Colors.**—In the case of a thin transparent lamina, such as a soap-film, the amplitudes of the disturbances reflected from the two surfaces are equal, and consequently completely destroy each other when the phase-difference is  $180^\circ$ . Inasmuch, however, as only a small percentage of light is reflected from each surface, the colors, though saturated, are not as intense as is desirable. If a plate of mica is pressed against a pool of molten selenium on a glass plate, and the whole allowed to cool under pressure, on stripping off the mica, films of mica of variable thickness will be left upon the surface of the selenium, which show Newton's colors of great beauty, arranged in mosaics. The patches of equal thickness being sharply bounded by straight lines, present an appearance similar to that of selenite-films under the polariscope. The selenium has a much higher refractive index than the mica, consequently the reflection at each surface is the reflection of rays incident from a rare to a denser medium, and the difference of phase is given by the difference of path alone; *i.e.* we do not have the loss of half a wave-length due to reflection under opposite conditions, as would be the case if the mica films were in air.

If the mica is cemented to the plate with sealing-wax or any of the common resinous cements, very little trace of the colors is to be seen, owing to the fact that the refractive indices of the two media being so nearly the same, practically no energy is reflected from the boundary. The use of selenium can be avoided by very lightly silvering the surface of the mica, which may then be cemented to the glass with any good laboratory cement, the metallic layer taking the place of the medium of high refractive index. This latter method is the best for the preparation of large mosaics suitable for lantern-projection. In the patches which show no color by reflected light, the interference may be detected with a small spectroscope, the spectrum appearing crossed by black bands, corresponding in position to the wave-lengths absent in the reflected light. Still more brilliant films can be prepared by first thickly silvering the mica, cementing the silvered side to the glass plate, and then stripping the mica off. The colors are scarcely visible, owing to the disproportionality between the amplitudes of the two interfering streams of light, but appear as soon as the upper surface of the mica is half-silvered, which can be done by immersing the plate in a silvering solution until the colors reach their maximum brilliancy. To obtain films which show the maximum brilliancy, it is

clear that the amplitude of the stream reflected from the first surface must have the largest possible value consistent with the fulfilling of the condition that it be completely destroyed by the disturbance coming from the second surface when the phase-difference is  $180^\circ$ .

In the case of transparent films, the wave-lengths absent in the reflected light appear in excess in the transmitted light, there being no destruction of energy. If the second surface is a perfect reflector, the energy thrown down upon it by interference at the first surface will all be returned. If we consider the upper metallic surface as non-absorbing, and work out the case by the method of multiple reflections, we find that no color will be produced, light of all wave-lengths being reflected with equal facility. The fact that brilliant colors appear, means that light is absorbed at one or both of the silvered surfaces, since this is the only way in which the energy of the absent wave-lengths can disappear. If we assume a certain percentage reflected and a certain percentage absorbed by the thin silver film, we find that the waves for which the phase-difference is  $180^\circ$  are compelled to make more transits through the film than those for which the phase-difference is  $360^\circ$ . It is possible to obtain experimentally a condition in which the former are almost completely absent in the reflected light, while the latter are reflected with scarcely any loss of intensity. To calculate the most favourable conditions, we should require data regarding the percentages reflected and transmitted by films of various thicknesses.

Another method is to substitute a thin film of collodion for the mica, half-silvering the film as before.

A sheet of glass can be silvered chemically, or procured by removing the varnish from the back of a piece of modern mirror-glass with alcohol. The silver film is then flowed with collodion diluted with three or four parts of ether. As soon as the film dries, colors appear, contrary to theory. These colors may be quite brilliant, and are due to diffraction, as will be shown presently. If the plate be now immersed in Brashear's silvering-bath, the colors will instantly disappear, owing to the fact that the collodion-film and the solution have nearly the same refractive index. As soon as the silver begins to deposit, the colors reappear and increase rapidly in intensity. The bath should be rocked, the process being similar to the development of a negative. A little experience will enable the moment of maximum brilliancy to be correctly judged, when the plate should be immediately removed from the solution, washed, and dried. It is well to provide the plate with a cover of glass mounted over it at an angle of  $20^\circ$ , the whole forming a prismatic box. The object of inclining the cover is to get rid of the light reflected from it, which would otherwise dilute the interference colors. Plates prepared in this way show a wonderful blaze of color and make excellent preparations for the lantern.

**Colors of Frilled Transparent Films on Metallic Surfaces.**—We will now consider a remarkable case of interference which appears to be essentially different from any of the cases which have been previously studied. The theory of thin film shows, as Lord Rayleigh points out in his article on "Wave-Theory of Light," that a transparent film on a perfectly reflecting surface shows no interference-colors. As has been

already pointed out, a thin film of collodion deposited on a bright surface of silver shows brilliant colors in reflected light. It, moreover, *scatters* light of a color complementary to the color of the directly reflected light. This is apparently due to the fact that the collodion film "frills," the mesh, however, being so small that it can only be detected with the highest powers of the microscope. Commercial ether and collodion should be used. If chemically pure ether obtained by distillation is used, the film does not frill, and no trace of color is exhibited.

In the cases of the transparent films with the first surface lightly silvered, the second heavily coated, the waves absent in the reflected light are absorbed by the metal, as has been already shown. In the present case these waves are scattered by the granular surface. If a spot on the film which appears purple by reflected light is illuminated with sunlight, it will be found that green light is scattered, not in all directions, but through a range corresponding to the size of the granulation, as in the case of mixed plates.

If the light is incident normally, the scattered light comes off through an angular range included between  $10^\circ$  and  $30^\circ$ , and again at an angle of nearly  $90^\circ$ , the latter being strongly polarized. Conversely, if the sunlight be incident at nearly  $90^\circ$ , strongly polarized light is scattered normally. Considerable difficulty has been found in explaining these colors satisfactorily. They appear to be saturated, i.e. certain wave-lengths are completely absent in the reflected light, and until the granulation was detected with the microscope it was impossible to make even a satisfactory hypothesis. Even now the polarization effects are difficult to account for.

At first sight it may seem as if the colors could be classed with the phenomena of mixed plates, their brilliancy and saturation reminding one of the appearances produced by laminary retardation. The films, however, show no color by transmitted light when deposited on glass, and the effective doubling of the retardation, by the reflection back through the film by the metal surface, can hardly account for the observed effects. Moreover, the energy stream reflected from the surface of the collodion appears to be essential, for if we employ light polarized perpendicular to the plane of incidence, and set the plate at the polarizing angle of collodion, so that no reflection occurs except at the metal surface, all trace of color disappears. If the angle of incidence is larger than the polarizing angle, the color of the reflected light changes to its complementary tint when the plane of polarization is made parallel to the plane of incidence. As has been shown in a preceding section, the effects at large angles of incidence involve the interference of two streams of light polarized in planes inclined at  $90^\circ$  to each other, which are complicated enough with monochromatic light and structureless films. For the present only normal incidence will be considered. Though there is no direct way of proving that, in this case, the light reflected from the collodion surface is an essential factor, there is strong indirect evidence.

If the film is wedge-shaped and sodium light is employed, the dark fringes seen at normal incidence move towards the thick edge of the wedge as the angle of incidence is increased, exactly as they do with thin films of the ordinary type. If the incident light is polarized

perpendicularly to the plane of incidence, the fringes gradually fade out, disappearing at the polarizing angle. This indicates that they are produced in the same way at normal incidence as at the polarizing angle, namely, through the agency of light reflected from the surface of the collodion.

If we consider some value of  $\lambda$ , for which the path-difference between the rays reflected from the collodion and metal surfaces amounts to an odd number of half-waves, the color corresponding to this wave-length will be weakened in the reflected beam owing to interference. In the case of transparent thin films the absent color appears in excess in the transmitted light, while in the present case it is thrown back through the film by the metal surface. It is thus clear that the colors which are weakened in the reflected light are made to traverse the frilled film a greater number of times than the colors for which the path-difference is an even number of half-waves.

This accounts for the fact that these colors are more strongly scattered by the granulations of the films.

A collodion surface only reflects about 5 per cent. of the incident energy; and it is found impossible to account for the strong colors seen in the reflected light, by compounding the feeble stream of light from the collodion with the powerful stream coming from the metal.

It appeared, however, that the observed effects could be accounted for, if the somewhat arbitrary assumption were made that the granulated surface reflected more strongly than a smooth surface. As has been said, the granulations are too small to interfere with the regular reflection of light, the scattering being selective, so to speak, *i.e.* confined to the waves which, owing to interference, are compelled to traverse the film a number of times.

The assumption above referred to appeared to be too arbitrary to make without some experimental evidence, and experiments were therefore made to determine the effect of the "frilling" of the film on its reflecting power. One of the faces of a 60° prism of crown glass was flowed with collodion of the same dilution as that used in the preparation of the colored films. It showed in reflected light interference-colors, which, however, were very much diluted with white light, owing to the small difference between the refractive indices of the two media. In working with the film on silver it was found that, if the colors did not appear at once, as soon as the film dried, they could be brought out by breathing on the film, the deposit of moisture being advantageous to the formation of the granulations. It was always possible to intensify the colors in this way. The film deposited on the surface of the prism was treated in this way, one half of it being screened from the deposit of moisture by a plate of glass. As soon as the moisture had evaporated, it was found that the reflecting power of the surface had been greatly increased, the film appearing almost as bright as a half-silvered surface.

The increase in the brilliancy of the reflected light was about three-fold, as was shown by covering the unfrilled portion with a sheet of thin glass, which about equalized the intensities. In other words, the frilled collodion-surface regularly reflects white light, of an intensity very nearly equal to that of light reflected from three glass surfaces.

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On examining the granular surface with polarized light, it was found that the angle of maximum polarization was in the neighborhood of  $63^\circ$ , which would make its refractive index about 1.96. The polarizing angle of the smooth collodion was about  $56^\circ$ , the corresponding refractive index being 1.48.

An attempt was made to determine whether the granulation gave rise to elliptical polarization, the abnormal value of the refractive index suggesting the properties of the surface-films, which play such an important part in the theory of elliptical polarization. No decisive results were obtained, for though the phenomenon was found, it seemed impossible to eliminate the component reflected from the collodion-glass surface, which, as has been shown, may, by interference with the component reflected from the air-collodion surface, give rise to an elliptical vibration.

The interferometer failed to show any change in the refractive index as the result of frilling, which indicates that the effect is confined to the surface. A film deposited on glass of such thickness as to produce a shift of half a fringe width (sodium light) was frilled by moisture, one half being protected by a glass plate. No shift was found at the line of demarcation, as would have been the case if the refractive index of the film had been raised from 1.48 to 1.96 throughout its entire thickness. These phenomena are worthy of further investigation.

**Stationary Light-Waves**—In all the cases of interference which we have thus far examined, the interfering wave-trains have been moving in the same direction. In acoustics we have cases of interference where the waves are moving in opposite directions. Interference under these conditions gives rise to the so-called stationary waves. If we send a train of waves along an elastic cord, one end of which is fastened, the waves are reflected from the wall, and running backward along the cord, interfere with the direct waves. The cord is at rest at points half a wave-length apart, vibrating in nodes (Fig. 124). We should expect something of the kind to occur



FIG. 124.

when light is reflected normally from a mirror, and the possibility of such an action was early recognized. Zenker in his *Lehrbuch der Photochromie* explains the colors sometimes exhibited in photographs of the spectrum taken on silver chloride, as due to the formation of layers of reduced silver between the nodal points of a system of stationary light-waves.

The existence of these stationary light-waves was first demonstrated in 1891 by Otto Wiener.<sup>1</sup> When rays of light are incident normally on a polished mirror the reflected rays travel back over the same course. If the light is monochromatic, we shall then have planes half a wave-length apart, parallel to the plane of the mirror where the ether is at rest. The condition is shown in a crude way in Fig. 125. We require now some device for recognizing the existence of these nodal

<sup>1</sup> *Wied. Ann.*, 40, page 203, 1890.

planes in the medium in front of the mirror. Wiener made use of an exceedingly thin photograph film, the thickness of which was less than one-twentieth of the wave-length of light. Suppose such a film in the position shown by the line  $AB$ . The plane of the film now coincides with the first nodal plane. There is no vibration of the ether within the film, and though two powerful streams of light are traversing it, it will be unaffected. It is, however, unnecessary to get the film exactly into one of these planes, a matter which would, of course, be exceedingly difficult. Suppose the film to make a small angle with the mirror occupying the position  $A'B'$ . It will now cut the nodal planes, some portions of it lying in the loops and other portions lying at the nodes. Those parts of the film which are in the loops will be acted on, the other places will be unaffected. Wiener coated a glass plate with a thin photographic film, placed the film side close to the mirror, at a very small angle, and allowed monochromatic light to pass through the film and suffer reflection from the mirror. On developing the film he found it blackened along lines corresponding to the points where it intersected the loops of the standing waves, while the intervening portions were quite clear. Increasing the angle of course caused the dark lines to become finer, since a greater number of planes were cut in a given distance.

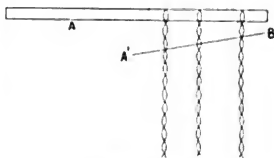


FIG. 125.

Some objections were raised against Wiener's demonstration, grounded on the fact that the layer of air between the mirror and the photographic plate would act in the same manner as the air film producing Newton's rings; in other words, the dark bands might be the result of interference between the light reflected from the mirror and the under surface of the photographic film. At first sight these objections seem quite plausible, since the maxima and minima formed in this manner would be identical in position with those obtained by Wiener. With a thin film we get a minimum when the thickness of the film is  $\frac{\lambda}{4}$  and the next minimum when it is  $\frac{3\lambda}{4}$ . Thus every time we increase the

thickness by half a wave-length we have destructive interference between the light reflected from the upper and lower surfaces. These points correspond to the nodal planes which are half a wave-length apart. The objections were satisfactorily answered by Wiener, who showed that such minima would be produced by the interference of a disturbance of large amplitude coming from the silver, and one of small amplitude coming from the collodion film used for photographing the minima and maxima. Interference fringes produced under these conditions would be of the same general character as those seen in the transmitted light of this film, *i.e.* the difference in intensity between the bright and dark bands would be very small.

Wiener finally eliminated all such action by putting a layer of benzole between the film and the metallic reflecting surface, thus making

the refractive index the same on both sides of the surface of the film, and doing away with the reflected component. The dark and bright bands were formed just as before, proving conclusively that they were caused by stationary waves. The presence of stationary waves can be shown independently of photography by employing a thin fluorescent film in place of the sensitized collodion. This experiment was performed by Drude and Nernst (*Wied. Ann.*, 45, page 460, 1892).

Another very beautiful experiment was performed by the same investigators. One half of a glass plate was coated with a film of silver, and the whole then coated with a fluorescent film only a small fraction of a wave-length in thickness. When this plate was illuminated with monochromatic light obtained from a spectroscope, it was found that the fluorescence was much stronger over the transparent portions of the plate than over the silvered portions. In the former case the film is traversed by a beam of light of which only a small percentage is reflected back through the film, consequently it fluoresces brightly. In the latter case we have stationary waves, and the film is located at the first node, which lies on the surface of the silver, a position in which fluorescence is impossible, since there is practically no vibratory motion at this point.

**Lippmann's Color Photographs.**—Zenker made the suggestion that the colors which sometimes appeared in photographs taken with chloride of silver, might be due to interference, the silver being deposited at the antinodes of a system of stationary light-waves. The layers thus formed would show the colors of thin films in reflected light. A process of direct color photography has been devised by Lippmann which is based upon this principle. The photographic plate is placed in the camera with the glass side facing the objective, and the sensitive film backed by a reflecting layer of mercury. This of course requires a special form of plate-holder.

A system of stationary waves is formed in the film as shown in Fig. 126, and the silver compound is acted upon only at the antinodes, which form planes parallel to the reflecting surface. On developing and fixing the plate in the usual manner, it is found that the film shows,

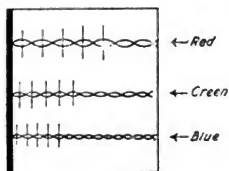


FIG. 126.

in reflected light brilliant colors, similar to the colors which illuminated it. The silver, instead of being reduced in a mass, uniformly distributed throughout the thickness of the film, is laid down in thin laminae, coinciding with the antinodal planes of the stationary light-waves. The distance between the laminae is equal to the half wave-length of the light which formed them, consequently they show the same color by interference in reflected light. The process

is not an easy one to carry out, and very few have been successful with it. Especially prepared plates must be used, as the grain of the commercial plates is too coarse to record the minute structure of the wave-system.

Some pictures in the possession of the author, for which he is indebted to the kindness of Professor Lippmann, testify to the perfection

to which the process has been brought during the past few years. In one, a view of a Swiss landscape, not only are the various flowers in an Alpine garden faithfully reproduced, but the various shades of green of the trees in the middle distance and foreground, and the blue haze which gives "atmosphere," are most faithfully reproduced.

Dr. Neuhauss of Berlin has also been very successful in repeating Lippmann's experiments, and has moreover accomplished the feat of making thin cross sections of the film, which show, under the microscope, the thin layers of silver which give rise to the color. A microphotograph of a similar section made by Mr. Senior is reproduced in Fig. 127.

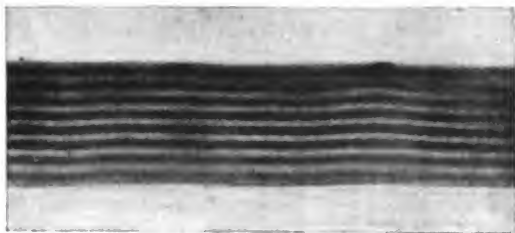


FIG. 127.

Some very remarkable results were obtained by Neuhauss in the course of some experiments made with a view of determining the action on the Lippmann film of two different spectrum colors. A continuous spectrum was projected upon a plate and an exposure made. The plate was then rotated through 180 degrees and a second exposure made, the red now falling on the end of the plate which had previously recorded the violet. On developing the plate a number of parallel dark bands parallel to the Fraunhofer lines appeared on the superposed part of the two spectra, as shown in Fig. 128. Dr. Zenker called them Talbot's bands, and Prof. Wiener refers to them as "Beats." It is, however, difficult to trace any analogy between the Neuhauss fringes and either of the above phenomena. An elaborate treatment of the subject has been given by Pfaundler (*Ann. der Physik*, 15, p. 371, 1904), based on the supposition of some sort of continuous interference between waves of different lengths. This treatment is obviously incorrect from start to finish, for no permanent phase relation could exist between radiations taken from different regions of two continuous spectra. Moreover, the exposures were made in succession, and not simultaneously.

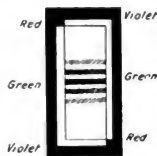


FIG. 128.

See Appendix C. for the action of mixed colors on the Lippmann film.

## CHAPTER VII.

### THE DIFFRACTION OF LIGHT.

THE rectilinear propagation of light depends, as we have seen, on the destructive interference within the region of the geometrical shadow, of the elementary waves which we may regard as originating on the wave-front. That this destruction is as complete as it is, depends on the shortness of the wave-length of light, and it is in this respect that sound differs from light. Sound we know does bend around corners, and unless we make our experiments on a very large scale, it is difficult to get any evidence of sound shadows. In the case of light, if we examine with sufficient care the propagation of the radiation by the edges of opaque obstacles, we shall find that it does bend into the region of the shadow to a slight extent, and that the bending is more pronounced for the long waves than for the short. That the bending is greater for the long waves may be shown by the following simple experiment. Sun or arc light is passed through a narrow slit, and a sharp straight edge of metal mounted in the narrow beam at a distance of a meter or so from the slit. If now we bring the pupil of the eye well within the shadow of the edge we shall see the edge strongly illuminated. This light which we see coming from the edge is the diffracted light, and it should be richer in long waves than in short. To show this we have only to view the illuminated edge through a direct vision prism. As we move our eye into the shadow the blue end of the spectrum of the diffracted light disappears first. If any doubt exists as to the reality of the phenomenon, it is easy to mount a filament of glass, just outside of the edge, which will give a spectrum of about the same intensity. By a little adjustment the red and yellow of the two spectra can be made the same, when it will be found that one shows the blue and violet distinctly while the other does not.

This simple bending of the light around the edges of obstacles is termed diffraction, though in the study of the subject we are concerned chiefly with the so-called fringes, or alternate bands of light and shadow which usually accompany diffraction. These fringes always appear just outside of the boundary of the geometrical shadow, and have in reality nothing to do with the bending of light around corners. They are termed diffraction fringes notwithstanding.

The diffraction of light was first observed by Grimaldi about the middle of the 17th century. Admitting sunlight into a darkened room

through a very small aperture, he observed that the propagation of the light by the edges of objects, did not obey strictly the laws of geometrical optics. The edges of the shadow were bordered by several rainbow tinted fringes, while in the case of very small objects similar fringes were found within the geometrical shadow. These fringes should not, however, be spoken of as diffraction fringes, though they owe their origin to diffraction. He also described the branching or crested fringes which appeared within the shadow at the corners of rectangular screens.

Newton repeated and improved upon the experiments of Grimaldi, using light of different colors, and found that the distance between the fringes decreased as the refrangibility of the light increased, and increased as the screen was removed to a greater distance from the object casting a shadow. He explained this phenomenon on his corpuscular theory as due to attractive or repulsive forces, which the edges of the obstacle exerted on the flying corpuscles.

The first attempt to bring the wave-theory to bear upon the subject was made by Young, who regarded the fringes as due to interference between the rays passing close to the edge, and rays reflected as grazing incidence. This explanation could scarcely be applied to the fringes found within the regions of shadow, and in the case of the external fringes the distances between successive maxima and minima are not such as would occur if the interference took place in the manner imagined by Young. The internal fringes he explained as due to the interference of inflected rays, without attempting to explain how the inflection took place, and in this he was in part correct, for the internal system we can regard as a set of interference fringes produced by two similar sources of light situated at the edges of the obstacle.

Fresnel made a series of experiments with slits having polished and blackened edges, and showed that the intensity of the fringes was independent of the nature of the edge.

He was the first to give the true explanation of the phenomena, regarding the maxima and minima as the result of the interference of the hypothetical secondary wavelets diverging in all directions from these portions of the wave-front not blocked off by the opaque screen.

The foundations of the theory of diffraction were laid by Fresnel, and though he treated only a few of the simpler cases, his fundamental equations were subsequently applied to, and solved for, much more complicated cases by Schwerd, Knochenhauer, and others.

We shall first examine some of the simpler cases by very elementary methods, and then take up the more rigorous treatments of Fresnel and Fraunhofer.

In commencing our study of diffraction it is best to dispense as much as possible with apparatus. Collimators and telescopes, which are so often employed for the subjective exhibition of diffraction effects, are unnecessary complications, and prevent us from obtaining the clear conception of the actual magnitude of the effects, which we get when we employ simply a brilliant source of light and a white screen, in combination with various obstacles which we place in the

path of the light. A narrow beam of sunlight, or the light from an arc lamp, should be focused with a lens on a pin-hole in a thin sheet of metal mounted over an aperture cut in a large screen of cardboard. The light diverging from the small source thus produced should be received on a screen of white Bristol board at a distance of two or three meters from the first screen, the best distance being found by trial. The room should be made as dark as possible, and if the arc-lamp is used it should be placed either in a box or in an adjoining room, and the light admitted into the room through a small hole against which the condensing lens is placed. Diffraction fringes will now appear around the edges of the obstacles placed midway between the source of light and the screen. Beautiful effects can be obtained by sprinkling a little lycopodium dust in the air, the shadow of each particle being surrounded with colored rings. Even the motes floating in the air produce colored fringes.

**Straight Edge.**—If an obstacle bounded by a straight edge is placed in the path of the rays, we find that a set of rain-bow colored fringes appear on the screen parallel to, and outside of the edge of the geometrical shadow, while within the geometrical shadow the illumination falls off gradually to zero, without showing any maxima and minima. If we examine the fringes carefully we shall see that they are not equidistant, as is the case with two small sources of light, but lie closer together as we recede from the edge.

We will now calculate the illumination at various points in the vicinity of the edge of the geometrical shadow, by dividing the wave-front, or as much of it as gets by the obstacle, into half-period elements with respect to the point in question. This method is one commonly employed in text-books, and is given merely as an example of a very elementary treatment. It is only applicable to a linear wave, *i.e.* a thin section of the actual wave which is represented by the circular arc *ST* in the diagram. In Fig. 129, *O* is the source of light, *AB* the

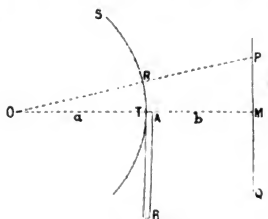


FIG. 129.

diffracting screen, *PQ* the screen on which the fringes appear; *P* is the point at which we are to determine the illumination, situated at a distance  $x$  from *M*, the edge of the geometrical shadow. The distance  $OA = a$  and  $AM = b$ . We divide the wave into half-period elements with respect to *P*, as shown in the figure. We have seen that the effect at *P* is chiefly due to a few elements on each side of the pole *R*, and that when the whole wave acts, the action of these

elements reduces by interference to a small portion surrounding the pole, approximately equal to the effect which would be produced by one half of the central element. In the case above it is clear that we have the entire half of the wave above the pole, and the portion *RA* below the pole. If *RA* contains an even number of half-periods they will destroy each other in pairs, and the illumination at *P* will be small, being due to *RS* alone. If *RA* contains an odd number of elements,

the one adjacent to the pole will remain over, after the others have interfered in pairs, and will add its effect to that produced by  $RS$ , and the illumination at  $P$  will be a maximum. As the point  $P$  moves away from  $M$ , the illumination will pass through alternate maxima and minima, the positions of which we can calculate as follows:

$$OP = a + b + \frac{x^2}{2(a+b)} \quad \text{and} \quad AP = b + \frac{x^2}{2b};$$

$$\therefore AP - RP = \frac{x^2}{2} \left( \frac{1}{b} - \frac{1}{a+b} \right).$$

If  $RA$  contains an even number of half-period elements  $AP - RP$  must equal an even number of half waves, therefore the condition for minimum illumination is

$$\frac{x^2}{2} \frac{a}{a(a+b)} = 2n \frac{\lambda}{2},$$

while if  $RA$  contains an odd number of elements, the condition for maximum illumination, we have

$$\frac{x^2}{2} \left( \frac{a}{b(a+b)} \right) = (2n+1) \frac{\lambda}{2}.$$

which gives us  $x = \sqrt{\frac{b(a+b)}{a}} 2n\lambda$  for the minima,

$$x = \sqrt{\frac{b(a+b)}{a}} (2n+1)\lambda \text{ for the maxima.}$$

The maxima are therefore spaced like the Fresnel zones, *i.e.* they lie nearer to each other as we recede from  $M$ , the distances being proportional to the square roots of the natural numbers.

We can see this at once, for, as we move  $P$  up, we expose the zones below the poles and the successive increments of  $x$  necessary to expose zone after zone will be proportional to the decreasing distances between the zones. The illumination within the shadow is due only to a portion of the wave above the pole, since for all points so situated the pole of the wave is intercepted by the screen. The effect of a wave thus reduces to approximately one half of the element adjacent to the edge, and as we pass deeper into the region of the shadow, this element is further removed from the zone, and consequently less effective. The illumination thus falls off gradually without passing through maxima and minima.

This treatment cannot be rigorously applied to the entire spherical wave, and we shall presently consider a better method which is due to Schuster.

**Circular Disc and Circular Aperture.**—The method is better adapted to the discussion of the distribution of the illumination produced by a small circular disc, since if we confine ourselves to the axis of the conical shadow, the disc exactly coincides with zones described on the wave-front with respect to points lying on the axis.

This is the celebrated problem of Poisson, who was led by theoretical considerations to the remarkable conclusion that the illumination along

the axis of the shadow of a small circular disc is the same as if the disc were removed, a prediction which was verified experimentally by Arago. It is not very difficult to repeat this experiment, provided it is tried on a large scale. The experiment has been already described in the chapter on The Rectilinear Propagation of Light.

Applying the Fresnel construction to this case we see that the illumination on the axis is due to the action of the entire wave with the exception of the zone or zones covered by the disc. These reduce by interference to approximately one half of the effect of the zone bordering the disc. Since the effect of the zones becomes less as we recede from the pole, it is clear that increasing the size of the disc, other things being equal, will reduce the illumination at a given point on the axis. To obtain the most striking results we should so proportion the distance of the screen to the size of the disc, that the latter covers only a very few zones. A small coin at a distance of 4 or 5 meters from the screen shows the effect well. If we make a small hole in the screen at the center of the shadow, and bring our eye close up to it, we shall see the rim of the coin brilliantly luminous, which shows that the light within the shadow is propagated as if coming from the edge of the obstacle.

This same effect can be seen within the shadow of a straight edge. It is often seen in nature on a large scale when the sun rises over the tops of tree-covered mountains, the tops of which are situated nearly above the observer, every branch, twig, and leaf shining with a silvery light, the small birds flying about appearing like specks of fire.

**Small Circular Aperture.**—In this case the illumination along the axis is due solely to the zones lying within the aperture. Suppose the point to be so situated that the aperture contains only the two central zones. The disturbances from these will completely destroy each other at the point, and the illumination will consequently be zero. If we bring the point a little nearer to the aperture the scale of the zones will be reduced, and the aperture will contain say three. The two outer ones will annul each other, and we shall have an illumination due to the outstanding central one. We thus see that the illumination is a maximum or minimum according as the aperture contains an odd or even number of zones.

We shall later on investigate the problem of the circular aperture more completely, the foregoing method being applicable only to the axis.

The effects which we have been considering belong to the so-called Fresnel class of diffraction phenomena. In this class the source of light and the screen are both at finite distances from the obstacle. The problem of determining the distribution of light and shadow becomes much simpler, if we consider the source of light and the screen removed to infinity. This means that we have plane-waves falling upon the aperture, and the secondary disturbances, which we consider as coming from the plane of the aperture, all start at the same time. In other words, the phase of the vibration is the same at every point of the aperture, which is not the case when the source of light is at a finite distance, the wave-fronts then being spherical. For a screen at infinity we have to consider the interference as taking place between parallel rays. This condition can be reproduced experimentally

if we place a lens behind the aperture, which brings every system of parallel secondary rays to a focus. Effects produced in this way are said to belong to the Fraunhofer class, since they were first studied and discussed by Fraunhofer, in connection with his experiments on the diffraction grating.

We will now consider the diffraction effects produced by a narrow slit and then by a number of parallel equidistant slits, when the incident rays are parallel, and the diffracted rays are brought to a focus by a lens. We shall investigate the subject in an elementary way, and then take up the more complete mathematical treatment of the subject.

**Narrow Slit.**—If the telescope of a spectrometer is directed down the collimator, and the diffracting aperture, in this case a slit, is placed between the two, we have the conditions specified for the Fraunhofer class, for the light focused upon the slit of the collimator is made parallel by the lens, which virtually removes the source to infinity, and the parallel diffracted rays are brought to a focus and interfere in the focal plane of the telescope, the maxima and minima resulting therefrom being examined with the eye-piece.

A still simpler way of getting the fringes is to place the slit before the eye, and view a distant lamp through it, the lens of the eye here performing the function of the telescope lens, uniting the parallel diffracted rays upon the retina.

In the following treatment, which is taken from Müller and Pouillet's *Lehrbuch der Physik*, we shall use a very elementary method, and yet arrive at an approximate numerical value for the illumination at various points of the diffraction pattern.

Let  $DC$  (Fig. 130) be the aperture upon which parallel waves are incident in a normal direction. The phase of the vibration will then be the same across the aperture, or along any line parallel to it, such as  $p, o$ . We will consider the normally diffracted ray bundle as divided into 16 elementary ray bundles, which are united by the lens at a point situated at the center of the diffraction pattern which we are to study. Let the amplitude produced at the focus by one of

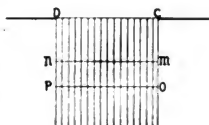


FIG. 130.

these elementary bundles be  $a$ , then the amplitude produced by all 16 will be  $16a$ , since they all arrive in the same phase, and the intensity of the illumination will be

$$I = 16^2 a^2 = 256a^2.$$

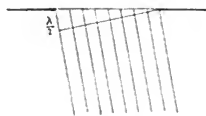


FIG. 131.

Consider next a bundle of parallel diffracted rays which leave the aperture at such an angle that the path-difference between the extreme rays is exactly one half wave-length, as shown in Fig 131. These extreme rays will then arrive at the focus with a phase-difference of  $180^\circ$  and destroy each other. The other elementary bundles destroy one another to a certain extent, and we determine the resultant effect in the following way. If the path-difference between the extreme bundles is  $\frac{\lambda}{2}$ , the

path-difference between any two adjacent bundles is  $\frac{\lambda}{32}$ . For the resultant amplitude of any pair, we have the expression

$$A = a\sqrt{2 + 2\cos 2\pi \frac{x}{\lambda}}, \text{ in which } \frac{x}{\lambda} = \frac{1}{32} \text{ and } \frac{2\pi}{32} = 11^\circ 15';$$

$$A = a\sqrt{2 + 2\cos 11^\circ 15'} = a\sqrt{3.962}.$$

Now the path-difference between two adjacent double bundles is  $\frac{\lambda}{16}$ , consequently the amplitude due to two adjacent double bundles will be

$$B = A\sqrt{2 + 2\cos(22^\circ 30')} = A\sqrt{3.848}.$$

This value  $B$  is then the amplitude produced by four adjacent elementary bundles.

In a similar way we find the effect of eight bundles to be

$$C = B\sqrt{2 + 2\cos 45^\circ} = B\sqrt{3.414},$$

and by combining the two groups of eight each, we get for the final amplitude

$$D = C\sqrt{2 + 2\cos 90^\circ} = C\sqrt{2}.$$

Substituting for  $C$ ,  $B$ , and  $A$  the values found, we get

$$D = a\sqrt{(3.962)(3.848)(3.414)2}.$$

The final intensity  $I_1 = D^2 = 104 \cdot a^2 = .406I$ .

The value found by calculus is

$$I_1 = .4053I,$$

which shows that even the crude subdivision which we have employed yields a fairly accurate result.

The intensity at the point at which these diffracted rays come together is therefore about .4 of the intensity at the center of the pattern.

In a direction such that the extreme rays have a path-difference of  $\lambda$ , we find either by combining two sets of 8 elements each with a path-difference of  $\frac{\lambda}{2}$ , or by pairing off the 1st and 9th, 2nd and 10th, etc., with path-differences of  $\frac{\lambda}{2}$ , that the resultant illumination is zero.

If the path-difference is  $\frac{\lambda}{3}$ , we may divide the bundle into three bundles, the path-difference between the extreme rays of each bundle being  $\frac{\lambda}{2}$ . Two of the three will mutually destroy each other, the resultant illumination being that which would be produced by the third acting alone. The amplitude produced by this third will evidently be  $\frac{1}{3}$  of the amplitude produced in the case where the path-difference between the extreme rays was  $\frac{\lambda}{2}$ , consequently the intensity is

$$I_3 = \frac{1}{9}I_1 = .045I.$$

With a path-difference of  $\frac{\lambda}{4}$  between the extreme rays, we again have zero illumination.

The diffraction pattern thus consists of a central bright band which is very intense, bordered by alternate dark and bright bands, the intensity of the latter decreasing very rapidly. As the width of the slit is made less, the angle of diffraction necessary to give the extreme rays a path-difference of  $\frac{\lambda}{2}$  becomes greater and the minima retreat from the center of the system, the fringes broadening.

This can be seen by holding before the eye a slit made of black paper, the width of which can be varied, and viewing a distant lamp through it: the same experiment can be made with a slit formed between the first and second finger. The fringes produced by diffraction through a single slit were termed by Fraunhofer spectra of the first class.

**Two Parallel Slits.**—We will now take up the case of diffraction by two similar parallel slits. They produce spectra of the first class in the same position, but we shall find that these maxima are broken up by a new set of minima which run through them. These minima are especially noticeable in the bright central maximum, and are produced by destructive interference between diffracted rays from the first slit and corresponding rays from the second.

Let us assume the slits  $AB$  and  $CD$  to be so narrow that in the direction represented in Fig. 132 the path-difference between the extreme rays  $A$  and  $B$ ,  $C$  and  $D$  is  $\frac{\lambda}{4}$ .

Rays diffracted in this direction will then be the rays which bring about the illumination of that portion of the central maximum lying midway between the center and the point where the illumination is  $\frac{1}{4}$  of its value at the center.

If but one slit were acting, the intensity would be about  $\cdot 7$ . If the slit  $CD$  is at such a distance from  $AB$  that the path difference between corresponding rays  $A$  and  $C$  is  $\frac{\lambda}{2}$ , the corresponding rays from

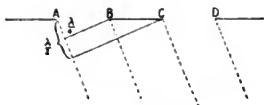


FIG. 132.

the two slits will destroy each other and the illumination will be zero.

It is in this way that the new minima are produced, and though they may be investigated in a manner similar to that employed in the case of a single slit, it will be best to postpone their more complete investigation until we come to the mathematical discussion of the grating.

**Graphical Method of Solution of Diffraction Problems.**—There is another method of solving problems in diffraction, which, though elementary, depends upon the application of results obtained from the mathematical treatment of the subject.

The graphical representation of the resultant of a large number of vibrations, of continuously varying phase and amplitude, was employed by Cornu in the discussion of diffraction problems. The resultant effect of a number of disturbances of different amplitude and phase can be represented graphically as the closing side of a polygon, the sides of which are proportional in length to the amplitudes produced by the

disturbances acting separately, and which make angles with a fixed line equal to the phases of the disturbances. For example, in Fig. 133, let

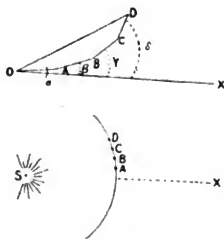


FIG. 133.

$OA$ ,  $AB$ ,  $BC$ , and  $CD$  be the amplitudes produced at a point by three disturbances which arrive simultaneously with phases represented by the angles  $\alpha$ ,  $\beta$ ,  $\gamma$ , and  $\delta$ . The resultant amplitude will be represented by the closing side  $OD$  of the polygon, the phase of the resultant being the angle  $DOX$ . In diffraction problems we have to determine the effect of a large number of disturbances of different phases coming from the portions of the wave-front which are not cut off by the diffracting screen. We can consider the effect of a complete wave at a point in front of it as made up

of a large number of small amplitudes, of variable phase and of constantly decreasing magnitude. Each point on the wave-front will produce at the point in question a disturbance of a certain amplitude and phase. For example, in the figure we may regard the points  $A$ ,  $B$ , and  $C$  on the wave-front as producing at the point  $X$  the amplitudes  $OA$ ,  $OB$ ,  $OC$ , and  $OD$  above referred to, with phases  $\alpha$ ,  $\beta$ , and  $\gamma$ . As we recede from the pole of the wave the phase angle of the secondary disturbances at  $X$  will increase continuously, owing to the increased distances of the successive points. The effect at  $X$  due to the disturbances coming from  $A$ ,  $B$ , and  $C$  will be represented by  $OD$ , the closing side of the polygon. In reality we have an infinite number of points on the wave-front. Let us consider the first Fresnel zone as divided into eight elements, each one of which produces at  $X$  unit amplitude. The effect of all acting simultaneously can be found by the construction given in Fig. 134. The first element, which is next to the pole, will produce the amplitude  $OB$ , the second  $BC$ , the third  $CD$ , etc.; the eighth element will produce the amplitude  $HI$ , the phase having turned through 180 degrees, since by Fresnel's construction the edge of the zone is half a wave-length farther away from the illuminated point than the pole. The resultant effect of the first half zone will therefore be represented by  $OI$ . The effects of the second Fresnel zone can be represented by continuing the construction from the point  $I$  to the point  $K$ , and as the amplitudes due to the successive elements decrease rapidly owing to the obliquity, the broken curve will assume the form of the spiral. If now we consider an infinite number of points on the wave-front, the phase will no longer change abruptly, but will vary continuously in passing from each point to the next. The amplitudes due to the successive elements being very small, the broken curve will now be smoothed out into a continuous one, as shown in Fig. 135.

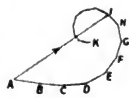


FIG. 134.

**Cornu's Spiral.**—The spiral shown in Fig. 135 was constructed by Cornu from the tables of Fresnel's integrals, which we shall consider subsequently. The effect of each Fresnel zone is represented by a half

turn of the spiral, and if we consider the action of the whole wave, the spiral will make an infinite number of turns, finally subsiding to asymptotic circles at  $J$  and  $J'$  of sensibly zero radius. The spiral  $OJ$  represents one half of the complete wave, and the spiral  $OJ'$  the other. The line  $JJ'$  joining the two asymptotic points represents the action of the complete wave. Any portion of the wave is represented by the corresponding arc of the spiral. The effect, for example, due to the second, third, and fourth zone on the one side of the pole will be found by joining the ends of the second, third, and fourth half turns of the spiral. It is possible by means of this spiral to plot graphically the distribution of light in the diffraction patterns formed when the wave is partially cut off by screens of various types. The spiral will be

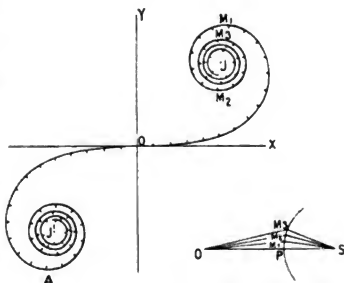


FIG. 135.

better understood after we have studied Fresnel's integrals, and it is introduced at this point merely because it furnishes an easy means of solving a number of the simpler problems in diffraction. A larger drawing of the spiral will be found on Plate IV. at the end of the book, which should be used in plotting diffraction patterns.

We will now consider a number of cases.

**CASE 1. Straight Edge.**—The elementary treatment has shown us that in this case we have a system of fringes of decreasing width, outside of the edge of the geometrical shadow, while within the edge the illumination falls off rapidly, without, however, passing through maxima and minima. Consider first the illumination outside of the edge as represented by the spiral.

Since our spiral represents *amplitudes*, we shall find our intensities by squaring the resultant amplitude lines. At the edge of the shadow the intensity will be represented by the square of the distance  $OJ$ , since one half of the complete wave is operative at this point. As we pass out from the edge, the lower part of the spiral begins to operate, and on reaching a distance such that the whole of the first or central Fresnel zone is exposed, the intensity will be found by squaring the line joining the point  $A$  with  $J$  (Fig. 135); i.e. it will be considerably greater than the intensity due to the entire wave, which is represented by the square of  $JJ'$ . If we represent the intensities as ordinates, our abscissae must be taken proportional to the distances, measured along the spiral from  $O$ , at which the point which we join with  $J$ , is located. To facilitate measurements, equal distances have been marked off on the spiral. The first maximum occurs at abscissa 14, the distance from  $O$  to the bottom of the spiral.

The intensity as we proceed outward will be represented by the square of the line joining  $J$  with a point which travels around the

lower half of the spiral. The intensity thus passes through maxima and minima, soon reaching a nearly constant value, owing to the small diameter of the convolutions, the first minimum occurring at abscissa 20.

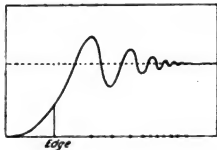


FIG. 137.

in this way is shown in Fig. 137 (dotted line represents illumination due to the complete wave).

**CASE 2. Narrow Slit.**—In this case the amplitude of the vibration is measured by an arc of the spiral, the length of which is proportional to the width of the slit. The intensity will be represented by the square of a line joining the extremities of a constant length of the spiral. Equal distances will be found marked off on the spiral. Suppose the width of the slit and its distance from the screen to be such that it subtends exactly one half of the central zone. The length of the arc which we are to employ is then equal to that of the first half turn of the spiral, namely, within the geometrical projection of the aperture, the arc will lie partly in the upper, and partly in the lower branch of the spiral. At the center it will be symmetrically placed; that is, with its center at  $O$ . As we proceed from the center, we push the arc of constant length along the spiral, squaring the line joining its extremities at regular intervals, plotting these values as ordinates, at abscissae corresponding to the distances advanced along the spiral as before.

It is at once apparent that the illumination at the center of the fringe system may be either a maximum or a minimum, according to the width of the aperture. If the aperture just covers the entire central zone, the illumination will be a maximum, and will have a larger value than that due to the whole wave, while it will be a minimum if the aperture covers two zones.

**CASE 3. Narrow Wire.**—This case is a little more complicated, for the effect of the wire is to cut out a constant arc of the spiral just the reverse of the condition in Case 2. The amplitude is the resultant of the two remaining portions of the spiral, which must be compounded, paying attention to the directions as well as the lengths of the lines joining the extremities of the curves. The direction is always found by measuring from  $J'$  to  $J$ . This can be seen from the following consideration: The effect due to one half of the wave is  $J'O$ , that due to the other half is  $OJ$ . The whole wave will produce an amplitude equal to the sum of these vectors. If we take their directions as measured from  $J'$  to  $O$ , and from  $O$  to  $J$ , the amplitudes will be added, and we shall have amplitude  $J'J$ . If, however, we measure from  $O$  to  $J'$  and from  $O$  to  $J$  the vectors will be opposed, and cancel each other. If this is borne in mind, no difficulty will be found in remembering how to determine the direction in which the amplitude lines point.

Suppose now that the wire cuts off one half off the central zone, i.e. one half turn of the spiral measured from  $O$ . At the center the first elementary distances on each side of  $O$  (see Fig. 138, dotted line) will be absent, and the resultant amplitude will be found by compounding  $J'b$  with  $aJ$ , as shown in the figure. At the edge of the geometrical shadow of the wire, we compound the line  $J'O$  with the short line which joins the highest point, " $d$ ," of the upper branch of the spiral with  $J$  (as shown in the lower part of the figure), the resultant amplitude being  $JX$ . Abscissae are laid off as before, proportional to the distances advanced along the spiral.

The intensity curve should be plotted in this manner for a wire covering say two zones. It will be found interesting to compare it with the curve obtained in the case of a straight edge. Regarding one side of the wire as a straight edge, we see that the exterior fringe system is complicated by the effects due to the exposed portion of the wave beyond the opposite edge.

CASE 4. Two parallel slits. The effects in this case are found by compounding the resultants of two arcs, of lengths proportional to the widths of the slits, separated by a distance proportional to the distance between them.

**Graphical Method for Fraunhofer Class.** If the diffracting apertures are illuminated by plane-waves, and the parallel diffracted rays are brought to a focus by a lens, Cornu's spiral obviously reduces to a circle. The amplitude due to  $n$  parallel equidistant slits may be found by compounding the resultant of  $n$  equal chords taken at regular intervals around the circle, *i.e.* by finding their geometrical sum.

Kimball (*Phil. Mag.*, July 1903) has worked out the case of the plane diffraction grating in this way, and shown the existence of the principal maxima, with the  $(n-2)$  secondary maxima between them.

**Mathematical Treatment of Fraunhofer Diffraction Phenomena.**—If a converging lens is placed behind the aperture upon which plane waves are falling, the lens transforms that portion of the plane-wave which gets through the aperture into a concave wave, which, if the laws of geometrical optics were followed, would collapse to a point at the focus.

We have then to determine the effect of a small portion of a spherical concave wave at a plane passing through the center of curvature of the wave. To do this we first get a general expression in the form of a double integral for the effect of a complete hemispherical wave, and then integrate this expression over the aperture (Fig. 139).

Let the center of the concave wave be at the origin of three rectangular coordinates. We are to determine the effect of a disturbance starting from the point  $P$  with coordinates  $x, y, z$  at the point  $M$  with coordinates  $\xi, \eta$ , and then the collective effect of all the dis-

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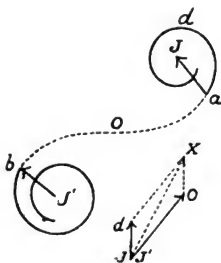


FIG. 138.

turbances coming from the entire wave-front. Call the distance  $MP = \rho$  and  $dx dy$  the element of the wave at  $P$ .

The amplitude at  $M$  produced by the secondary disturbances from the area  $dx dy$  will be  $\kappa dx dy \sin 2\pi \left( \frac{t}{T} - \frac{\rho}{\lambda} \right)$ , in which  $\kappa$  is a coefficient depending on the inclination of the surface element  $dx dy$  to  $\rho$  and on the distance of  $P$  from  $M$ . Since we limit the area of the wave, by blocking off the greater part we can regard the inclination as the same for all portions considered, and  $\kappa$  therefore becomes a constant.

The collective effect of all points  $P$  at  $M$ , taking into account their mutual interference, is

$$a = \iint \sin 2\pi \left( \frac{t}{T} - \frac{\rho}{\lambda} \right) dx dy$$

(in which  $a$  is the displacement).

Let  $OP = R$ , then

$$R^2 = x^2 + y^2 + z^2,$$

$$\rho^2 = (x - \xi)^2 + (y - \eta)^2 + z^2 = R^2 - 2x\xi - 2y\eta + \xi^2 + \eta^2,$$

or, if the surface of the wave utilized be confined to the immediate neighborhood of the  $z$  axis, approximately

$$\rho = \sqrt{R^2 + \xi^2 + \eta^2} - \frac{x\xi + y\eta}{\sqrt{R^2 + \xi^2 + \eta^2}},$$

the variables  $x$  and  $y$  occurring in the last term.

This gives us for the resultant displacement

$$a = \iint \sin 2\pi \left( \frac{t}{T} - \frac{\sqrt{R^2 + \xi^2 + \eta^2}}{\lambda} + \frac{x\xi + y\eta}{\lambda \sqrt{R^2 + \xi^2 + \eta^2}} \right) dx dy,$$

and since  $\sin(a + b) = \sin a \cos b + \cos a \sin b$ ,

$$a = \sin 2\pi \left( \frac{t}{T} - \frac{\sqrt{R^2 + \xi^2 + \eta^2}}{\lambda} \right) \iint \cos 2\pi \frac{x\xi + y\eta}{\lambda \sqrt{R^2 + \xi^2 + \eta^2}} dx dy \\ + \cos 2\pi \left( \frac{t}{T} - \frac{\sqrt{R^2 + \xi^2 + \eta^2}}{\lambda} \right) \iint \sin 2\pi \frac{x\xi + y\eta}{\lambda \sqrt{R^2 + \xi^2 + \eta^2}} dx dy.$$

Let  $\Phi = 2\pi \left( \frac{t}{T} - \frac{\sqrt{R^2 + \xi^2 + \eta^2}}{\lambda} \right).$

Let  $A = \iint \cos 2\pi \frac{x\xi + y\eta}{\lambda \sqrt{R^2 + \xi^2 + \eta^2}} dx dy.$

Let  $B = \iint \sin 2\pi \frac{x\xi + y\eta}{\lambda \sqrt{R^2 + \xi^2 + \eta^2}} dx dy,$

then  $a = A \sin \Phi + B \cos \Phi.$

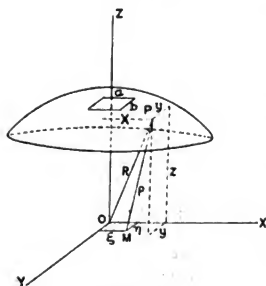


FIG. 139.

Let  $\theta = \text{arc tang } \frac{B}{A}$ , i.e.  $\frac{B}{A} = \tan \theta$ , and let  $C = \sqrt{A^2 + B^2}$ , then if we multiply  $A \sin \phi + B \cos \phi$  by  $\frac{\sqrt{A^2 + B^2}}{C}$  and substitute for the  $\frac{A}{C}$  in each term  $\cos \theta$  we get at once as the equivalent

$$\sqrt{A^2 + B^2} \sin \phi \cos \theta + \cos \phi \sin \theta,$$

or

$$a = C \sin (\phi + \theta),$$

in which  $C$  is the amplitude, of which the square represents the intensity of the illumination.

$$I = C^2 = A^2 + B^2.$$

Substituting,

$$I = \left( \iint \cos 2\pi \frac{x\xi + y\eta}{\lambda \sqrt{R^2 + \xi^2 + \eta^2}} dx dy \right)^2 + \left( \iint \sin 2\pi \frac{x\xi + y\eta}{\lambda \sqrt{R^2 + \xi^2 + \eta^2}} dx dy \right)^2,$$

or, since  $\xi$  and  $\eta$  are small in comparison with  $R$ ,

$$I = \left( \iint \cos 2\pi \frac{x\xi + y\eta}{R\lambda} dx dy \right)^2 + \left( \iint \sin 2\pi \frac{x\xi + y\eta}{R\lambda} dx dy \right)^2.$$

**Diffraction by a Rectangular Aperture.**—Suppose we have a small rectangular aperture of length  $a$  and width  $b$ , so placed that its sides are parallel to  $x$  and  $y$  axes, and the  $z$  axis passes normally through its center. To determine the intensity of the illumination we integrate the above expression between the limits  $+\frac{a}{2}$  and  $-\frac{a}{2}$ ,  $+\frac{b}{2}$  and  $-\frac{b}{2}$ .

$$I = \left( \int_{-\frac{a}{2}}^{+\frac{a}{2}} \int_{-\frac{b}{2}}^{+\frac{b}{2}} \cos 2\pi \frac{x\xi + y\eta}{R\lambda} dx dy \right)^2 + \left( \int_{-\frac{a}{2}}^{+\frac{a}{2}} \int_{-\frac{b}{2}}^{+\frac{b}{2}} \sin 2\pi \frac{x\xi + y\eta}{R\lambda} dx dy \right)^2,$$

and for brevity writing  $\int^a$  and  $\int^b$  in place of the above,

$$\begin{aligned} & \left( \int^a \cos 2\pi \frac{x\xi}{R\lambda} dx \int^b \cos 2\pi \frac{y\eta}{R\lambda} dy - \int^a \sin 2\pi \frac{x\xi}{R\lambda} dx \int^b \sin 2\pi \frac{y\eta}{R\lambda} dy \right)^2 \\ & + \left( \int^a \sin 2\pi \frac{x\xi}{R\lambda} dx \int^b \cos 2\pi \frac{y\eta}{R\lambda} dy + \int^a \cos 2\pi \frac{x\xi}{R\lambda} dx \int^b \sin 2\pi \frac{y\eta}{R\lambda} dy \right)^2. \end{aligned}$$

If we integrate these terms we find that

$$\int_{-\frac{a}{2}}^{+\frac{a}{2}} \sin 2\pi \frac{x\xi}{R\lambda} dx = 0, \quad \int_{-\frac{b}{2}}^{+\frac{b}{2}} \sin 2\pi \frac{y\eta}{R\lambda} dy = 0,$$

$$\int_{-\frac{a}{2}}^{+\frac{a}{2}} \cos 2\pi \frac{x\xi}{R\lambda} dx = \frac{R\lambda}{\pi\xi} \sin \pi \frac{a\xi}{R\lambda},$$

$$\int_{-\frac{b}{2}}^{+\frac{b}{2}} \cos 2\pi \frac{y\eta}{R\lambda} dy = \frac{R\lambda}{\pi\eta} \sin \pi \frac{b\eta}{R\lambda},$$

or 
$$I = \frac{R^4 \lambda^4}{\pi^4 \xi^2 \eta^2} \sin^2 \pi \frac{a \xi}{R \lambda} \sin^2 \pi \frac{b \eta}{R \lambda},$$

and finally 
$$I = a^2 b^2 \frac{\sin^2 \pi \frac{a \xi}{R \lambda}}{\pi^2 a^2 \xi^2} \cdot \frac{\sin^2 \pi \frac{b \eta}{R \lambda}}{\pi^2 b^2 \eta^2} \cdot \frac{R^2 \lambda^2}{R^2 \lambda^2}.$$

We thus see that the intensity at the point  $x, y$  is dependent on two variables of the form  $\frac{\sin^2 u}{u^2}$ .

To find the maxima and minima we differentiate this expression with respect to  $u$  and equate to zero.

If 
$$S = \frac{\sin^2 u}{u^2}, \quad \frac{dS}{du} = \frac{\sin u}{u} \cdot \frac{u \cos u - \sin u}{u^2} = 0,$$

which falls into two equations

$$\frac{\sin u}{u} = 0 \quad \text{and} \quad \frac{u \cos u - \sin u}{u^2} = 0.$$

The first of these two equations gives the position of the minima, the second that of the maxima.

The second equation takes the form  $u \cos u = \sin u$ ,  $u = \tan u$ .

This last equation can be solved graphically by plotting the curves  $y = x$  and  $y = \tan x$  (Fig. 140). The latter equation is represented by a

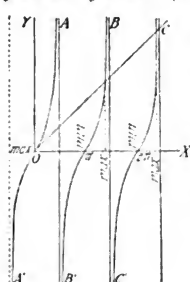


FIG. 140.

family of curves, having for asymptotes  $x = \pm \frac{\pi}{2}$ ,  $x = \pm \frac{3\pi}{2}$  and  $x = \pm \frac{5\pi}{2}$ , etc. The points at which the former curve cuts the various members of the family satisfy the equation  $x = \tan x$  and consequently  $u = \tan u$ . The first or central maximum occurs at  $u = 0$ , the second between  $\pi$  and  $\frac{3}{2}\pi$ , the third between  $2\pi$  and  $\frac{5}{2}\pi$ , etc. The successive values of  $u$  were calculated by Schwerd and found to be

$u_0 = 0$	$u_4 = 4.477\pi$
$u_1 = 1.430\pi$	$u_5 = 5.482\pi$
$u_2 = 2.459\pi$	$u_6 = 6.484\pi$
$u_3 = 3.471\pi$	$u_7 = 7.486\pi$

It is apparent from the diagram that the roots approach the limit  $(2n+1)\frac{\pi}{2}$ , i.e. the points of intersection come nearer to the asymptotes

as we recede from the origin, the 7th maximum from the center being at  $7.486\pi$  and the asymptote at  $7.5\pi$ . The minima occur where  $u = m\pi$ , exclusive of the case where  $m = 0$ , which is the value which gives the central maximum. The values of the maxima corresponding to odd multiples of  $\pi$  can be calculated by taking  $u = 1.5\pi, 2.5\pi, 3.5\pi \dots$  in the expression  $\frac{\sin^2 u}{u^2}$ . We find these to be in the ratio  $1, \left(\frac{2}{3\pi}\right)^2, \left(\frac{2}{5\pi}\right)^2 \dots$ , i.e.

if the illumination of the central maximum be taken as unity the illuminations of the succeeding maxima will be  $\frac{1}{25}$ ,  $\frac{1}{36}$ ,  $\frac{1}{49}$ . We are now in a position to discuss the complete diffraction pattern produced by the rectangular aperture. There will be complete darkness in all places for which one of the factors of the form  $\frac{\sin^2 u}{u}$  in equation  $I = a^2 b^2$ , etc., has the value 0, or when,  $m$  being a whole number,  $\frac{\pi a \xi}{R \lambda} = m \pi$  or  $\frac{\pi b \eta}{R \lambda} = m \pi$ . There will therefore be two systems of dark regions of which the equations are  $\xi = \frac{m R \lambda}{a}$  and  $\eta = \frac{m R \lambda}{b}$ , the former parallel to the  $y$  axis, the latter parallel to the  $x$  axis. If we bear in mind the fact that we oriented the aperture with its side  $a$  parallel to the  $x$  axis, and that  $\xi$  is the value in the direction of this same axis, we see that the distances between the parallel interference minima, are inversely as the width of the aperture measured in a direction perpendicular to them, and directly as the distance  $R$  and the wave-length  $\lambda$ . These dark bands form a network of rectangular meshes similar in shape to the aperture, but rotated through  $90^\circ$  with respect to it (Fig. 141).



FIG. 141.

**Introduction of Angular Measure.**—The investigations which are to follow are much simplified by the introduction of angular measure

into the expression for the action of a concave spherical wave. If  $R$  be taken infinitely large the concave wave becomes plane and the  $x, y$  plane or screen moves off to infinity.

The elementary diffracted rays which meet at the screen emerge from the aperture parallel to one another. Let  $\alpha$  and  $\beta$  (Fig. 142) represent the angles which these rays make with  $x$  and  $y$ . These angles will differ but little from  $90^\circ$  since the angle of diffraction is small.

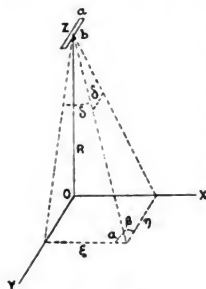


FIG. 142.

$$\cos \alpha = \frac{\xi}{\sqrt{R^2 + \xi^2 + \eta^2}}$$

$$\cos \beta = \frac{\eta}{\sqrt{R^2 + \xi^2 + \eta^2}}$$

The expression for the intensity now becomes

$$I = \left( \iint \cos 2\pi \frac{x \cos \alpha + y \cos \beta}{\lambda} dx dy \right)^2 + \left( \iint \sin 2\pi \frac{x \cos \alpha + y \cos \beta}{\lambda} dx dy \right)^2.$$

Let  $\delta$  and  $\delta' =$  the small complementary angles which the rays make with the planes  $xz$  and  $yz$ , and we have

$$I = \left( \iint \cos 2\pi \frac{x \sin \delta + y \sin \delta'}{\lambda} dx dy \right)^2 + \left( \iint \sin 2\pi \frac{x \sin \delta + y \sin \delta'}{\lambda} dx dy \right)^2.$$

This expression integrated over a rectangular aperture measuring  $a \times b$  gives for the resultant intensity for the point where the rays making angle  $\delta$  and  $\delta'$  with the planes  $xy$  and  $xz$ ,

$$I = a^2 b^2 \frac{\sin^2 \pi \frac{a \sin \delta}{\lambda}}{\pi^2 \frac{a^2 \sin^2 \delta}{\lambda^2}} \cdot \frac{\sin^2 \pi \frac{b \sin \delta'}{\lambda}}{\pi^2 \frac{b^2 \sin^2 \delta'}{\lambda^2}},$$

either factor becoming 0 for  $\sin \delta = \frac{m\lambda}{a}$  or  $\sin \delta' = \frac{m\lambda}{b}$ .

**Diffraction by a Single Slit.**—If we let one dimension of our small rectangular aperture become large, we have the condition of a narrow slit, the diffraction by which we will now investigate as an introduction to the study of the diffraction grating. In this case we can substitute for the luminous point a luminous line, parallel to the slit, without altering the diffraction pattern, a device which enables us to use much more light, though we do not thereby increase the illumination at any given point to any great degree. With a point source of light the diffraction pattern is reduced to a series of maxima and minima, distributed along a line which is perpendicular to the slit. With a linear source of light the maxima are extended in a direction parallel to the slit, the minima appearing as dark bands.

We will express the intensity by the equation in which the position of the illuminated point is defined by the angle of diffraction  $\delta$ , and since, if the slit be parallel to the  $y$  axis, ( $\eta=0$ ), the diffraction will only occur in directions parallel to the  $x$  axis, we can put  $\delta'=0$ . This makes the second factor equal unity, and we can write the expression for the intensity

$$I = a^2 \frac{\sin^2 \pi \frac{a \sin \delta}{\lambda}}{\pi^2 \frac{a^2 \sin^2 \delta}{\lambda^2}} = a^2 \frac{\sin^2 u}{u^2}.$$

It will be seen that  $b^2$  has been omitted. This is because varying  $b$  does not change the *distribution* of light in the maxima. It, however, affects the intrinsic intensity.

The minima of this function, as we have seen, are given by  $u = m\pi$  where  $m$  is a whole number (not for  $m=0$  however).

The intensity is zero for all directions  $\delta$  for which  $\sin \delta = \frac{m\lambda}{a}$ , or if  $\delta$  is small the directions for zero illumination

$$\delta = \frac{m\lambda}{a} \text{ MINIMA.}$$

The maxima lie in directions given by  $\sin \delta = \frac{u_n \lambda}{\pi a}$ , in which  $u_n$  is one of the roots of the equation  $u = \tan u$ , or for small values of  $\delta$ ,

$$\delta = \frac{u_n \lambda}{\pi a} \text{ MAXIMA.}$$

The maxima are the spectra of the 1st class which we have already investigated in an elementary way.

If white light is employed, the central maximum is white, the other maxima colored, owing to the fact that their position is a function of the wave-length of the light, the red maxima being farther apart than the blue.

**Diffraction by two Parallel Slits.**—This case, which we have already studied by elementary methods, is the next step which we shall take in the development of the theory of the diffraction grating.

The width of the slits we will call  $a$ , and the distance between them  $b$ .

The diffracted rays coming in a parallel direction from a slit at angle  $\delta$  with the normal give a resultant intensity

$$A^2 = a^2 \frac{\sin^2 \pi \frac{a \sin \delta}{\lambda}}{\pi^2 \frac{a^2 \sin^2 \delta}{\lambda^2}},$$

in which expression  $A$  is the amplitude.

Parallel rays coming from corresponding parts of the two slits have a path-difference of  $(a+d) \sin \delta$ , and the vibrations when brought together by the lens will have a phase-difference  $\frac{2\pi(a+d) \sin \delta}{\lambda}$ .

We have seen in the chapter on Interference that the resultant intensity of two streams of light of amplitude  $A$ , with phase-difference  $\epsilon$ , is  $I = 2A^2 + 2A^2 \cos \epsilon$ .

The resultant intensity in this case is therefore

$$I = 2A^2 \left( 1 + \cos 2\pi \frac{(a+d) \sin \delta}{\lambda} \right) = 2A^2 2 \cos^2 \pi \frac{(a+d) \sin \delta}{\lambda},$$

and substituting  $A^2$  from above,

$$I = 4a^2 \frac{\sin^2 \pi \frac{a \sin \delta}{\lambda}}{\pi^2 \frac{a^2 \sin^2 \delta}{\lambda^2}} \cdot \cos^2 \pi \frac{(a+d) \sin \delta}{\lambda}.$$

This expression contains two variable factors, one of form  $\frac{\sin^2 u}{u^2}$ , which we have already investigated, the other giving equidistant minima equal to zero, given by the equation

$$\sin \delta = \frac{(2n+1)\lambda}{2(a+d)} \quad \text{MINIMA,}$$

and maxima given by  $\sin \delta = \frac{n\lambda}{a+d} \quad \text{MAXIMA,}$

which expressions simply state that in the first case the rays coming from homologous parts of the two slits meet with a path-difference of an uneven number of half wave-lengths, and in the second case with an even number.

The intensity will be zero when either of the two variable factors is zero, i.e. when

$$\sin \delta = m\lambda,$$

$$\sin \delta = \frac{(2n+1)\lambda}{2(a+d)}.$$

The minima given by the first expression are the diffraction minima of a single slit which we have already studied, the second are interference minima resulting from the meeting of homologous rays from the two slits: they are chiefly noticeable in the central maximum of the first class system, and the maxima which lie between them were called by Fraunhofer spectra of the 2nd class. They are the spectra yielded by the diffraction grating.

If now we increase the number of slits we shall find that we have in addition spectra of a 3rd class, which, however, disappear entirely if the number of slits is very large.

**Diffraction by any Number of Parallel Equidistant Slits. (Diffraction Grating).**—If we have to find the resultant of a large number of vibrations of different phase, the method is the following. First,

resolve the lines representing the amplitudes into their projections on  $OX$  and  $OY$ . Let  $OA$  (Fig. 143) represent the amplitude and  $\epsilon$  the phase. Then  $y = OA \sin \epsilon$  and  $x = OA \cos \epsilon$ .

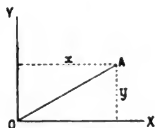


FIG. 143.

$$OA^2 = (OA \cos \epsilon)^2 + (OA \sin \epsilon)^2.$$

If now we have a large number of lines  $OA'$ ,  $OA''$ ,  $OA'''$ , etc., with phases  $\epsilon'$ ,  $\epsilon''$ ,  $\epsilon'''$ , etc., the resultant will have projections on  $OX$  and  $OY = \Sigma OA \cos \epsilon$  and  $\Sigma(OA \sin \epsilon)$ , and the resultant intensity will be

$$R^2 = [\Sigma(OA \cos \epsilon)]^2 + [\Sigma(OA \sin \epsilon)]^2.$$

The diffraction grating in its typical form consists of a large number of parallel equidistance slits ruled through an opaque film, such as a film of smoke on a glass plate.

We will consider the elementary rays which leave the grating at angle  $\delta$ , the rays which leave one of the slits unite into a resultant disturbance of which the amplitude is  $A$ . There are as many such disturbances as there are slits, and they differ in phase by a constant quantity which we will call  $\epsilon$ . The resultant intensity is

$$I = (A \cos \epsilon + A \cos 2\epsilon + A \cos 3\epsilon \dots)^2 + (A \sin \epsilon + A \sin 2\epsilon + A \sin 3\epsilon \dots)^2,$$

$$I = A^2 [(\cos \epsilon + 2 \cos 2\epsilon \dots)^2 + (\sin \epsilon + \sin 2\epsilon \dots)^2],$$

$$I = A^2 \left[ \left( \frac{\sin \frac{n\epsilon}{2} \cos \frac{(n+1)\epsilon}{2}}{\sin \frac{\epsilon}{2}} \right)^2 + \left( \frac{\sin \frac{n\epsilon}{2} \sin \frac{(n+1)\epsilon}{2}}{\sin \frac{\epsilon}{2}} \right)^2 \right],$$

$$I = A^2 \frac{\sin^2 \frac{n\epsilon}{2}}{\sin^2 \frac{\epsilon}{2}}.$$

Substituting in this expression the values found in the last section for  $A^2$  and  $\epsilon$ ,

$$A^2 = a^2 \frac{\sin^2 \pi \frac{a \sin \delta}{\lambda}}{\pi^2 \frac{a^2 \sin^2 \delta}{\lambda^2}},$$

$$\epsilon = \frac{(a+d) \sin \delta}{\lambda} \cdot 2\pi,$$

gives us for the resultant intensity in the direction  $\delta$  for the diffraction grating

$$I = a^2 \frac{\sin^2 \pi \frac{a \sin \delta}{\lambda}}{\pi^2 \frac{a^2 \sin^2 \delta}{\lambda^2}} \cdot \frac{\sin^2 n \pi \frac{(a+d) \sin \delta}{\lambda}}{\sin^2 \pi \frac{(a+d) \sin \delta}{\lambda}} \dots \dots \dots (1)$$

This expression contains two variable factors, the first of which we have already investigated. The illumination will be zero when either one of these factors is zero. The second factor contains the quantity  $n$ , the number of lines in the grating. If we put  $n=2$  this factor takes the form of the factor which we obtained in the expression for two slits. To find the maxima and minima of this factor we will put

$$\pi \frac{(a+d) \sin \delta}{\lambda} = z.$$

The factor then takes the form  $\frac{\sin^2 n z}{\sin^2 z}$ .

To find the value of  $z$  which makes this a maximum or minimum we will differentiate the expression and equate to zero.

$$\frac{\sin n z (n \sin z \cos n z - \cos z \sin n z)}{\sin^3 z} = 0,$$

which is satisfied when  $\frac{\sin n z}{\sin z} = 0, \dots \dots \dots (2) \text{ MINIMA}$

or when  $\frac{n \sin z \cos n z - \cos z \sin n z}{\sin^2 z} = 0, \dots \dots \dots (3) \text{ MAXIMA}$

From the first expression we get

$$z = \frac{k}{n} \pi,$$

in which  $k$  is a whole number not divisible by  $n$ . If  $k$  is divisible by  $n$  the expression takes the indeterminate form  $\frac{0}{0}$ , which, if we examine by substituting for the numerator and denominator their differential coefficients, we shall find does not satisfy the expression (2). Substituting this value of  $z$  in the second factor of (1) reduces it to 0 and gives us minima.

To determine the values of  $z$  which give the minima, we write

$$z = \pi \frac{(a+d)\sin\delta}{\lambda} = \frac{k\pi}{n},$$

$$\sin\delta = \frac{k\lambda}{n(a+d)}.$$

We will now examine expression (3), which is satisfied if  $z = m\pi$ .

Substitution of this value in (3) gives us  $\frac{0}{0}$ . Determining the true value by differentiation gives us the value 0, i.e. satisfies (3). We now substitute this value in the second factor of (1), which takes the indeterminate form.

Differentiating  $\left(\frac{\sin nz}{\sin z}\right)^2$  to evaluate this form gives

$$\left(\frac{n \cos nz}{\cos z}\right)^2 = n^2.$$

The second factor of (1) therefore gives equidistant maxima for directions for which  $z = m\pi$ , namely,

$$\sin\delta = \frac{m\lambda}{a+d}.$$

These maxima are the spectra which are given by the diffraction grating of many lines, and are called the principal maxima. If the grating has but few lines other maxima and minima appear between the principal maxima called spectra of the 3rd class. These occur for values of  $z$  not equal to  $m\pi$ , in which case equation (3) takes the form

$$n \sin z \cos nz = \cos z \sin nz,$$

or

$$n \tan z = \tan nz,$$

which equation expresses the position of other maxima and minima.

We will now determine the roots of this equation which lie between  $z = 0$  and  $z = \pi$ .

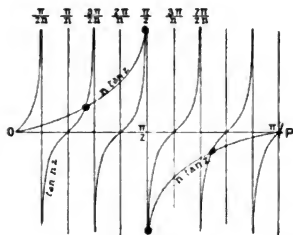


FIG. 144.

While  $z$  increases from 0 to  $\frac{\pi}{2n}$

both  $\tan nz$  and  $n \tan z$  increase, the first more rapidly; consequently there is no root in this interval, as will be seen by reference to Fig. 144, which is for  $n=5$ . Let  $z$  increase further from  $\frac{\pi}{2n}$  to  $\frac{3\pi}{2n}$ :  $\tan nz$

passes from  $-\infty$  to  $+\infty$  in this interval, while  $n \tan z$  continues to

increase. The root is given by the intersection of the curves.

The other roots are found at the points indicated in the figure. At first sight it appears as if  $n-1$  roots existed; in reality there are only  $n-2$ , since for an odd value of  $n$  the two occurring between  $\frac{\pi}{2} - \frac{\pi}{2n}$  and

$\frac{\pi}{2} + \frac{\pi}{2n}$  have the values  $\frac{\pi}{2}$ , while if  $n$  be even no root occurs between

$\frac{\pi}{2} - \frac{\pi}{2n}$  and  $\frac{\pi}{2} + \frac{\pi}{2n}$ .

The principal maxima fall at 0 and  $\pi$ , consequently there exist between them  $n - 1$  secondary maxima.

For a six-line grating the roots occur as shown in Fig. 145, there being 4 secondary maxima between the principal maxima. The intensities of the secondary maxima are found from  $\tan nz = n \tan z$  as follows.

Writing this expression in the form

$$\frac{\sin nz}{\cos nz} = \frac{n \sin z}{\cos z}.$$

Squaring, 
$$\frac{\sin^2 nz}{\cos^2 nz} = \frac{n^2 \sin^2 z}{\cos^2 z},$$

$$\frac{n^2 \sin^2 z}{1 - \sin^2 z} = \frac{\sin^2 nz}{1 - \sin^2 nz},$$

$$\frac{\sin^2 nz}{\sin^2 z} = \frac{n^2}{1 + (n^2 - 1) \sin^2 z}.$$

For multiples of  $\pi$  this expression becomes  $n^2$ , which we have seen before

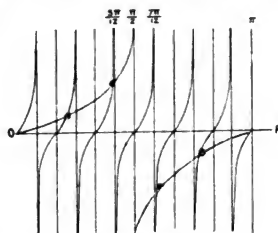


FIG. 145.

gives us the measure of the intensity of the principal maxima. For other values of  $z$ , since then  $\sin z$  differs from 0, the value decreases as  $n$  increases. The expression also shows that the secondary maxima are not of equal intensity, but decrease as we pass away from the principal maxima in either direction.

The intensity curves for 1, 2, 3, 5, and 6 slits are shown in Fig. 146, from which we see that by increasing the number of lines of the grating we throw more light into the principal maxima, the secondary maxima decreasing in intensity and becoming more crowded together. We also cause the curves of the principal maxima to become steeper, *i.e.* the light is concentrated more and more nearly into a line. When we have a very large number

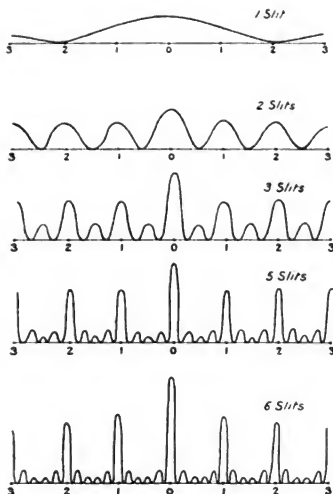


FIG. 146.

of lines the principal maxima are exceedingly bright and narrow when

monochromatic light is employed, and the secondary maxima disappear entirely. The principal maxima then constitute the narrow spectrum lines seen with the grating.

**Absent Spectra.**—The principal maxima will be absent at places for which the first of the two variable factors in the complete expression for the grating has the value 0. These are the places at which the diffraction by a single slit or line of the grating produces zero illumination. It is clear that if the disturbance here, due to a single line, is zero, the resultant of all the lines will also be zero. If the width of the lines is small, however, the points where the spectra are absent are too far removed from the central maximum to appear in the field.

**Verification of Results.**—The results which we have deduced for the diffraction grating can be easily verified by experiment. A piece of thin plate-glass is smoked over a flame, and one edge moistened with alcohol. The alcohol spreads over the film, and on drying leaves it compact enough to enable us to rule lines through it with a sharp steel point by means of the dividing engine.

If no dividing engine is available it is not difficult to fit up an arrangement by which the plate may be advanced through equal distances by turning a screw provided with a roughly graduated head, while the ruling is done with a needle point mounted on a pivoted arm, arranged to slide back and forth along a fixed line. The distance between the lines should be as nearly as possible equal to the width of the lines. A dozen or so lines will be found sufficient. The plate thus ruled should be provided with a cover, also of thin plate-glass (ordinary window-glass will not do), to prevent injury to the film, and a movable slide of thin black paper so arranged that the lines can be covered or exposed in succession. If we mount the plate on the table of a spectrometer and illuminate the slit with sunlight, which has been passed through a sheet of dense ruby glass, we can verify in succession all of the results which we have deduced.

If all of the slits but one are covered we shall see the broad spectra of the 1st class, the central one being by far the brightest; on uncovering another slit, this central maximum, as well as the lateral maxima, appear furrowed by narrow dark bands, the bright bands between them being the spectra of the 2nd class. If three lines be uncovered the faint 3rd class maxima appear, one between each pair of 2nd class maxima. As we proceed with the uncovering process we shall see the secondary maxima crowd in between the principal maxima, until, when all the lines are exposed, they are too faint and too close together to be detected.

If a piece of dense cobalt glass, which transmits the extreme red and blue only, be substituted for the ruby glass, a series of red maxima and blue maxima will be seen simultaneously, the latter being closer together than the former.

**The Laminary Grating.**—If we consider the opaque strips of the grating which we have just studied replaced by transparent strips of such a thickness that some one wave-length in the spectrum suffers a retardation of  $\frac{\lambda}{2}$ , we have a type of grating which was first studied

by Quincke. The mathematical treatment of a grating of this type is somewhat long and involved, and, as the chief peculiarities of the grating can be seen by elementary considerations, had best be omitted. These gratings have the peculiar property of failing to show the central image when light of the specified wave-length is used. If we employ white light, the central image, which is white in the case of the ordinary grating, appears colored owing to the absence of the wave-lengths in the immediate vicinity of the one for which the retardation by the thin lamina is  $\frac{\lambda}{2}$ .

The reason of this is very easy to see. Let  $BC$ ,  $DE$ , and  $FG$  be the retarding lamina (Fig 147). The central image is formed by the normally diffracted rays, *i.e.* the rays for which  $\delta=0$ . In the case of the ordinary grating these disturbances all arrive at the focus of the lens in the same phase, and produce a maximum. In the case of the laminary grating there is a second set of disturbances from the laminae which reach the focus half a wave-length behind the others, or unretarded ones, which they destroy. Thus the disturbance  $AA''$  is destroyed by  $BB'$  and so on. For waves for which the

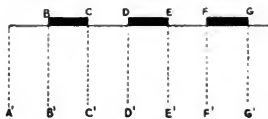


FIG. 147.

retardation is not exactly  $\frac{\lambda}{2}$  the destructive interference will not be so complete, and these will appear in the central image, though with reduced intensity. The difference in the retardation is of course due to the dispersion of the lamina, and if the latter be somewhat thick, we may have several values of  $\lambda$  distributed along the spectrum, for which the retardation is  $(n+1)\frac{\lambda}{2}$ . These wave-lengths will consequently be absent, and the intermediate ones, for which the retardation is an even number of half wave-lengths, will be present. The illumination will be greater for these wave-lengths than in the case of the ordinary grating, since twice as many elements are operative in producing it.

We know, however, that luminous vibrations cannot be actually destroyed by interference, consequently the light which fails to make its appearance in the central image must be looked for somewhere else.

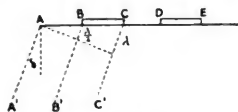


FIG. 148.

It is not difficult to see that this light will be found in the lateral spectra. Suppose  $AA'$  and  $CC'$  to be the rays diffracted in such a direction that their path-difference at the focus is  $\lambda$ , and consider this value of  $\lambda$  the one absent in the central image (Fig. 148). The ray  $BB'$ , were it not for the retardation, would arrive at

the focus with a path-difference of  $\frac{\lambda}{2}$  with respect to  $AA'$ , consequently it would destroy it. If the laminae were removed, the rays traversing the spaces which they occupied would destroy the rays passing through

the adjoining spaces, and there would be no illumination in this direction; in other words, the diffraction effects would disappear, the grating having been reduced to a piece of clear glass. But the ray  $BB'$ , in addition to its  $\frac{\lambda}{2}$  path-difference due to the angle  $\delta$ , has an additional path-difference of  $\frac{\lambda}{2}$  due to the retardation in the lamina, consequently it will arrive at the focus in condition to reinforce the disturbance along  $AA'$ . The spectrum of the first order will consequently be much brighter (for this wave-length) than if the laminae were opaque; in other words, the light which is absent in the central image appears here.

For some other value of  $\lambda$  it may happen that the disturbances coming from the laminae will be retarded by such an amount that they will destroy the disturbances coming from the clear spaces, in the spectrum of the first order, consequently light of certain colors may be absent in the spectra as well as in the central image.

Gratings of this description may be made by depositing silver on glass by the chemical method (a wedge-shaped film is preferable), and then ruling lines through the film perpendicular to the thin edge of the wedge. By covering the plate with iodine crystals, the silver is transformed into a transparent iodide, and we have a laminary grating, the retardation increasing progressively as we recede from the thin edge. With a grating of this description we can verify the above results. A candle flame viewed through it appears brilliantly colored, the color varying according to the portion of the grating held before the eye, and certain colors will be found to be absent in certain spectra, and present in excess in others.

Similar appearances are sometimes found with photographic copies of gratings ruled on glass made with bichromatized gelatine on glass. The writer has prepared gratings in this way, the laminae of which gave a half-wave retardation for sodium light.

A sodium flame could not be seen *directly* through it at all, though the lateral spectra were very brilliant.

The same thing may happen with a reflection grating if the depth of the groove is  $\frac{\lambda}{4}$ , and, in fact, most of the gratings ruled on speculum metal show more or less color in the central image. This color is sometimes erroneously referred to oxidation, and is regarded, as a bad feature. Quite the reverse is the case, these gratings giving brilliant spectra for obvious reasons.

**Concentration of Light into a Single Spectrum.**—By making the laminae of prismatic form it is possible to throw all of the light into a single spectrum, as has been shown by Lord Rayleigh. The angles of the elementary prisms must be such as would cause refraction of the incident rays in the direction of the diffracted rays of the spectrum into which the light is to be thrown. Gratings of precision have not yet been made on this principle, but the possibility of its experimental realization has been demonstrated by Mr. Thorp of Manchester, who punched a series of parallel saw-tooth grooves on a soft metal surface by means of a die. The surface was then flowed with a solution of

celluloid in amyl acetate, which upon solidification was stripped off in the form of a film, bearing an accurate cast of the metal surface. Fully ninety per cent. of all the light was concentrated into the first order spectrum on one side. The same thing has been accomplished in Michelson's remarkable echelon grating, which will be treated in another chapter.

**Parallel Slits of Equal Width, but not Equidistant.**—To find the intensity for diffraction angle  $\delta$ , we compound the effects of the slits, assuming the phases to vary irregularly, instead of remaining constant, as is the case with the diffraction grating.

The intensity due to a single slit is

$$A^2 = a^2 \frac{\sin^2 \pi \frac{a \sin \delta}{\lambda}}{\pi^2 \frac{a^2 \sin^2 \delta}{\lambda^2}}.$$

If we have  $n$  slits,  $\epsilon$  the phase corresponding to each, and  $A$  the amplitude, we have for the illumination

$$I = [\Sigma(A \cos \epsilon)^2] + [\Sigma(A \sin \epsilon)^2],$$

$$\begin{aligned} I &= A^2 [(\Sigma \cos \epsilon)^2 + (\Sigma \sin \epsilon)^2] \\ &= A^2 [\Sigma \cos^2 \epsilon + 2 \Sigma \cos \epsilon \cos \epsilon' + \Sigma \sin^2 \epsilon + 2 \Sigma (\sin \epsilon \sin \epsilon')] \\ &= A^2 [n + 2 \Sigma (\cos \epsilon \cos \epsilon' + \sin \epsilon \sin \epsilon')] = A^2 (n + 2 \Sigma \cos (\epsilon - \epsilon')), \end{aligned}$$

$$I = A^2 \cdot n + 2A^2 \Sigma \cos (\epsilon - \epsilon').$$

Substituting,

$$I = na^2 \frac{\sin^2 \pi \frac{a \sin \delta}{\lambda}}{\pi^2 \frac{a^2 \sin^2 \delta}{\lambda^2}},$$

an expression which shows us that the distribution of the light is the same as with a single slit, but of  $n$ -fold intensity. The maxima and minima of the 2nd and 3rd class are absent.

**The Plane Grating.**—Diffraction gratings for spectroscopic apparatus are usually ruled on a reflecting surface of speculum metal. If the surface is plane the case is analogous to that of the transparent grating. The formula for the grating we have already deduced in considering the diffraction fringes of the 2nd class, due to two parallel slits. It is

$\sin \theta = \frac{n\lambda}{a+b}$ , in which  $\delta$  is the angle of

diffraction,  $n$  the order of the spectrum, and  $(a+b)$  the grating constant. The condition of things is represented in Fig. 149, from which it is apparent that the disturbances which originate at the grating elements, when a plane-wave is incident normally upon it, pass through the point  $P$  in succession. It is thus apparent that even if but a single wave or

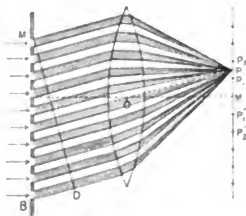


FIG. 149.

pulse struck the grating we should have a periodic disturbance at  $P$ . The grating is thus able to manufacture, as it were, light of a definite wave-length or color, a circumstance which will be considered more fully in the chapter on White Light. The optical paths from what we may call the diffracted wave-front  $MD$  to the point  $P$  are equal, and since the path-difference between two adjacent diffracted streams for the first order spectrum is  $\lambda$ , the path-difference between the extreme rays is  $m\lambda$ ,  $m$  being the number of lines in the grating. As we have seen, if we employ monochromatic light from a slit, made parallel by a lens, we shall have at  $P$  a sharp maximum accompanied by very faint maxima, which lose in intensity and crowd closer to the principal maximum as we increase the number of lines in the grating. The path-difference between the extreme rays for the position of the two minima immediately adjacent to the principal maximum is  $(m \pm 1)\lambda$ , an expression which we shall make use of in considering the resolving power of the grating. If  $n$  is the order of the spectrum, the path-differences for the principal maxima and adjacent minima are  $mn\lambda$  and  $(mn \pm 1)\lambda$ . The formula for the grating shows us that the position of the diffracted image depends upon the value of  $\lambda$ .

The dispersive power of the grating is represented by

$$\frac{d\theta}{d\lambda} = \frac{n}{(a+b) \cos \theta},$$

which shows us that the dispersion increases with the order of the spectrum, and that it is inversely proportional to  $(a+b)$ , the grating constant. For small values of  $\theta$ ,  $\cos \theta = 1$  approximately, and the spectrum is normal, *i.e.* equal increments of wave-length correspond to equal increments of  $\theta$ . In the higher orders of spectra, however, the dispersion increases with  $\theta$  and therefore with  $\lambda$ , and the spectra are more drawn out at the red than at the violet end, exactly the reverse of what we have in the case of prismatic spectra. On this account the grating should be mounted on the table of the spectrometer so as to stand normal to the observing telescope, for in this position  $\theta = 0$ , for the center of the spectrum and the other values of  $\theta$  are small.

The general formula for a reflecting grating, for any angle of incidence  $i$ , is

$$(a+b)(\sin i \pm \sin \theta) = m\lambda.$$

**Overlapping Spectra.**—The formula for the grating shows us that the spectra of the different orders overlap, for by doubling  $n$  and halving  $\lambda$  we have the same value of  $\theta$ . This overlapping gives no trouble in the visible region, if we limit ourselves to the first or even the second order, but in photographic work it must be taken into account, for the ultra-violet of the second order is superposed on the visible region of the first order, wave-length 25 of the second coinciding with 50 of the first. This overlapping is often of use, as we shall see when we come to the consideration of the concave grating; if it is desired to eliminate it, color screens or prismatic analysis must be resorted to: glass cuts off practically everything below  $\lambda = 32$ , consequently a glass lens or plate is all that is necessary when working in the first order spectrum.

**Resolving Power of Gratings.**—Since the diffraction grating is used largely in place of a prism, for the formation of spectra and the examination of spectrum lines, it is of importance to determine upon what the resolving power of the grating depends. Lord Rayleigh<sup>1</sup> has shown that in order to separate two spectrum lines, the distances between the central maxima of their diffraction images must be at least as great as the distance of the first minima from the central maximum.

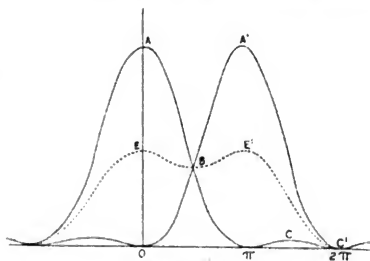


FIG. 150.

If this condition is as represented in Fig. 150, the resultant illumination (dotted) being given by summing the ordinates of the two curves, the lines will not appear clearly separated, but the duplicity of the line can be recognised from the slight shading down the centre. Let  $AB$  (Fig. 151) be the grating and  $BC$  the direction of the diffracted rays, which form, for wave-length  $\lambda$ , the central maximum of the spectrum of the  $m$ th order. As we have seen, the central maxima in each spectrum are accompanied by secondary maxima which decrease in intensity, and become crowded together as we increase the number of lines of the grating. The first minimum at  $\pi$  on each side of a central maximum will obviously lie closer to the central maximum of the spectrum line, if we employ a large number of lines in the grating. In other words, the intensity curve becomes steeper, the light crowding together more nearly into a geometrical line. As an expression for the resolving power we require the change in wave-length

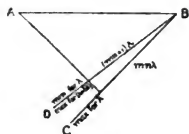


FIG. 151.

necessary to shift the central maximum into the position of the first minimum. For the central maximum for wave-length  $\lambda$ , the retardation between the disturbances coming from the edges of the grating is  $mn\lambda$ . The first minimum to the left for this wave-length is represented by the line  $BD$ , which will be the direction of the central maximum for some other wave-length slightly greater than  $\lambda$ , say  $\lambda + \delta\lambda$ . The retardation of the extreme rays of the first minimum is  $(mn + 1)\lambda$  or  $mn(\lambda + \delta\lambda)$ . Since these two are equal, we can write

$$mn(\lambda + \delta\lambda) = (mn + 1)\lambda, \text{ or } \frac{\delta\lambda}{\lambda} = \frac{1}{mn}.$$

For the  $D$  lines of sodium  $\frac{\delta\lambda}{\lambda} = \frac{1}{1000}$ , so that in order to resolve them in the first order spectrum, we must utilize at least 1000 lines of

<sup>1</sup> "Wave-Theory," *Ency. Brit.*, vol. xxiv.

the grating. They may be separated in the second order spectrum with 500 lines, etc. This treatment is due to Lord Rayleigh.

It is especially to be noticed that the resolving power of the grating does not depend upon the closeness of the ruling, but merely upon the number of lines. Let us take for example a grating one inch in width, ruled with 1000 lines, which in the first order spectrum will barely resolve the sodium lines. Suppose now we interpolate an additional 1000 lines, making them bisect the original spacing. The spectra of odd order will disappear by interference, the energy being thrown into the spectra of even orders, which increase in brilliancy, the gain being fourfold since the amplitudes are doubled.

The resolving power in each spectrum is exactly the same as it was before, since what is now the first order spectrum was previously the second order. If one half the grating is cut away, leaving 1000 lines in half an inch, the dispersion will not be altered while the brightness and the resolving power are halved. The sodium lines are now just barely resolvable in the first order spectrum. If the grating had been cut in halves before the interpolation of the second ruling, the sodium lines would have been just barely separated in the second order spectrum, 500 lines only operating. This spectrum is identical in position with the first order spectrum in the second case. The advantage of ruling the lines close together is two-fold. In the first place, for a given aperture of telescope and collimator, we can bring more lines to bear, and consequently increase the resolving power; in the second place, we concentrate the light into fewer spectra, and obtain increased brilliancy. As Lord Rayleigh remarks, "There is clearly no theoretical limit to the resolving power of gratings even in spectra of given order, but it is possible that, as suggested by Rowland, the structure of natural spectra may be too coarse to give opportunity for resolving power much higher than those now in use. However this may be, it would be possible with the aid of grating of given resolving power to construct artificially of white light mixtures of slightly different wave-length whose resolution or otherwise would discriminate between powers inferior and superior to the given one." We can easily investigate the relation between resolving power and number of lines. Illuminate the slit of the spectrometer with the light of a sodium flame, viewing the spectrum by means of a small grating, the aperture of which can be contracted in the horizontal direction by means of vertical opaque screens. If we are working in the spectrum of the first order, with a Rowland grating of 14,000 lines to the inch, it will be found that the *D* lines run together when the width of the aperture is about one-fourteenth of an inch. Turning the telescope so as to view the lines in the second order, we shall find that they are clearly resolvable. If a grating with coarser ruling is used, the minimum width of the aperture consistent with resolution will be increased in a corresponding degree.

**Intensity of Grating Spectra.**—The intensity of grating spectra can be calculated in the case of gratings made up of opaque and transparent intervals. As gratings of this type are seldom or never used, such calculations are of little practical value. In the case of gratings ruled on speculum metal the distribution of light in the spectra of

different orders is very irregular, depending upon the form of the groove. The following method of measuring the intensity, used by the author in determining what percentage of the total incident light appeared in the very bright, first order spectrum of a particular grating, may be of interest, as the measurements are very easily made.

The measuring apparatus, or photometer, consisted of a pair of Nicol prisms (one mounted in a graduated circle), a small piece of silvered glass, and a bright and uniform sodium flame. The silvered glass can be made by dissolving the varnish from the back of a piece of modern mirror, and polishing with rouge. It is mounted vertically at an angle of  $45^\circ$  with the axes of the Nicols, and covers the lower half of the field (Fig. 152). The soda flame is immediately behind the polarizing prism, and the grating stands to one side, as shown in the figure. By turning the grating, the central, or any one of the lateral (spectral) images of the flame can be viewed in the silvered mirror, immediately in contact with the image of the flame seen through the Nicols, and by turning one of them the intensities can be accurately adjusted. We first set the graduated Nicol in the zero position, and then turn the other Nicol to the position of extinction. The intensity of the restored light for a given angle measured from this position, is proportional to the square of the sine of the angle. The central image can be located easily by watching for the reflection of the flame in the unruled portion of the surface. The results obtained are recorded in the following table, eight spectral images having been measured :

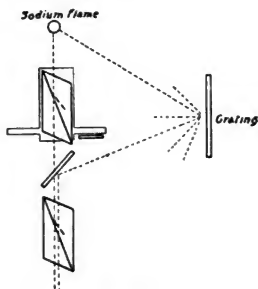


FIG. 152.

Fourth Spectrum	Third Spectrum	Second Spectrum	First Spectrum	Central Image	First Spectrum	Second Spectrum	Third Spectrum	Fourth Spectrum
0.073	0.057	0.20	0.31	0.16	0.98	0.096	0.032	0.01

The numbers given are the squares of the sines of the angles, and represent the intensities of the images as fractional parts of the light transmitted through the first Nicol. The intensity of the first spectrum on the right is as great as the sum of all the others, together with the central image (0.94), which amounts to saying that half of the total light reflected is found in one spectrum.

It is frequently stated that a Nicol reduces the intensity of unpolarized light by one half. The reduction is obviously greater than this on account of the reflections at the two oblique surfaces, and to a slight extent by the balsam film. In the present case the surfaces of the prism were slightly dull, and I doubt if the intensity of the transmitted light was much over 40 per cent. of the original intensity.

Calling the intensity of the soda flame 100, we get the intensities of the spectra by multiplying 40 by the fractions given in the table. The sum of these intensities (eight spectra and central image) is 75.6, which agrees fairly well with Rubens's determination of the reflecting power of speculum metal for yellow (70 per cent.). This indicates that the ruling of the surface interferes in no way with the *total* reflection, which is what might be expected. The interesting point is that half of the total light is found in one spectrum. If speculum reflects 70 per cent., this means that we have 35 per cent. of the light in the first order spectrum, or about one-third of the original amount.

**False Lines due to Periodic Errors or "Ghosts."**—If the illumination is sufficiently intense, the bright lines of the spectrum are usually seen accompanied by companion lines, symmetrically placed to the right and left of the principal lines. These fainter companions are termed "ghosts," and are due to periodic errors of ruling. Very elaborate mathematical treatments of their cause have been given by Rowland and others, which hardly fall within the scope of this book. The following simple method of regarding the effects of periodic errors will, however, make the reasons for the occurrence of the spurious lines clear, though we cannot determine the number of the lines or their relative intensities, as can be done by the more rigorous mathematical methods.

Suppose that some irregularity of the ruling occurs at regularly recurring intervals, due to some slight imperfection of the dividing engine. For the sake of simplicity we will suppose that the irregularity consists in the introduction of an extra line at points half a millimeter apart, a fault that would not be likely to occur in practice, of course. The grating would thus be the equivalent of two superposed gratings, one with a fine spacing, namely, that which the machine is ruling, the other with a 0.5 mm. spacing. Consider now the effect of superposing two gratings. If we view a slit illuminated with sodium light through the fine spaced grating we shall see the central image accompanied by widely separated spectral images. On introducing the second grating, with lines ruled at wide intervals, the central image and the lateral spectral images will all be accompanied by lateral spectra, which lie close together on account of the coarseness of the ruling, and are very faint on account of the small value of the ratio of the width of the line to that of the space between the lines. In other words, the second grating forms spectra not only of the slit (central image), but also of the lateral spectral images.

The errors which actually occur in the process of ruling are, of course, much more complicated. For example, the spacing of the lines may alter gradually in a periodic manner. We can say, however, that, in general, the effect of any periodic irregularity in the ruling will be to produce spectra at angular distances similar to those produced by a grating with a spacing equal to the spacing of the periodic error. In the words of Rowland, "Each periodic error produces the same ghosts in the same place as if it were the only error, while others are added which are the ghosts of ghosts."

**Abbé's Diffraction Theory of Microscopic Vision.**—The diffraction theory of microscopic vision was proposed about thirty years ago by Ernst Abbé. It may be briefly stated as follows. For the production of a truthful image of an illuminated structure by a lens, it is necessary that the aperture be wide enough to transmit the whole of the diffraction pattern produced by the structure. If but a portion of the diffraction pattern is transmitted, the image will differ from the object, and will correspond to an object the entire diffraction pattern of which is identical with the portion passed by the lens. If the structure is so fine, or the lens aperture so narrow, that no part of the diffraction pattern is transmitted, the structure will be invisible, no matter what magnification is used. Abbé devised a number of interesting experiments to illustrate the theory. By means of suitably perforated screens placed within or above the objective of a microscope, one or more of the diffraction spectra produced by a glass grating (the object viewed through the microscope) were cut off, and the appearance of the grating shown to be modified in a most remarkable manner. If all of the spectra were screened off, all trace of the lines vanished. It is sometimes assumed that there is an essential difference between microscopic and ordinary vision, and that the phenomena of diffraction play no part in the latter. Whatever difference there may be between the two cases, arises, however, from the relative size of the objects involved, and from the special methods of illumination employed with the microscope, as has been shown by A. B. Porter, who devised the following ingenious experiment for demonstrating that the images of periodic structures formed by the naked eye itself are due to diffracted light. Light from an arc lamp or the sun passes through a pin-hole in a screen and is focused by means of a photographic lens on a cardboard screen, about 30 cms. from the lens—immediately in front of which a piece of wire gauze having about thirty wires to the cm. is placed. The diffraction pattern produced on the screen by the wire gauze consists of a central image with a large number of radiating spectra surrounding it. There are two sets of spectra at right angles to each other, formed by the two sets of wires, with two intermediate sets also at right angles, but rotated through forty-five degrees with respect to the others. These latter may be regarded as the spectra of spectra, and are always seen when a source of light is viewed through a pair of crossed diffraction gratings. By cutting small holes in the screen we may transmit any portion of the diffraction pattern and allow it to enter the eye. If the screen is pierced by a hole only large enough to transmit the central image, the wire gauze is quite invisible. If a narrow slit is used which transmits only the central image and the horizontal line of spectra, the vertical wires alone are seen; if the slit is turned vertically so as to transmit the vertical line of spectra, the horizontal wires alone are visible. If the slit is turned at an angle of forty-five degrees so that the diagonal set of spectra are transmitted, neither the vertical nor horizontal wires are seen, but a very real-looking set of wires appears running diagonally in a direction perpendicular to the slit. Such a set of wires would, if acting alone, give rise to the spectra transmitted. If the card is pierced with three pin-holes which transmit the central image and two second order spectra, a set of vertical wires is

seen, twice as close together as the actual wires. By cutting two slits in the screen at right angles to each other, and arranging them so as to transmit the diagonal spectra, we see the gauze as if rotated through forty-five degrees.

These experiments are similar to the ones devised by Abbé, and furnish a very easy and convenient means of illustrating his theory.

**Number of Spectra and Distribution of Intensity.**—The intensity of grating spectra has been treated by Lord Rayleigh ("Theory of Diffraction Gratings," *Phil. Mag.*, xlvii., pp. 81-93, 193-205, 1874; also "Wave-Theory," *Encycl. Brit.*). In the case of gratings consisting of opaque and transparent parts of widths  $d$  and  $a$ , it can be shown that the intensity in any spectrum of order  $m$  is given by

$$B_m : B = \frac{1}{m^2 \pi^2} \sin^2 \frac{a m \pi}{a + d}$$

in which  $B$  is the intensity of the image produced by the lens in the absence of the grating, and  $B_m$  the intensity of the spectrum of order  $m$ . Since the sine can never exceed unity, the utmost intensity attainable under the most favorable circumstances is only  $\frac{1}{m^2 \pi^2}$  of the

original light. In the first order, this may amount to  $\frac{1}{\pi^2}$  or  $\frac{1}{10}$ , when the opaque and transparent parts are of equal width. If  $d = a$ , the formula reduces to

$$B_m : B = \frac{\sin^2 \frac{1}{2} m \pi}{m^2 \pi^2},$$

and the spectra of even order disappear. This can be shown by means of a grating formed by winding fine copper wire on a brass frame. Two strands should be wound side by side, and pressed close together. One of the ends is then soldered to the frame, and the other wire unwound. The remaining wire is now soldered to the frame, and the wires on one side cut away. It is best to make the frame in such a way that it can be expanded by means of screws, after the winding is completed; in this way the wires are drawn taut. On viewing a slit backed by a sodium flame through such a grating, the 2nd, 4th, 6th, etc., spectra will be found wanting. If the grating is turned a little so as to alter the ratio of  $a$  to  $d$ , these missing spectra at once appear. If the transparent intervals are small in comparison to the total interval  $(a + d)$ , we have  $B_m : B = \{a/(a + d)\}^2$ , except for spectra of very high order. In this case the spectra are all of equal intensity. Gratings made by photographing black and white drawings of parallel lines with various ratios of  $a$  to  $d$  are useful for purposes of illustration.

The subject of the distribution of the light in the spectra has recently been investigated by A. B. Porter, and some extremely interesting effects found which were verified by experiment (*Phil. Mag.*, 1905).

The case of a grating formed by opaque bars and transparent intervals was first examined by Fourier's theorem. The amplitude of the transmitted light is in this case represented by a square-topped curve,

which, by Fourier's theorem, may be represented by an infinite series of cosine terms. If the edges of the opaque bars are not sharply defined, *i.e.* if they are shaded slightly, the analysis shows that the higher harmonic terms in the series are absent, and since each one of these gives a spectrum, the higher orders of spectra are absent. This was verified experimentally by making a contact print on a photographic plate of a grating with 400 very sharply-ruled black lines to the inch; one edge of the grating was in contact with the plate, while the other was separated from it by means of a piece of paper. The blurring thus increased progressively across the plate. The original grating showed the first 35 orders of spectra of a sodium flame, while the print showed but three, when the flame was viewed through the edge which was in contact with the original. As the eye was moved along the print, the 2nd and 3rd order spectra rapidly faded away, the first order only being yielded by the end which was separated from the original. Applying this principle to Abbé's theory, we see at once that the sharpness of the edges of the images of a series of black lines depends upon the transmission of the spectra of high orders. If only the first order spectra are passed the lines appear greatly blurred. If four or five orders were passed, the images became sharper and less blurred, but a fine dark line appeared down the centre of each. This was predicted by Porter from curves drawn representing the summation of the first five terms of the Fourier series, and subsequently verified by experiment. The result is rather remarkable in showing that a falsification of the image may result from an improvement of the lens. When 8 spectra were transmitted, two dark lines appeared running down the centre of each bright band. The results were verified by examining a grating under a microscope, the objective of which was furnished with an iris diaphragm, by which any desired number of spectra could be cut off. Monochromatic light was used, and the experiments show that some caution must be used in interpreting minute details in micro-photographs obtained with monochromatic light.

Porter has also examined the effects of the spectra produced by structure underlying the structure upon which the microscope is focused. It was found that when monochromatic light was employed, the presence of a second grating, a short distance below the one under observation, in certain cases caused a complete obliteration of the lines over the greater part of the field. It is well known that when two gratings are superposed, with their lines parallel and separated by a short distance, the spectra formed by the double grating are intersected by transverse shadow bands. If under certain specified conditions these shadow bands lie in the yellow of the first order spectra, and the iris diaphragm of the objective is contracted until only the first order spectra are passed, we shall have no spectra at all when the microscope is illuminated with yellow light, and the first order spectra when it is illuminated with light of some other color. In Porter's experiment, two gratings of 3000 lines to the inch, separated by a space 1 mm. thick, were placed upon the stage of the microscope which was focused upon the upper grating. The instrument was illuminated with monochromatic light furnished by a spectroscope, and it was found that the

lines completely disappeared when the illumination was with yellow light, though they were distinctly visible with all other colors.

These experiments are very interesting, as they were all made with the microscope under what would be considered normal working conditions, with central illumination and circular diaphragms centered on the optic axis. "Nevertheless," as Porter says, "when certain relations existed between the aperture of the lens and the coarseness of structure of the object, images were formed which were utterly false in their smaller details, and other images were profoundly modified by the presence of structure lying entirely beyond the focal plane. It therefore seems that a working knowledge of the phenomena and laws of diffraction might well form a part of the equipment of everyone who uses the microscope and attempts to interpret its indications."

Abbé's theory has sometimes been lightly treated, as most of his anomalous appearances of images were produced by diaphragms of peculiar form, slits, crosses, triangles, etc., which are never met with in practice. Porter's experiments were, however, all made with circular diaphragms under normal conditions, and the effects observed may frequently occur in practice, and be falsely interpreted. The reader should refer to the original paper for other interesting cases, and a more complete treatment than the scope of this book permits.

**Concave Gratings.**—To obtain sharply focused spectra by means of the plane diffraction grating, two lenses are required: one to render the light parallel before its incidence upon the grating, the other to unite the parallel diffracted rays in a focus. The brilliant discovery was made by Rowland that gratings ruled on concave spherical surfaces would of themselves furnish focused spectra, excelling in sharpness those obtained by means of lenses. This discovery marked an epoch in the history of spectroscopy, for by dispensing with the lenses, and the absorbing action which they exerted on the ultra-violet, the region of short waves could be explored with an accuracy never before attainable, and the gratings ruled upon Professor Rowland's machine have become the standard instruments for spectroscopic work throughout the world. They combine the image-forming power of concave mirrors with the spectrum-producing power of gratings. A marked advantage of the concave grating lies in the fact that the superposed spectra of different orders are all in focus, which is not the case with plane gratings and lenses, owing to the fact that complete achromatization can never be obtained. It is thus possible to measure the relative wave-lengths with great accuracy. An ultra-violet line of wave-length 2950 of the second order spectrum will be photographed nearly in coincidence with the *D* lines of sodium, and its wave-length can be very accurately measured relatively to these lines. This method of coincidences was originated by Rowland. But the greatest advantage of all is the fact that, when properly mounted, the concave grating yields spectra which are truly normal, *i.e.* spectra in which the distances between the lines are proportional to their wave-lengths.

Various methods of mounting the concave grating have been devised. That due to Rowland is the following: The theory of the grating, which we shall take up presently, shows that if the grating and the illuminated slit are both situated on a circle, the diameter of which

is equal to the radius of curvature of the grating, the spectra of different orders will all be in focus upon the same circle. The spectra are normal along that portion of the circle diametrically opposite the grating, consequently if a photographic plate is placed at this point and bent to the radius of curvature of the circle, the photographic image will be everywhere in focus, and the spectrum will be normal. To pass from one part of the spectrum to another we have only to move the slit around on the circle, a method sometimes employed. With fixed sources of light, such as the image of the sun formed by a lens in combination with a heliostat, this is impossible, and Rowland devised the following extremely ingenious mechanical device, by which the camera and grating could be moved, with reference to a fixed slit, so as to comply with the required conditions.

Two tracks  $AB$ ,  $AC$  (Fig. 153) are rigidly mounted on fixed beams, so as to meet accurately at a right angle. On these tracks roll a pair of carriages which support a trussed tube of iron, the length of which is equal to the diameter of the large circle, i.e. the radius of curvature of the grating. One of the carriages carries the camera, the other the grating  $G$ , while the slit is permanently mounted above the point where the rails meet. As the camera is moved away from the slit, the grating is drawn towards it, the three always remaining on the circumference of the circle, with the grating and camera always at opposite ends of a diameter. The grating is turned into such a position that its center of curvature coincides with the center of the photographic plate.

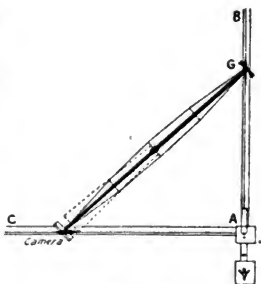


FIG. 153.

Full and explicit directions for mounting and adjusting the grating will be found in Kayser's *Handbuch der Spectroscopie*, vol. i.

**Theory of the Concave Grating.**—The theory of the concave grating has been treated by Rowland, Mascart, and others. The following is due to Runge, who specifies definite conditions regarding the position of the grating, slit, and spectrum, and then investigates the nature of the ruling necessary to produce sharply focused spectra; for example, should the lines be equally spaced along the *arc* of the grating or along the chord? The manner of ruling adopted by Rowland gives equal spacing along the chord, for, as the diamond point moves back and forth along a fixed line, the concave surface, which lies flat upon the moving carriage of the dividing engine, is advanced through equal distances by means of the screw.

Suppose that we have a small source of monochromatic light at the point  $A$  (Fig. 154), and wish to determine the resultant at  $A'$  of the disturbances coming from the different elements of the concave surface  $GP$ .

Let  $P$  be any point on the concave surface. An image of  $A$  will be formed at  $A'$  whenever the disturbances arising at every point  $P$  reach

$A'$  in the same phase. This condition is fulfilled if  $AP + PA' = \text{Const.}$ , or if the curved surface is a portion of an ellipsoid of revolution having  $A$  and  $A'$  as foci. Construct now a series of confocal ellipsoids, under the condition that the constant distance  $AP + PA'$  for each increases by  $\frac{\lambda}{2}$  for each successive surface. These ellipsoids will cut the spherical

surface  $G$  up into zones, in such a manner that the disturbances from any two adjacent ones will reach  $A'$  in opposite phase. If  $AP$ ,  $A'P$  and the radius of curvature of  $G$  are large, the zones will have practically the same width, and the resultant effect of all at  $A'$  will be zero. If every other zone is blotted out, or if a line is ruled on every other one, so as to get rid of this destructive interference, we shall have illumination at  $A'$ . Runge then shows that with any other wave-length differing even but slightly from the one

considered, we shall have zero illumination at  $A'$ . This is merely an explanation of the formation of the spectrum and can be omitted.

Consider the spherical surface of radius  $\rho$  as fixed with its vertex at the origin of co-ordinates  $x, y, z$ , and tangent to the  $yz$  plane. Its equation in this position is

$$x^2 + y^2 + z^2 - 2\rho x = 0.$$

Let the points  $A$  and  $A'$  lie in the  $xy$  plane, and let their coordinates be  $a, b$  and  $a', b'$  (Fig. 155). We require expressions for  $AP$  and  $A'P$ , for it is the sum of these quantities which enters into our expression for the illumination at  $A'$ .

The coordinates of  $P$ , a point on the spherical surface, are  $x, y$ , and  $z$ .

Then  $\overline{AP}^2 = (x - a)^2 + (y - b)^2 + z^2$ ,  
or writing  $r^2$  for  $(a^2 + b^2)$ ,

$$\overline{AP}^2 = r^2 - 2ax - 2by + x^2 + y^2 + z^2.$$

From the equation of the spherical surface we have

$$2x = \frac{x^2 + y^2 + z^2}{\rho},$$

and substituting in the last equation for  $2ax$ , the value given above, i.e.  $\frac{a}{\rho}(x^2 + y^2 + z^2)$ ,

$$\overline{AP}^2 = r^2 - 2by + \left(1 - \frac{a}{\rho}\right)y^2 + \left(1 - \frac{a}{\rho}\right)z^2 + \left(1 - \frac{a}{\rho}\right)x^2.$$



FIG. 154.

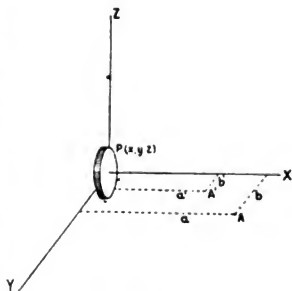


FIG. 155.

From the equation of the sphere,  $x$  is of the second order with respect to  $y$  and  $z$ ; the above equation reduces to (neglecting 3rd order terms)

$$AP = r - \frac{b}{r}y + \frac{a}{2r}\left(\frac{a}{r^2} - \frac{1}{\rho}\right)y^2 + \frac{1}{2r}\left(1 - \frac{a}{\rho}\right)z.$$

A similar equation is obtained for  $A'P$ , and the sum may be written

$$AP + PA' = r + r' - \left(\frac{b}{r} + \frac{b'}{r'}\right)y + \left[\frac{a}{2r}\left(\frac{a}{r^2} - \frac{1}{\rho}\right) + \frac{a'}{2r'}\left(\frac{a'}{r'^2} - \frac{1}{\rho}\right)\right]y^2 + \left[\frac{1}{2r}\left(1 - \frac{1}{\rho}\right) + \frac{1}{2r'}\left(1 - \frac{1}{\rho}\right)\right]z^2.$$

We can simplify this equation by imposing certain conditions. If we limit the vertical aperture of the mirror sufficiently, the terms in  $z^2$  may be neglected. This condition is fulfilled if the ruled lines are short, and in practice the ratio of the length of line to the radius of curvature is never made greater than a certain amount, say  $\frac{1}{50}$ .

We can get rid of the terms in  $y^2$  by proper disposition of  $A$  and  $A'$ . The condition to be fulfilled is that

$$\frac{a}{2r}\left(\frac{a}{r^2} - \frac{1}{\rho}\right) + \frac{a'}{2r'}\left(\frac{a'}{r'^2} - \frac{1}{\rho}\right) = 0,$$

which will occur if  $r^2 = a\rho$  and  $r'^2 = a'\rho$ , that is if  $A$  and  $A'$  are situated on a circle, the center of which is on the  $x$ -axis at a distance  $\rho/2$  from the origin.

This is the condition already specified in the treatment of the mounting of the grating.

The equation now reduces to

$$AP + A'P = r + r' - \left(\frac{b}{r} + \frac{b'}{r'}\right)y.$$

Under the conditions specified  $r + r'$  are independent of the position of the point  $P$  on the surface of the grating; we need only consider the term  $\left(\frac{b}{r} + \frac{b'}{r'}\right)y$  in determining the illumination at  $A'$ .

Call  $e$  the distance between the  $y$  coordinates of the lines  $n$  and  $(n+1)$ , or the distance between their adjacent zones; there will be illumination at  $A'$  when the path-difference between the streams of light from the two zones amounts to a whole number of wave-lengths. This condition is obviously represented by

$$\left(\frac{b}{r} + \frac{b'}{r'}\right)(y + e) - \left(\frac{b}{r} + \frac{b'}{r'}\right)y = m\lambda,$$

or 
$$e\left(\frac{b}{r} + \frac{b'}{r'}\right) = m\lambda.$$

The consecutive values of the  $y$  coordinates of the lines must therefore differ by a constant amount; in other words, the spacing of the grating must be equal when measured along a chord of the arc, and not along the arc itself. The manner of ruling the gratings insures this, as has been pointed out.

We will now prove that the spectrum is normal when the image is formed at a point lying on the normal to the grating, i.e. when  $A'$  lies on the  $x$  axis.

Under this condition  $b' = 0$ , and our equation reduces to

$$e \frac{b}{r} = m\lambda, \text{ or since } \frac{b}{r} = \sin i,$$

$$e \sin i = m\lambda \quad \text{or} \quad \sin i = \frac{m\lambda}{e}.$$

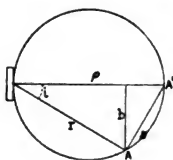


FIG. 156.

The distance  $AA'$  (Fig. 156) of the spectrum line from the source is, when  $b' = 0$ , given by  $AA' = \rho \sin i$ , or by substituting  $\frac{m\lambda}{e}$ ,

$$AA' = \frac{\rho m\lambda}{e}.$$

This equation shows us that  $AA'$  increases in proportion to  $\lambda$ , or that the spectrum is normal.

**Diffraction by a Circular Aperture.**—This case is of especial interest in connection with the theory of optical instruments.

Let  $R$  be the radius of the aperture, and  $\theta$  the angle of diffraction of parallel rays which meet at  $M$ , the focus of the lens.  $AB$  is a diameter of the aperture and  $ON$  the normal at the center.

Let the displacement at  $M$  due to an area  $\rho d\phi d\rho$  at  $A$  be expressed by  $\sin 2\pi \frac{t}{\lambda} \rho d\phi d\rho$  and let  $OP = \rho$  and  $\angle AOP = \Phi$  be the coordinates of a point  $P$  of the aperture. The path-difference between the rays leaving  $A$  and  $H$  is  $AH \sin \theta$ .

A ray leaving  $P$ , parallel to the other three rays, will have the same path-difference with respect to  $A$  as has the ray from  $H$ , the foot of the perpendicular, let fall from  $P$  upon  $AO$ . The displacement due to the disturbance from  $P$  is therefore expressed by

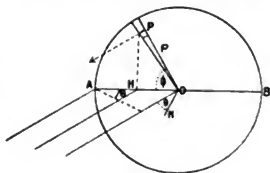


FIG. 157.

$$\rho \sin 2\pi \left( \frac{t}{\lambda} - \frac{AH \sin \theta}{\lambda} \right) d\phi d\rho,$$

$\rho d\phi d\rho$  being the area of the surface element at  $P$ , and since

$$AH = R - \rho \cos \phi,$$

we can write the above expression

$$\rho \sin 2\pi \left( \frac{t}{\lambda} - \frac{R \sin \theta}{\lambda} + \frac{\rho \cos \phi \sin \theta}{\lambda} \right) d\phi d\rho,$$

which being the sine of the sum of two quantities can be treated in the same way as the expression of similar form which we developed in determining the effect of the concave spherical wave.

The resultant obtained by integrating over the whole aperture is then

$$\left( \int_0^{2\pi} \int_0^R \rho \cos 2\pi \frac{\rho \cos \Phi \sin \theta}{\lambda} d\Phi d\rho \right)^2 + \left( \int_0^{2\pi} \int_0^R \rho \sin 2\pi \frac{\rho \cos \Phi \sin \theta}{\lambda} d\Phi d\rho \right)^2.$$

The second integral is zero, for the elements of it, arising from any two points situated at equal distances on opposite sides of  $PO$ , are equal and of opposite sign.

The intensity is therefore

$$I = \left( \int_0^{2\pi} \int_0^R \rho \cos 2\pi \frac{\rho \cos \Phi \sin \theta}{\lambda} d\Phi d\rho \right)^2.$$

This expression is integrated with respect to  $r$  by parts, and with respect to  $\Phi$  in series, the final result being

$$I = (\pi R^2)^2 \left\{ 1 - \frac{1}{2} \left( \frac{m}{1} \right) + \frac{1}{3} \left( \frac{m^2}{2} \right)^2 - \frac{1}{4} \left( \frac{m^3}{2 \times 3} \right)^2 + \frac{1}{5} \left( \frac{m^4}{2 \cdot 3 \cdot 4} \right)^2 \right\},$$

in which  $m$  is defined by  $2m = \frac{2\pi R}{\lambda} \sin \theta$ .

This result was obtained by Airy (*Camb. Phil. Trans.*, page 283, 1834). The series is convergent for all values of  $m$ , and becomes alternately positive and negative as  $m$  increases. The intensity is therefore zero for certain values of  $m$ , i.e. for certain values of  $\theta$ . We have in consequence a series of concentric bright and dark rings. The angle  $\theta$  corresponding to any bright or dark ring is found by ascertaining the corresponding value of  $m$  in the series, and equating it to

$$\frac{\pi R \sin \theta}{\lambda} \quad \text{or} \quad \sin \theta = \frac{m\lambda}{\pi R}$$

an equation which shows that the deviation  $\theta$  for any ring is proportional to  $\lambda$ , and inversely as the radius of the aperture.

The diameters of the rings and central spot consequently become less as the aperture is increased in size. It is on this account that the images of the fixed stars appear smaller in telescopes of large aperture than in smaller instruments.

The following table gives the values of  $\frac{m}{\pi}$  for the first few maxima and minima:

	$\frac{m}{\pi}$	Intensity.
1st Max.,	0	1
1st Min.,	0.61	0
2nd Max.,	0.81	0.0174
2nd Min.,	0.116	0
3rd Max.,	1.333	0.0041
3rd Min.,	1.619	0

**Resolving Power of Telescope.**—The images of two stars can be seen separated if the central spot of the diffraction pattern of one falls at or beyond the first minimum (*i.e.* dark ring) of the image of the other. Let  $R$  be the radius of the telescope's aperture. The diffraction angle  $\theta$  for the first minimum is given by

$$\sin \theta = 0.61 \frac{\lambda}{R}.$$

The angular distance between two stars must therefore be greater than  $\theta$ , as defined above, if they are to be seen separated, *i.e.* we must have the angular separation  $\Phi > 0.61 \frac{\lambda}{R}$  (writing  $\phi$  for  $\sin \phi$ ).

Calling  $R = 0.0056$  mm. and expressing  $\Phi$  in minutes, we have  $\Phi > \frac{1.17'}{R}$ . A telescope of 200 mm. will therefore resolve a double star with an angular separation of  $0.117' = 7''$ . The equation shows us that the angular separation of two stars which can be separated by a given lens is roughly equal to the angle subtended by the wavelength of light at a distance equal to the diameter of the lens.

**Babinet's Principle.**—This principle is one which is applied to complementary diffraction screens, by which we mean a pair of screens in which the transparent portions of one are replaced by opaque portions in the other, and *vice versa*. An example would be a metal plate with a number of small circular apertures and a glass plate with metal discs of similar size and distribution. Babinet's principle states that the diffraction patterns are the same in each case. This we can see from the following considerations:

In the case above the illumination at a point  $M$  on the screen, where the parallel diffracted rays of diffraction angle  $\delta$  from the collection of circular apertures come together, is represented by the sum of the squares of two integrals taken over the areas of the apertures. This we will call  $A_1^2 + B_1^2$ . In the same way the illumination at the same point due to the collection of discs is  $A_2^2 + B_2^2$ . If the two sets of disturbances act simultaneously, *i.e.* if the wave is disturbed by no screen, the illumination is zero, provided the point  $M$  is situated at some point not coincident with the point at which the wave comes to a focus; in other words, no diffraction effects are produced. This means that the resultant of one set of disturbances is able to exactly destroy the resultant of the other set, or

$$(A_1 + A_2)^2 + (B_1 + B_2)^2 = 0, \text{ or that } A_1 = -A_2 \text{ and } B_1 = -B_2.$$

The illumination is therefore the same in the two cases, and the only result of changing the screens is to alter the resultant phase by  $180^\circ$ .

The principle of Babinet cannot be applied universally to all diffraction problems, for example the circular aperture and disc in the Fresnel class, one of which gives maxima and minima along the axis, the other only a maximum. Its application is restricted to points lying outside of the projection of the aperture, where the illumination due to the whole aperture is zero. As an illustration of a case in which it can be applied, suppose we have a large aperture  $AB$

(Fig. 158) filled with small circular discs. The illumination at points in the regions  $CD$  and  $EF$  remains the same, when circular apertures are substituted for the discs, but alters in the region  $DE$ , the projection of the large aperture. Strictly speaking we cannot apply the principle quite up to the points  $D$  and  $E$ , for if we get very near them we are in a region where the illumination due to the whole aperture is not zero, owing to diffraction by its edges. The case above figured belongs to the Fresnel class. If we place a lens behind the aperture, we can apply the principle to all points lying outside of the system of small diffraction rings formed by the open aperture and the lens. If the aperture is fairly large and the lens of short focus the ring system is exceedingly small, and the principle applies everywhere except at the image of the source of light thrown by the lens, which is sensibly a point.

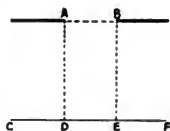


FIG. 158.

**Diffraction by Two Small Apertures.**—If we have two small circular apertures of the same size and close together we shall have interference between the disturbances exactly as in the case of the two parallel slits. The intensity due to a single aperture may be represented by

$$i = [f(\delta, \delta')]^2,$$

in which  $\delta$  and  $\delta'$  are the two angular co-ordinates which determine the direction of the diffracted ray.

If  $b$  is the distance between the centers of the apertures (Fig. 159), and consequently the distance between any two homologous points, and  $\phi$  is the angle between the diffracted ray and a plane perpendicular to the line joining the points, the path-difference between the rays will be  $b \sin \phi$ , and the intensity

$$I = 2[f(\delta, \delta')]^2 \cdot \left(1 + \cos 2\pi \frac{b \sin \Phi}{\lambda}\right).$$

There will thus be a system of circular maxima and minima, crossed by a system of parallel dark strips perpendicular to the line joining the points, the position of which is given by

$$2\pi \frac{b \sin \Phi}{\lambda} = (2m + 1)\pi \quad \text{or} \quad \sin \Phi = \left(m + \frac{1}{2}\right) \frac{\lambda}{b}.$$

By Babinet's principle we may substitute for the apertures two small circular discs, without changing the distribution of intensity in the diffraction pattern. In this case, however, the diffused light and the intensity of the illumination at the center make it difficult or impossible to see the rings and fringes.

**Diffraction by a Large Number of Irregularly Arranged Circular Apertures or Discs.**—In this case the phases between the parallel disturbances from homologous points vary in an irregular manner, and we have on the whole as much reinforcement as destructive interference in any given direction, the case being similar to the parallel, but not

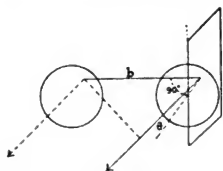


FIG. 159.

equidistant slits. The illumination at any point is the same as that produced by a single aperture multiplied by the number of apertures.

**Halos.**—The halos which are sometimes seen surrounding the sun or moon are due to diffraction by small drops of water, which by Babinet's principle will produce the same effects as small circular apertures of the same size. The smaller the drops the larger the halos, but we distinguish between the diffraction halos, which are always close to the sun, and the large rings due to ice spicules floating in the air. These halos can be imitated by viewing a candle flame or other source of light, through a glass plate, on the surface of which lycopodium has been dusted, or better, by viewing the light through a large glass flask, wet on the inside and connected with an air pump. On partially exhausting the flask with one or two strokes of the pump a cloud forms in the flask, and the light is seen to be surrounded by brilliantly colored rings.

When the halo is produced by particles at a great distance, as is the case in the atmospheric phenomena, instead of by particles immediately in front of the lens of the eye or telescope, the complete ring system as seen is of course not produced by each individual particle.

The production of the colored ring is illustrated in Fig. 160. The broad arrow indicates the direction in which the sun is seen by an eye

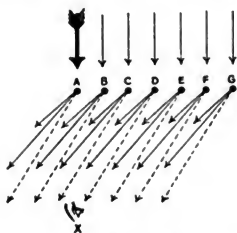


FIG. 160.

at X. *A, B, C, D*, etc., are small globules of water. The dotted arrows represent the directions of the diffracted rays, giving the first maximum to the left of the central maximum for the blue rays, the long solid arrows the directions of the diffracted rays for the green, and the short arrows for the red. It is obvious from the diagram that the particle *D* will send blue light to the eye, the particle *E* green light, and the particle *F* red light. The phenomenon in space will be represented by rotating the diagram on *AX* as an axis. Each particle

of water thus forms an infinitesimal element of the halo. If the particles vary in size in different parts of the sky, the angles of diffraction will vary also, and we may thus have a halo which is not a perfect circle.

In the same way a cobweb in the sunshine sends approximately monochromatic light to the eye, the color depending on its angular position, and a plane diffraction grating at a distance of eight or ten feet from the eye appears illuminated in light of a uniform color.

**Young's Eriometer.**—The dependence of the diameter of the halo on the size of the diffracting particles was utilized in an ingenious piece of apparatus devised by Young for measuring the diameters of fibres, or small particles of any sort. It consists of a metal plate with a small hole 5 mm. in diameter, surrounded by a circle of smaller holes about 1 cm. in radius. The plate is placed in front of a lamp flame, and viewed through the particles or fibres to be measured, which are best spread out on a glass plate. The halo surrounding the central aperture

can be brought into coincidence with the circle of small holes by varying the distance of the screen, which can be done by sliding the plate carrying the particles along a graduated rod, on the end of which the diffracting screen is mounted, the distance varying inversely as the diameter of the halo, which in turn varies inversely as the diameter of the particles. The constant of the instrument is determined by making an observation with particles of known size. If  $d$  is the distance between the screen and the particles of known radius  $r$ , when the halo is in coincidence with the ring, and  $d'$  is the distance for particles of unknown radius  $r'$ , we have

$$\frac{r'}{d'} = \frac{r}{d} \text{ or } r' = r \frac{d'}{d}.$$

**Effect of moving one of the Two Apertures in the Direction of the Source.**—This case is of especial interest in connection with the study of the so-called diffusion rings, which we shall take up next. Suppose the screen with the two circular apertures to be divided in two along a line perpendicular to the line uniting the centers of the apertures at its middle point, and the two halves displaced in the direction of the incident rays by a distance  $a$ . Call  $2\epsilon$  the distance between two homologous points  $A, B$  of the apertures,  $\Phi$  the angle between the incident ray and  $AB$ , and  $\chi$  the angle between the diffracted ray and  $AB$ . Let  $D$  be the position of  $B$  before it was moved forward through the distance  $a$ . We require the path-difference between the parallel diffracted rays  $AE$  and  $BF$ . When the incident wave-front reaches  $AD$ , the secondary disturbance leaves  $A$ , travelling along  $AE$  and reaching a point  $E$  (so situated that  $AE = a$ ) at the moment when the incident wave reaches  $B$ . Let fall a perpendicular from  $B$  upon  $AE$  meeting it at  $H$ , which may be above or below  $E$  according as  $\chi$  is greater or less than  $\phi$ . If  $\phi = \chi$  the two points will coincide, since then the rt. triangles  $DAB$  and  $AHB$  will be equal. The path-difference between the parallel diffracted rays is evidently  $HE$ , or

$$AE - AH = a - AH = 2\epsilon(\cos \Phi - \cos \chi).$$

If  $\Phi$  and  $\chi$  are small this is approximately equal to

$$\epsilon(\chi^2 - \Phi^2).$$

Substituting this in the expression obtained in the last article we get

$$I = 2[f(\delta, \delta')]^2 \left( 1 + \cos 2\pi \frac{\epsilon(\chi^2 - \Phi^2)}{\lambda} \right).$$

The minima will be given by  $2\pi \frac{\epsilon(\chi^2 - \Phi^2)}{\lambda} = (2m + 1)\pi$

$$\text{or } \chi^2 = \Phi^2 + \frac{(2m + 1)\lambda}{2\epsilon}.$$

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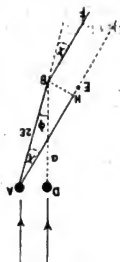


FIG. 161.

This equation represents concentric circles surrounding a point on the prolongation of  $AB$ . These minima of course correspond to the minima described in the last article. When the two apertures are side by side the minima are practically vertical straight lines. Shifting one of the sources in the direction of the incident light rays causes the minima to become arcs of circles, the center of the concentric system coming nearer to the image of the source as the apertures are further displaced. If the apertures were in line, *i.e.* if  $\Phi = 0$ , the image of the source would be at the center of the system.

This will be the case treated in the next article, in which, however, the apertures are replaced by small opaque particles. The linear and circular minima, which we have discussed in this and the preceding article, should be compared with the minima produced by two similar sources in directions at right angles to, and parallel with the line joining them.

**Diffraction by Small Particles on the Surface of a Mirror.**—The so-called diffusion rings observed when a small source of light is viewed in a silvered glass mirror, the front surface of which is slightly dimmed with a deposit of dust, such as lycopodium, are in reality diffraction phenomena. They are sometimes erroneously attributed to the interference of diffused light, and Stokes was the first to treat them as diffraction effects.

We have here a case of the interference of the secondary disturbances from a particle interfering with the reflected secondary disturbances from the same particle, the path-difference depending on the thickness of the glass plate and its refractive index. By a suitable arrangement of the apparatus employed for viewing them, they may be brought under the Fraunhofer class of diffraction phenomena. The method is due to Lommel. We require the normal incidence of parallel rays upon the surface of the mirror, and a lens or telescope for the purpose of rendering concave the reflected wave-front.

Sunlight is concentrated upon the slit of a spectrometer, which should be wide open. The parallel rays emerging from the collimator are reflected from a piece of plane-parallel glass, placed on the table of the instrument at an angle of  $45^\circ$  (Fig. 162), against the dusted mirror, and by this back through the inclined reflector into the telescope, in which the image of the source is seen surrounded by brilliant colored rings.

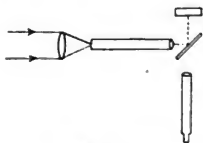


FIG. 162.

The investigation can be simplified by considering the glass plate absent, *i.e.* by reducing the diffracting system to a reflecting surface with a large number of small particles lying in a plane parallel to, and in front of it. We may further simplify the case by considering the reflecting surface absent, and a second layer of particles, absolutely identical with the first, occupying the position of the reflected image of the first layer. The source of light we consider the vertical image of the actual source, seen behind the reflecting surface. The case as it now stands is a source of light at an infinite distance, two parallel equidistant layers of small particles identical with each other, and a

lens for bringing the parallel diffracted rays to a focus where they interfere. If the distance between the layer of dust and the reflecting surface is  $\epsilon$ , the distance between the two layers in the modified case is obviously  $2\epsilon$ . Every particle in one layer has a similar neighbor in the direction from which the light comes, and we will consider the particles so small in comparison to the angular diameter of the light source, that they do not shade their neighbors to any sensible degree. Let  $A$  and  $B$  be two particles at distance  $2\epsilon$  (Fig. 163). We are to investigate the mutual interference of the diffracted disturbances in the direction of the dotted lines, making an angle  $\chi$  with the incident rays. The path-difference will be the same as in the case treated in the last article,  $2\epsilon(\cos\phi - \cos\chi)$ . For normal incidence  $\phi = 0$ . If  $I_1$  is the intensity of the field in the direction  $\chi$ , due to a single layer of particles, the intensity when both layers are present will be

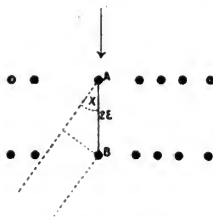


FIG. 163.

$$I = 2I_1 \left( 1 + \cos 2\pi \frac{2\epsilon(\cos\phi - \cos\chi)}{\lambda} \right).$$

The position of the maxima and minima, due to the second variable factor, are given by

$$2\epsilon(\cos\phi - \cos\chi) = m \frac{\lambda}{2},$$

odd values of  $m$  giving minima, even maxima

$$\cos\chi = \cos\phi - \frac{m\lambda}{4\epsilon}.$$

The maxima and minima are concentric circles, the common center of which is in the direction of the normal to the reflecting surface. The white central maximum for which  $m=0$  is given by  $\chi=\phi$ . This means that if we incline the mirror, the center will move to one side and eventually disappear, the fringes becoming approximately straight.

**Fresnel Diffraction Phenomena.**—In the preceding section we have discussed various diffraction problems under the simplified conditions of light source and screen at infinity. We will now proceed with the more general treatment of cases in which the source and screen are both at finite distances from the diffracting aperture, and no lenses are used for rendering the rays parallel or convergent. We cannot now treat the phase as the same at all points in the plane of the diffracting aperture, nor can we solve the problems by determining the resultant of parallel disturbances as in the Fraunhofer class.

Fresnel discussed only the diffraction patterns produced by screens bounded by straight lines of infinite length, such as straight edges, wires and slits. He first showed that the relative intensities at different points on the projection screen, along a line perpendicular to the diffracting edge, could be determined by considering only the

secondary disturbances coming from a circular section of the wave-front, the problem reducing itself to the discussion of the resultant of an infinite number of disturbances from a limited portion of a linear circular wave.

In Figure 164,  $AB$  and  $CD$  are the sections of the projection screen and spherical wave-front respectively,  $F$  is the section of the diffracting screen, and  $O$  the source of the light. The relative illumination along  $AB$  will be the same whether we take the resultant of the disturbances from the circular section of the wave which is not intercepted by the screen  $F$ , or the disturbances from that portion of the complete spherical wave which is not screened off.

We have then to determine the effects at a point  $P$  of disturbances coming from points  $M$ ,  $M'$ ,  $M''$ , etc. Let the distance from  $O$  to the edge of  $F$  be  $a$ , and from  $F$  to the screen  $b$ , and let  $ds$  be a small element of the wave at  $A$ . If the

displacement at  $A$  be proportional to  $\sin 2\pi \frac{t}{T}$ , that at  $P$ , contributed by  $ds$ , will be  $\sin 2\pi \left( \frac{t}{T} - \frac{b}{\lambda} \right) ds$ , while an element at  $M$  will contribute a displacement represented by  $\sin 2\pi \left( \frac{t}{T} - \frac{b+\delta}{\lambda} \right) ds$ , in which  $b+\delta = MP$ .

The displacement at  $P$  due to the simultaneous action of all the elements  $ds$  of the circular arc will be

$$\int \sin 2\pi \left( \frac{t}{T} - \frac{b+\delta}{\lambda} \right) ds,$$

and the intensity (compare article on Diffraction by Parallel Slits)

$$I = \left( \int \cos 2\pi \frac{\delta}{\lambda} ds \right)^2 + \left( \int \sin 2\pi \frac{\delta}{\lambda} ds \right)^2,$$

in which we have resolved each disturbance into two rectangular components, which are separately added.

If we can confine our attention to points not far removed from  $A$  we can write  $\delta = \frac{s^2(a+b)}{2ab}$ , as can be easily shown by considering  $a$  and  $b$  as the longer sides of two right triangles similar to the small triangles which have the side  $s$  in common;  $\delta$  is then equal to the sum of the short sides of the small triangles.

This gives us for the intensity

$$I = \left[ \int \cos \pi \frac{(a+b)s^2}{ab\lambda} ds \right]^2 + \left[ \int \sin \pi \frac{(a+b)s^2}{ab\lambda} ds \right]^2.$$

Writing  $\frac{\pi(a+b)s^2}{ab\lambda} = \frac{\pi}{2} v^2$ , which gives us

$$s = v \sqrt{\frac{ab\lambda}{2(a+b)}} \quad \text{and} \quad ds = \sqrt{\frac{ab\lambda}{2(a+b)}} dv,$$

the expression for the intensity reducing to

$$I = \frac{ab\lambda}{2(a+b)} \left[ \left( \int \cos \frac{\pi}{2} v^2 dv \right)^2 + \left( \int \sin \frac{\pi}{2} v^2 dv \right)^2 \right].$$

The two integrals occurring in this expression are known as the Fresnel integrals. Integrating them between certain values of  $v$ , gives us the resultant of the secondary disturbances from a corresponding portion of the wave-front,  $v$  varying with  $s$  the distance of the wave-front elements from the pole of the wave, the latter taken with reference to the point at which we are determining the illumination. The values of these integrals between 0, and upper limits of various values, have been evaluated by different methods by Fresnel, Knochenhauer, Cauchy, and Gilbert, and the results given in tables. As we gradually increase the upper limit, the values of the integrals pass through maxima and minima, approaching  $\frac{1}{2}$  as a limit, as we see from substitution in the formula

$$\int_0^{\infty} \sin mx^2 dx = \int_0^{\infty} \cos mx^2 dx = \sqrt{\frac{\pi}{8m}},$$

which gives 
$$\int_0^{\infty} \cos \frac{\pi}{2} v^2 dv = \int_0^{\infty} \sin \frac{\pi}{2} v^2 dv = \sqrt{\frac{\pi}{4m}} = \frac{1}{2}.$$

Fresnel's method of integration was as follows:

Since the absolute value of the integral remains the same when the upper limit changes sign, it was sufficient to integrate between 0 and  $+v$ . Assuming the value of the integral to be known between the limits 0 and  $i$  we deduce the expression for the value between  $i$  and  $i+t$ , where  $t$  is a small fraction of the unit, for example 0.1. Writing

$$v = i + \frac{t}{2} + u,$$

where  $u$  is a variable which increases from  $-\frac{t}{2}$  to  $+\frac{t}{2}$ , we have

$$\int_i^{i+t} \cos \frac{\pi}{2} v^2 dv = \int_{-\frac{t}{2}}^{+\frac{t}{2}} \cos \frac{\pi}{2} \left( i + \frac{t}{2} + u \right)^2 dv.$$

Fresnel found for this the value

$$\begin{aligned} \int_{-\frac{t}{2}}^{+\frac{t}{2}} \cos \frac{\pi}{2} \left[ i^2 + it + \frac{t^2}{4} + 2 \left( i + \frac{t}{2} \right) u \right] & \text{(in which } u^2 \text{ has been neglected as small)} \\ &= \frac{1}{\pi \left( i + \frac{t}{2} \right)} \left[ \sin \frac{\pi}{2} \left( i + \frac{t}{2} \right) \left( 1 + \frac{3t}{2} \right) - \sin \frac{\pi}{2} \left( i + \frac{t}{2} \right) \left( i - \frac{t}{2} \right) \right]. \end{aligned}$$

An expression was developed in the same way for the other integral, and with these formulae Fresnel calculated his table for  $t=0.1$ , and  $i$  (in succession) = 0, .1, .2, .3, etc., getting values for  $\int_0^1$ ,  $\int_1^2$ ,  $\int_2^3$ , etc., which by addition give  $\int_0^1$ ,  $\int_0^2$ ,  $\int_0^3$ .

The values of the integrals are usually given in the form of a table, thus:

TABLE OF FRESNEL'S INTEGRALS (Gilbert).

$v$	$\int_0^v \cos \frac{1}{2} \pi v^2 dv$	$\int_0^v \sin \frac{1}{2} \pi v^2 dv$	$v$	$\int_0^v \cos \frac{1}{2} \pi v^2 dv$	$\int_0^v \sin \frac{1}{2} \pi v^2 dv$
0.0	0.0000	0.0000	2.6	0.3389	0.5500
0.1	0.0099	0.0005	2.7	0.3926	0.4529
0.2	0.1999	0.0042	2.8	0.4675	0.3915
0.3	0.2994	0.0141	2.9	0.5624	0.4102
0.4	0.3975	0.0334	3.0	0.6057	0.4963
0.5	0.4923	0.0647	3.1	0.5616	0.5818
0.6	0.5811	0.1105	3.2	0.4663	0.5933
0.7	0.6597	0.1721	3.3	0.4057	0.5193
0.8	0.7230	0.2493	3.4	0.4385	0.4297
0.9	0.7648	0.3398	3.5	0.5326	0.4153
1.0	0.7799	0.4383	3.6	0.5880	0.4923
1.1	0.7638	0.5365	3.7	0.5419	0.5750
1.2	0.7154	0.6234	3.8	0.4481	0.5656
1.3	0.6386	0.6863	3.9	0.4223	0.4752
1.4	0.5431	0.7135	4.0	0.4984	0.4205
1.5	0.4453	0.6975	4.1	0.5737	0.4758
1.6	0.3655	0.6383	4.2	0.5417	0.5632
1.7	0.3238	0.5492	4.3	0.4494	0.5540
1.8	0.3363	0.4509	4.4	0.4383	0.4623
1.9	0.3945	0.3734	4.5	0.5258	0.4342
2.0	0.4883	0.3434	4.6	0.5672	0.5162
2.1	0.5814	0.3743	4.7	0.4914	0.5669
2.2	0.6362	0.4556	4.8	0.4338	0.4968
2.3	0.6268	0.5525	4.9	0.5002	0.4351
2.4	0.5550	0.6197	5.0	0.5636	0.4992
2.5	0.4574	0.6192	$\infty$	0.5000	0.5000

Knochenhauer developed the integrals in series by partial integration, thus:

$$\begin{aligned}
 \int_0^v \cos \frac{\pi}{2} v^2 dv &= v \cos \frac{\pi}{2} v^2 + \pi \int_0^v v^2 \sin \frac{\pi}{2} v^2 dv, \\
 \int_0^v v^2 \sin \frac{\pi}{2} v^2 dv &= \frac{v^3}{3} \sin \frac{\pi}{2} v^2 - \frac{\pi}{3} \int_0^v v^4 \cos \frac{\pi}{2} v^2 dv, \\
 \int_0^v v^4 \cos \frac{\pi}{2} v^2 dv &= \frac{v^5}{5} \cos \frac{\pi}{2} v^2 + \frac{\pi}{5} \int_0^v v^6 \sin \frac{\pi}{2} v^2 dv, \\
 &\dots\dots\dots
 \end{aligned}$$

his final expression giving the integral in the form of the sum of two convergent series:

$$\begin{aligned}
 \int_0^v \cos \frac{\pi}{2} v^2 dv &= \cos \frac{\pi}{2} v^2 \left( v - \frac{\pi^2 v^5}{1.3.5} + \frac{\pi^4 v^9}{1.3.5.7.9} \dots \right) \\
 &+ \sin \frac{\pi}{2} v^2 \left( \frac{\pi v^3}{1.3} - \frac{\pi^3 v^7}{1.3.5.7} + \dots \right).
 \end{aligned}$$

The convergence becomes less as  $v$  increases, consequently the expression can only be used for small values of the upper limit.

Cauchy in a somewhat similar way developed the integral in series which were convergent for large values of  $v$ .

Turning back now to the expression for the illumination we see that it consists of the sum of the squares of two integrals. The two integrals, therefore, represent the components along two rectangular axes of the resultant amplitude. The illumination is thus represented by the square of a line joining the origin with a point, the coordinates of which are the two integrals. Taking  $\xi$  and  $\eta$  as the coordinates of the point for different values of  $v$  we will investigate the curve along which the point moves as  $v$  varies. This geometrical discussion of the equation is due to Cornu, and the curve is known as Cornu's Spiral. By its aid the classical problems of diffraction can be solved in a geometrical manner, the intensity curve of the diffraction pattern being plotted from measurements made on the spiral.

**Cornu's Spiral.**—Let

$$\xi = \int_0^v \cos \frac{\pi v^2}{2} dv, \quad \eta = \int_0^v \sin \frac{\pi v^2}{2} dv.$$

The curve passes through the origin, since for  $v=0$ ,  $\xi$  and  $\eta$  also equal zero. Changing the sign of  $v$  does not change the values of  $\xi$  and  $\eta$ , but only their sign; the curve is therefore symmetrical about the origin.

The tangent to the curve makes an angle  $\tau$  with the  $\xi$  axis given by

$$\tan \tau = \frac{d\eta}{d\xi} = \tan \frac{\pi v^2}{2} \text{ or } \tau = \frac{\pi v^2}{2}.$$

At the origin, where  $v=0$  the curve is parallel to the  $\xi$  axis. For  $v=1$  or  $s=1$  it is parallel to the  $\eta$  axis, for  $s^2=2$  again parallel to the  $\xi$  axis, and for  $s^2=3$  parallel to the  $\eta$  axis.

The radius of curvature is given by

$$\rho = \frac{ds}{d\tau} = \frac{1}{\pi v} = \frac{1}{\pi s}.$$

For  $v=0$  the radius is infinite, and the curve has a point of inflection, as  $v$  increases the radius decreases, the curve having the form of a double spiral, which winds about the asymptotic points  $P$  and  $P'$ , which corresponds to the values of the integrals when the upper limits are

$$+\infty \text{ and } -\infty.$$

We can construct the curve by employing the equations for the radius of curvature, and the angle which the tangent makes with the  $\xi$  axis.

A small element of the curve at distance  $\xi$  from the origin corresponding to  $s=0.1$  has a radius of curvature  $\rho = \frac{1}{\pi s} = \frac{10}{\pi}$ , the center of curvature lying in a direction from the point such that it makes the same angle with the  $\eta$  axis as the tangent makes with the  $\xi$  axis, viz.,

$$\tau = \frac{\pi s^2}{2} = 0.01 \frac{\pi}{2}.$$

On this circle we lay off the arc  $s = \cdot 1$ . We proceed in the same way for points corresponding to  $s = 0\cdot 2$ ,  $s = 0\cdot 3$ , etc., and thus build up the entire curve. This somewhat laborious geometrical method is not necessary, for in Fresnel's table of integrals we have successive values of  $\xi$  and  $\eta$  and can plot the curve at once by taking them as ordinates and abscissae. In this way the curve shown in Plate IV. at the end of the book was constructed. This curve we have already used in the solution of certain diffraction problems. We will now examine the method employed by Fresnel.

**Diffraction by a Straight Edge.**—In the elementary treatment we have seen that the illumination within the geometrical shadow falls off gradually without showing maxima and minima, while outside of the edge of the shadow we have maxima and minima, which decrease in distance, and become more nearly of the same intensity as we recede from the edge, until finally we have uniform illumination. We will now apply Fresnel's expression to a point  $P$  within the geometrical edge of the shadow. The integration must be taken from the edge of the screen to infinity. The pole of the wave with reference to  $P$  is cut off by the screen, and if  $S$  represents the arc from the pole to the edge, the expression for the intensity is obviously

$$I = \left[ \int_s^\infty \cos \frac{\pi(a+b)s^2}{ab\lambda} ds \right]^2 + \left[ \int_s^\infty \sin \frac{\pi(a+b)s^2}{ab\lambda} ds \right]^2,$$

or introducing the quantity  $v$  as before,

$$I = \frac{ab\lambda}{2(a+b)} \left[ \left( \int_v^\infty \cos \frac{\pi}{2} v^2 dv \right)^2 + \left( \int_v^\infty \sin \frac{\pi}{2} v^2 dv \right)^2 \right],$$

in which

$$V = \sqrt{\frac{2(a+b)}{ab\lambda}} S.$$

Let  $x$  equal the distance of the point  $P$  from the edge of the geometrical shadow (Fig. 165),

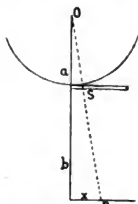


FIG. 165.

$$x = \frac{a+b}{a} S = \sqrt{\frac{(a+b)b\lambda}{2a}} V.$$

$$\text{Write } \int_0^x \cos \frac{\pi}{2} v^2 dv = C_r, \quad \int_0^x \sin \frac{\pi}{2} v^2 dv = S_r.$$

We have shown that the integrals taken between 0 and  $\infty$  are both equal to  $\frac{1}{2}$ , therefore

$$I = \frac{ab\lambda}{2(a+b)} \left[ \left( \frac{1}{2} - C_r \right)^2 + \left( \frac{1}{2} - S_r \right)^2 \right].$$

The value of the quantity within the brackets was determined by Fresnel for different values of  $V$  (which corresponded to certain values of  $S$  and  $x$ ) by the method which we have already seen. It was found that the illumination fell off rapidly without passing through maxima and minima, becoming 0 as soon as  $V$  reached any considerable value. In the same way the illumination outside of the edge of the shadow was found to depend on the quantity

$$\left( \frac{1}{2} + C_r \right)^2 + \left( \frac{1}{2} + S_r \right)^2.$$

The values calculated from the Fresnel tables for this sum showed that it passed through maxima and minima, the increment of  $x$  necessary for the change becoming less as the distance from the edge of the shadow increased.

The geometrical solution of this and kindred problems by the aid of the Cornu spiral which we have already considered is much more convenient and direct however.

**Diffraction by Thin Laminae.**—Laminary diffraction phenomena of the Fraunhofer class have already been briefly discussed in connection with the theory of the diffraction grating. We will now examine the case of diffraction by a straight edge, when the screen, instead of being opaque, consists of a thin transparent lamina of thickness  $\epsilon$  and refractive index  $n$ . The path-difference between two rays, one passing *by* the edge and the other *through* the edge of the lamina, is  $(n-1)\epsilon$ . The phase-difference  $\Delta$  will be given by

$$\frac{\Delta}{2\pi} = \frac{(n-1)\epsilon}{\lambda}, \quad \Delta = 2\pi \frac{(n-1)\epsilon}{\lambda}.$$

If this is an odd multiple of  $\pi$ , the edge of the geometrical shadow of the lamina will be in total darkness, for the disturbances from homologous points of the lamina and the clear space reach points situated on the edge of the shadow with a path-difference of half a wave-length, and mutually destroy one another. There will in addition be interference fringes both within and without the edge of the shadow, which we shall investigate presently.

If  $\Delta$  equals an even multiple of  $\pi$  not only will there be no minimum at the edge of the shadow, but the fringes will disappear, the illumination being the same as if the lamina were absent. Owing to the dispersion in the lamina the former condition may hold for one color, the latter for another, and we may have the fringes very distinct with blue light, and scarcely visible with red.

Assuming that we are working with monochromatic light and a lamina giving a half-wave retardation, we can easily construct the intensity-curve with the help of the Cornu spiral.

The whole wave is utilized in this case, the portion of the spiral representing the part passing through the lamina being rotated through  $180^\circ$ . For  $x=0$  the upper half of the spiral is rotated around the origin through angle  $\pi$  and comes into coincidence with the lower half. The vector sum of the lines from the asymptotic points to the origin is zero, since they are equal and oppositely directed, and the illumination is therefore 0 at the center of the system.

As we increase  $x$  we have our moving point tracing out maxima and minima as in the case of the opaque straight edge, but we shall find that the effect of adding the vector of the portion of the wave which was before left out, and is now utilized after a rotation of  $180^\circ$ , is to increase the intensity at the maxima and decrease it at the minima. For example, with an opaque screen the

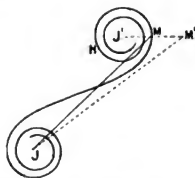


FIG. 166.

amplitude would be  $JM$  for the value of  $x$  giving the first maximum. With the lamina screen the portion of the spiral  $MJ$  is used. The natural direction of the vector is towards  $J'$ , but the half-wave retardation directs it away from  $J'$ , the resultant being  $JM'$ , which is larger than  $JM$ . If our point has moved around to  $N$ , the first minimum, the direction of the smaller vector after rotation is  $J'N$ , which, being opposed to  $JN$ , means a subtraction, and a smaller value of the amplitude.

It is obvious that the fringes are similar on both sides of the central minimum, for the above construction applies in whichever direction we take  $x$ .

These fringes are easily observed: split off the thinnest possible film of mica, and cut the edges straight with a pair of sharp scissors.



FIG. 167.

Mount it on the end of a straight strip of metal or a thin card, with its edge accurately in line with that of the card. Concentrate sun or arc light on a large pin-hole in a piece of tin foil, allowing the divergent beam to enter a long dark-room. The screen should be mounted about 4 meters from the source, and the fringes viewed on a sheet of white paper 3 or 4 meters behind the screen. A photograph of the fringes is shown in Fig. 167.

**The Colors of Mixed Plates.**—Interference colors of this type were discovered by Young, and described in the *Philosophical Transactions* for 1802, and were subsequently studied by Sir David Brewster. Verdet and other writers on Optics have classified them with Newton's thin-film colors, and have given treatments which are not very rigorous, and fail to show where the energy goes to.

The colors are very easily obtained by pressing a little white of egg between two pieces of plate-glass, separating the plates and squeezing them together a number of times so as to form a froth. The plates are to be pressed firmly together with a rotary sliding motion just before the froth becomes sticky, enclosing a film made up of air and albumen in the form of a mosaic. The colors are best seen by holding the plate towards a distant window or other bright source of light on a dark field. Certain wave-lengths will be found to be absent in the directly transmitted light. Young's explanation was that the path-difference between a ray passing through an air-space and one passing through the albumen was an odd number of half wave-lengths for such colors as failed to appear in the transmitted light. Neither Young nor subsequent writers, so far as I have been able to find, show what becomes of these absent colors, though both Young and Brewster observed the colored fringes which appeared in the dark background to one side of the source of light. Brewster published a paper in the *Philosophical Transactions* for 1837 in which he referred the colors to diffraction,

though his treatment was not very complete, and concerned chiefly the case of diffraction by a transparent lamina bounded by a straight edge. Verdet objected to this explanation on the ground that the colors are independent of the size and arrangement of the air-bubbles, depending only on the thickness of the albumen-film and the angle of incidence. The interference phenomena of mixed plates are easily explained by the elementary theory of diffraction, and they should be classed with laminary diffraction effects, and not with thin film interferences, as is usually the case.

In Fig. 168 let  $AA$  represent the glass plates with the albumen and air elements between them. We will assume the thickness of the albumen such that green light suffers a retardation of  $\frac{\lambda}{2}$  in traversing it. If

$B$  is the lens of the eye, and parallel rays traverse the plate, the secondary disturbances represented by the dotted lines (normally diffracted rays) will be brought to a focus at  $E$ ; that is, the reduced paths of all these rays are equal and the disturbances arrive at  $E$  in the same phase, if there be no retardation. The disturbances coming from the albumen elements are retarded, however, and reach  $E$  half a wave-length behind the disturbances coming from the air elements. The two sets destroy each other at this point, and green light will

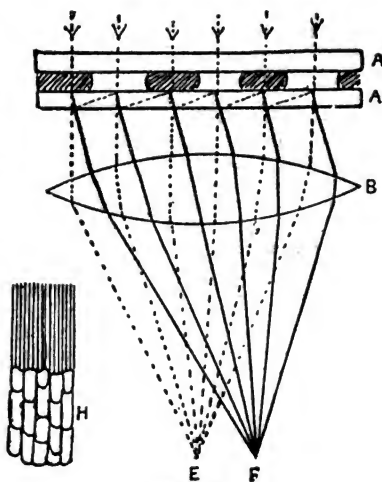


FIG. 168.

not be represented here. In general, light will be absent at this point if the retardation of a ray passing through albumen with respect to one passing through the adjacent air-space is  $(2n+1)\frac{\lambda}{2}$ . If the film is fairly thick, this condition may hold for a number of colors in the spectrum, which will consequently be absent in the image of any source of light seen through the plate. The question now is: What becomes of this energy? In the case of thin film interferences, the wave-lengths absent in the transmitted light appear in the reflected. This is not the case with mixed plates, which show little or no color by reflection. If we refer to Fig. 169, it is clear that if we take parallel rays diffracted in an oblique direction, the phase-difference introduced by the retardations in the mosaic may be compensated by the obliquity, the agreement of phase being more or less complete for green light in the point  $F$ . The

case is analogous to a laminary grating, which yields colored central images, the absent wave-lengths appearing in the spectra. Mixed plates throw the light absent in the direct image into a halo or ring, which is seen to surround the source of light.

Laminary diffraction phenomena, which we have just discussed, and mixed plates belong to the same class, the case being best defined as laminary diffraction by a great number of irregularly distributed transparent disks. If the patches of the mosaic were of uniform size, the halo would be fairly sharply defined and separated from the direct image by a dark space, which would become wider as the size of the elements of the mosaic decreased. Though it is easy to obtain very perfect halos in some cases, separated by a dark area of considerable size, the variation in the size of the elements usually causes the halo to take the form of a disk, the center of which is occupied by the direct image.

If the plates are held close to the eye and a distant lamp-flame viewed through them, the flame will, for example, appear purple and the surrounding halo green. If a small sodium flame is employed, parts of the mosaic will show it much blurred, and surrounded by a halo, while other parts, where the retardation is a whole number of half-waves, show it perfectly sharp and distinct. The distribution of the light in the halo depends on the form of the elements of the mosaic. By pressing the plates firmly together and sliding one over the other, the circular air-bubbles can be deformed into ellipses. The light in the ring will be more or less concentrated on opposite sides of the halo. If the ellipses were drawn out indefinitely, we should pass over to the grating, and the points of concentration would become first-order spectra, the rest of the halo disappearing.

A very curious and interesting example of this concentration of light in a halo was observed by the author when copying some diffraction gratings on bichromatized albumen. The original grating was ruled on glass, 14,400 lines to the inch, a spacing so fine that copies were only obtained with considerable difficulty.

Some of the films were found to have frilled in the process of washing, the buckling of the film following the grooves of the grating to a certain extent. The albumen surface was seen by the microscope to have frilled into oval patches of varying length, but of fairly constant width, the width being equal to three lines of the original grating. In Fig. 168, *H*, we have a diagram illustrating this condition. This plate, when held before the eye showed a ring of wide aperture surrounding a brilliant source of light, with four distinct concentrations, two very bright, and two quite faint. The appearance reminded one most forcibly of a solar halo, with parhelia or mock suns. A photograph of this curious diffraction pattern was made by directing a camera towards a brilliant point source of light, and placing one of the frilled plates before the lens. This photograph is reproduced in Fig. 169.

The arrangement of the colors in the "mock-suns" produced in this way is, however, exactly the opposite of the arrangement in the real ones, which makes it appear doubtful if there is any connexion between the two. It is possible, however, that the usual

treatment of parhelia could be improved by considering diffraction as well as refraction, as has been done in the case of the rainbow.

**Talbot's Bands.**—A curious type of interference bands was described by Talbot<sup>1</sup> in 1837. If, when viewing a continuous spectrum in a spectroscope, we place a thin piece of glass or mica in front of one half of the pupil of the eye, with its edge towards the red end of the spectrum, a set of extremely black bands appear, crossing the spectrum parallel to the Fraunhofer lines. If, however, the thin plate be turned with its edge pointing towards the violet, no trace of the bands is to be seen.

Talbot gave an imperfect explanation of the bands on the elementary principles of interference. The thin plate retards certain colors an odd number of half wave-lengths. These waves arrive at the retina in



[FIG. 169.]

a condition to interfere destructively with the waves which enter the uncovered portion of the pupil, the lens of the eye bringing them to the same point, consequently these colors are absent in the spectrum. The distinctness of the bands depends on the thickness of the plate, and Talbot's explanation neither accounts for this nor for the apparently remarkable circumstance that the bands do not appear at all when the plate is turned with its edge the other way, covering the other half of the pupil, a change which should produce no effect on the interference theory; neither is it quite clear what becomes of the light. The correct explanation was first given by Airy.<sup>2</sup>

He showed that it was necessary to take into account the diffraction maxima and minima as well as the interference minima. As it is only the minima parallel to the Fraunhofer lines that affect the result, we

<sup>1</sup> Gilbert's *Ann.*, 1837.

<sup>2</sup> Airy, *Pogg. Ann.*, liii., lviii.; *Philos. Trans.*, 1840, vol. ii.; 1842, vol. i.

can simplify the problem by discussing the distribution of the illumination along a line parallel to the spectrum. In Fig. 170 let  $S$  be a point of the spectrum,  $S'$  its image on the retina. We know that the image

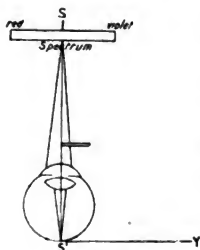


FIG. 170.

$S'$  consists in reality of a central maximum with maxima and minima on each side. Talbot's treatment does not take account of these diffraction effects, but assumes that  $S'$  is identical with  $S$ . If the diffraction minima and the interference minima are the same distance apart and in step, it is evident that they will be very distinct. If they are out of step they will be less distinct, the maxima of one set falling upon the minima of the other, producing more nearly uniform illumination.

We will first consider the interference effects. Suppose the lamina to have such a thickness that the wave-length at  $S$  is retarded

a half wave-length or an odd multiple of  $\frac{\lambda}{2}$ . Suppose also the rest of the spectrum to be absent, i.e. reduced to a monochromatic line at  $S$ .

If the lamina were absent the lens of the eye forms at  $S'$  an intense central maximum, with a faint lateral maximum on each side at such a distance that the path-difference between the extreme rays is  $2\lambda$ , the minimum between representing a path-difference of  $\lambda$  between the extreme rays.

Suppose  $AB$  (Fig. 171) represent the aperture, the lines  $aa$  represent the directions of rays forming the central maximum:  $bb$  the directions giving a path-difference of  $\lambda$  for the extreme rays, and  $\frac{\lambda}{2}$  between the

center and either edge as shown. This is the direction of the first minimum to the left. Now introduce the lamina  $CB$  with its half-wave retardation. The illumination in the direction  $aa$  will vanish, since the disturbances from the lamina destroy those from the uncovered portion. The conditions for a maximum are now fulfilled for the directions  $bb$ , since the path-difference between the ray  $Ab$  and the parallel ray from the edge  $C$  of the lamina is  $\lambda$ , the same being true for homologous rays from  $AC$  and  $CB$ , which we can consider as two apertures. On this side of the central maximum

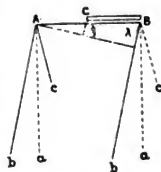


FIG. 171.

we add the retardation of the lamina to the path-difference due to the obliquity of the rays. On the other side of the central maximum we subtract the retardation of the lamina, and obtain for the homologous rays in direction  $Ac$  and  $Bc$  the path-difference zero. The result of the introduction of the lamina is the division of the central maximum into two maxima which occupy the positions of the minima which originally existed on each side of the central maximum. This can be confirmed by experiment. If the spectrometer is illuminated with monochromatic light and the glass plate inserted before the pupil of the eye it will be found, if things have been

so arranged that the spectrum line in question occupies the position of one of the minima, that the line appears double. A mercury or helium tube makes the best source of light. The soda flame is less satisfactory, owing to the fact that we have a double line to start with.

This experiment shows us that the light which disappears from the spectrum in the Talbot bands is merely shifted slightly to the right and left of the position which it normally occupies. We will next consider the part played by diffraction, considering the effect at various points of the disturbances from two adjacent apertures, one of which introduces a definite retardation. We can make use of the equation which we developed for two parallel slits, which gives for the illumination of a point at distance  $y$  from the center of the system,

$$I = 4a^2 \left( \frac{\sin \omega}{\omega} \right)^2 \cos^2 \pi \frac{(a+d) \sin \delta}{\lambda}, \text{ in which } \omega = \frac{\pi y a}{\lambda c},$$

$\delta$  equals the angle of diffraction for the point, and  $c$  the radius of curvature of the concave wave (Fig. 172). Now  $\sin \delta = \frac{y}{c}$ , and the additional retardation, which results from the distance  $d$  between the apertures in the case of the two parallel slits, is in this case the retardation of the lamina, *i.e.* this portion of the retardation is independent of  $\delta$ , consequently we may write the above expression in the form

$$I = \left( \frac{\sin \omega}{\omega} \right)^2 \cos^2 \left( \omega \pm \frac{R}{2} \right), \text{ in which } R = \frac{a(n-1)}{\lambda}.$$

This is the expression for the illumination produced by a point of the spectrum located on what we may call the axis of the system. Consider now some adjacent point of the spectrum, which would, in the absence of the lamina, be brought to a focus at a point situated at distance  $\eta$  from the axis. Our expression for the illumination at  $y$  due to light coming from that part of the spectrum for which the focus point is  $\eta$ , is obtained by substituting  $y - \eta$  for  $y$  in the above expression, which gives as the more general expression for  $\omega$ ,

$$\omega = \frac{\pi a}{\lambda c} (y - \eta).$$

Owing to the dispersion of the lamina  $R$  is a function of  $\eta$  as well as  $\omega$ , and over a small region of the spectrum this function can be regarded as linear. If the spectrum lies on the focal plane with its violet end towards the positive direction of  $y$  we can write

$$R = \Phi \eta, \text{ in which } \Phi \text{ is a constant.}$$

Our expression for the intensity now becomes

$$I = \left( \frac{\sin \omega}{\omega} \right)^2 \cos^2 \left( \omega \pm \frac{\Phi \eta}{2} \right), \text{ in which } \omega = \frac{\pi a y}{\lambda c} - \frac{\pi a \eta}{\lambda c},$$

$$\omega \pm \frac{\Phi \eta}{2} = \omega \mp \left( \frac{\Phi y}{2} + \frac{\Phi \eta}{2} \right) \pm \frac{\Phi y}{2},$$



FIG. 172.

and if we write  $\frac{\Phi \lambda c}{\pi a} = C$ , and substitute for  $\phi$  the value given by this expression ( $C$  can be regarded as a constant since  $\lambda$  only varies a small amount in the region considered),

$$\omega \pm \frac{\Phi \eta}{2} = \omega \left( \mp \frac{\pi a y}{2 \lambda c} C + \frac{\pi a \eta}{2 \lambda c} C \right) \pm \frac{\phi y}{2} = \omega \mp \omega \frac{C}{2} \pm \frac{\phi y}{2}.$$

We have now for the intensity

$$I = \left( \frac{\sin \omega}{\omega} \right)^2 \cos^2 \left[ \omega \left( 1 \mp \frac{C}{2} \right) \pm \frac{\phi}{2} y \right].$$

The resultant illumination at  $y$  due to the superposition of the diffraction images of all the adjacent points of the spectrum will be the integral of the above expression,

$$I = \int_{-\infty}^{+\infty} \left( \frac{\sin \omega}{\omega} \right)^2 \cos^2 \left[ \omega \left( 1 \mp \frac{C}{2} \right) \pm \frac{\phi}{2} y \right] \delta \omega,$$

$$I = \frac{\pi}{2} + \cos(\phi y) \times \frac{1}{2} \int_{-\infty}^{+\infty} \left( \frac{\sin \omega}{\omega} \right)^2 \cos[\omega(2 \mp C)] \delta \omega.$$

Writing  $R'$  for the retardation of the lamina for light of wave-length corresponding to the point  $y$  of the image of the spectrum, and writing

$$\int_{-\infty}^{+\infty} \left( \frac{\sin \omega}{\omega} \right)^2 \cos[\omega(2 \mp C)] \delta \omega = \text{const.},$$

we get for total intensity  $I = \frac{\pi}{2} + \frac{\text{const.}}{2} \cdot \cos R'$ .

This expression shows us that the *position* of the bands depends only upon  $R'$ .

If the thin plate is introduced from the red side we have the  $+$  sign in the expression for the constant,

$$\text{Const.} = \int_{-\infty}^{+\infty} \left( \frac{\sin \omega}{\omega} \right)^2 \cos(2 + C) \omega \delta \omega = 0 \quad \text{and} \quad I = \frac{\pi}{2}.$$

If the plate is introduced from the violet side we have

$$\begin{aligned} \text{Const.} &= \frac{C}{2} \pi, & \text{for } C < 2, \\ &= \left( 2 - \frac{C}{2} \right) \pi, & \text{for } 2 < C < 4, \\ &= 0, & \text{for } 4 < C. \end{aligned}$$

For values of  $C$  between 0 and 2,  $I = \frac{\pi}{2} + \frac{\pi}{2} \frac{C}{2} \cos R'$ ,

$$2 \text{ and } 4, \quad I = \frac{\pi}{2} + \frac{\pi}{2} \left( 2 - \frac{C}{2} \right) \cos R',$$

$$\text{for } C > 4, \quad I = \frac{\pi}{2}.$$

The bands are most distinct when  $C = 2$ ,  $\frac{\phi\lambda c}{\pi u} = 2$ .

The above treatment, while perfectly sound, involves so much mathematics, that the mind is apt to lose track of the physical significance of the treatment, especially as regards the circumstance that the bands only appear when the plate is inserted on the violet side of the spectrum. A much simpler, and more intelligible, explanation has been given by Schuster, which is based upon the modern idea that white light consists of irregular impulses, the *periodicity* or regularity of light in the spectrum resulting from the dispersing apparatus which produces it. This idea was first advanced by Lord Rayleigh in considering the action of the diffraction grating when analyzing white light; a matter which we shall investigate more in detail when we come to the chapter on White Light. Suppose we have a source of white light at  $S$  (Fig. 173) which is emitting non-periodic impulses.

Let one of these impulses fall upon the grating. If our source is at a great distance, or if we use a collimating lens, secondary waves will leave the various elements of the grating at the same moment. Obviously we can choose three points, designated "Blue," "Green," and "Red" respectively, such that the separate impulses from the grating elements will pass through them (in succession) with the periodicity of blue, green, and red light. We thus see why the red, with its slow periodic impulses, is further removed from the normal to the grating than the blue, or, in other words, *why the grating constructs red light at a point further removed*.

Our light will not be monochromatic at the three points unless we add a lens to the system, for the inclination of different parts of the grating to the lines joining them with the points will be different; that is, different parts of the grating construct different colors at a given point. With the lens added, we have, however, monochromatic light at points lying in the focal plane of the lens.

If now we retard the impulses coming from one half of the grating half a wave-length, they will destroy the impulses coming from the other half, *provided* the two sets traverse the point simultaneously. Clearly we must introduce our plate on the *red* side if we are to accomplish this, for if we introduce it on the other side, we retard a set of impulses which is already behind the set with which we wish to make it interfere. This is equivalent to introducing it on the blue side, if we put the plate between the spectrum and the eye, as can be easily seen by constructing a diagram illustrating the formation of the image of the spectrum. The rays cross at the point, and the bundle which we must retard, which was originally on the red side, is now found on the opposite side. If the retarding plate is placed between the grating and the telescope lens of the spectroscope it must be introduced from the *red* side, instead of the blue, as is easily proven by experiment. The best thickness of the plate is

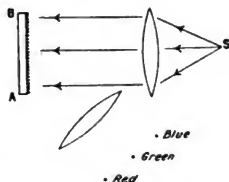


FIG. 173.

such as will divide the whole series of impulses into two equal portions, which arrive at the point in pairs; *i.e.* an impulse from the near edge of the grating and one from the central element should reach the point simultaneously. If  $N$  be the total number of lines of the grating, the best retardation is therefore  $\frac{1}{2}N\lambda$ , and the plate should intercept exactly one half of the beam. The value  $\lambda$  here means the wave-length of the light which the grating constructs at the point. If the retardation is greater or less, some of the impulses arrive either too soon or too late to interfere with others, and the bands are not as clearly defined. The impulses coming from the various elements of the grating need not even be considered as "to-and-fro" to account for interference in the manner supposed. Consider them all in one direction, *i.e.* half-waves, and let the retardation be such as to cause one set to fit exactly half way between those belonging to the second set. The resulting disturbance at the point would have a periodicity twice as great as it had before; there is therefore light at the point, but it is light which belongs to the overlapping spectrum of the second order. As regards the wave-length  $\lambda$  under consideration there is darkness.

It will be remembered that, in the previous treatment, we saw that the bands were most distinct when the interference maxima were the same distance apart as the diffraction maxima. The same thing can be shown by Schuster's treatment. Quoting from his paper:

"If at a certain point of the spectrum corresponding to wave-length  $\lambda$  there is a *maximum* of light, the relative retardation of the two interfering impulses must be equal to  $m\lambda$ ,  $m$  being an integer, the next adjoining band towards the violet will appear at wave-length  $\lambda'$  such that

$$m\lambda = (m+1)\lambda'.$$

"Hence for the distance between the bands

$$\frac{\lambda - \lambda'}{\lambda'} = \frac{1}{m},$$

with the best thickness of the interposed plate  $m = \frac{1}{2}N$ , and hence

$$\frac{\lambda - \lambda'}{\lambda'} = \frac{2}{N},$$

where  $\lambda'$  in the denominator may with sufficient accuracy be replaced by  $\lambda$ . If  $\lambda''$  be that wave-length nearest to  $\lambda$  at which there is a minimum of light, it follows that

$$\frac{\lambda - \lambda''}{\lambda} = \frac{1}{N}.$$

This equation shows us that, *under the conditions of* "best thickness" the difference between the wave-lengths of a maximum and its neighboring minimum, divided by  $\lambda$ , is equal to  $\frac{1}{N}$ . We shall find the same to be true for the diffraction maxima and minima.

"If a linear homogeneous source of light of wave-length  $\lambda$  be

examined by means of a grating, the central image extends to a wave-length  $\lambda$ , such that

$$\frac{\lambda - \lambda_1}{\lambda} = \frac{1}{N},$$

where  $N$  as before is the total number of lines on the grating."

In this expression  $\lambda - \lambda_1$  is the distance of the first minimum from the central maximum, of the diffraction pattern of a linear source of monochromatic light seen with a grating of  $N$  lines, which we have fully considered on page 170.

Schuster also shows that if we are using a plate which is not of the "best thickness," we may restore the original contrast between the maxima, by screening off the portions of the beam which are not interfering.

In the case when the spectrum is formed by a prism, the method is not so obvious, as there is apparently no periodic structure to build up the colored light. There is, however, no especial difficulty in this case, as we shall see when we take up the subject of White Light.

## CHAPTER VIII.

### INTERFERENCE SPECTROSCOPES AND THE RESOLUTION OF SPECTRAL LINES.

IN the chapter on Diffraction we have discussed the action of the diffraction grating, and we will now take up the subject of the more recently devised spectroscopes, which should have been treated in the chapter on Interference, but which have been postponed for the reason that certain points cannot be well understood without previously considering the theory of the grating. We will begin with the Michelson interferometer, which is, perhaps, the best known type.

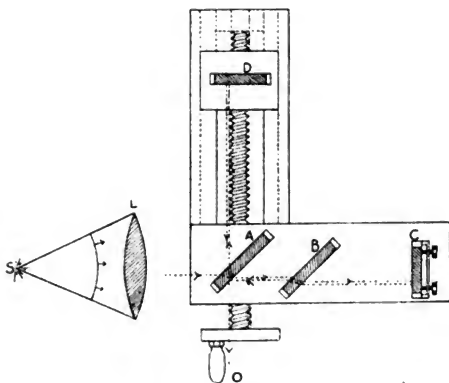


FIG. 174.

**Michelson's Interferometer.**—The essential parts of this instrument are four plates of glass arranged as shown in Fig. 174. Plates *A* and *B* are cut from the same piece of glass accurately plane parallel. Both may be transparent or *A* may be half silvered on the surface opposed to *B*. Plates *C* and *D* are heavily silvered on their front surfaces. Plate *D* is mounted on a carriage arranged so that it can be moved

along parallel ways by means of a screw. The action of the apparatus is as follows: Light from a source  $S$  made parallel by a lens falls upon the plate  $A$ , the beam being divided into two portions by the half-silvered surface. One portion is reflected to the mirror  $D$ , the other transmitted through  $B$  to the mirror at  $C$ , which is fixed in position. The mirror  $D$  returns the light to  $A$ , a portion of it escaping through the half-silvered film and entering the observer's eye, which is located at  $O$ . The light reflected back from  $C$  is in part reflected from the silver film and enters the eye over the same path. If the path-difference is an odd number of half-wave lengths, these two streams will interfere destructively and we shall have darkness. The path-difference between the two rays can be altered by moving the mirror  $D$  by means of the screw. Consequently the point in question upon the half-silvered surface will appear alternately bright and dark as the carriage is moved along the ways. The plate  $B$  is not essential, and its object will be explained presently. We can get a better idea, perhaps, of the action of the instrument in the following way: The mirror  $C$  is seen by reflection in the half-silvered film in coincidence with the mirror  $D$ , if the optical paths are the same. The instrument is thus the equivalent of two parallel reflecting surfaces, the distance between which can be varied. The phenomena presented by the interferometer are thus similar to those shown by thin films, the difference lying in the fact that in the present case we may make the distance between the reflecting surfaces as great or as small as we please.

If our source of light is a point or narrow line, *e.g.* a vacuum tube, a lens must be employed as shown in the diagram. If, however, we use a broad source, such as a sodium flame, the lens may be dispensed with.

The plate  $B$  is called the compensator, and is introduced to make the two optical paths symmetrical. In its absence it is obvious from the diagram that one of the interfering beams which enters the eye has traversed the plate  $A$  three times, while the other has passed through it but once; the double transit of the latter ray through the compensator makes the two paths optically equivalent. The compensator has also another use, for by turning it slightly we can increase or diminish the optical path, thus compensating for and measuring a change produced in the other path, as, for example, by the introduction of a thin film, the refractive index of which we wish to determine.

**Use of the Interferometer.**—The following very explicit directions for using the interferometer are taken from Mann's *Manual of Advanced Optics*, in which various experiments with the instrument are described in detail.

**Adjustment.**—"Measure roughly the distance from the silver half-film upon the rear of the plate  $A$  to the front of the mirror  $C$ . Set the mirror  $D$ , by turning the worm wheel, so that its distance from the rear of  $A$  is the same as that of  $C$  from  $A$ . This need not be done accurately. It is suggested because it is easier to find the fringes when the distance between the mirror  $D$  and the virtual image of the mirror  $C$  is small. This distance will hereafter be called the distance between the mirrors.

"Now place a sodium burner, or some other source of monochromatic light at  $S$ , in the principal focus of a lens of short focus. It is not

necessary that the incident beam be strictly parallel. Hold some small object, such as a pin or the point of a pencil, between  $L$  and  $A$ ."

A pin hole in a card is preferred by the author, as the vertical and horizontal adjustments can be made with greater precision.

"On looking into the instrument from  $O$ , three images of the small object will be seen. One image is formed by reflection at the front surfaces of  $A$  and  $D$ ; the second is formed by the reflection at the rear surface of  $A$  and the front surface of  $D$ ; the third is formed by reflection from the front surface of  $C$  and the rear surface of  $A$ . Interference fringes in the monochromatic light are found by bringing this third image into coincidence with either of the other two by means of the adjusting screws upon which the mirror  $C$  rests. If, however, it is desired to find the images in white light, the second and third of these images should be brought into coincidence, because then the two paths of the light in the instrument are symmetrical, *i.e.* each is made up of a given distance in air and a given thickness of glass. When the paths are symmetrical, the fringes are always approximately arcs of circles as described above. If, however, the first and third images are made to coincide, then the two optical paths are unsymmetrical, *i.e.* the path from  $A$  to  $C$  has more glass in it than from  $A$  to  $D$ , and in this case the fringes may be ellipses or equilateral hyperbolae, because of the astigmatism which is introduced by the two plates  $A$  and  $B$ . It is quite probable that the fringes will not appear when the two images of the small objects seem to have been brought into coincidence. This is simply due to the fact that the eye cannot judge with sufficient accuracy for this purpose when the two are really superposed. To find the fringes then it is only necessary to move the adjusting screws slightly back and forth. As the instrument has here been described, the second image lies to the right of the first.

"Having found the fringes the student should practise adjustment until he can produce at will the various forms of fringes. Thus the circles appear when the distance between the mirrors is not zero, and when the mirror  $D$  is strictly parallel to the virtual image of  $C$ . The accuracy of this adjustment may be tested by moving the eye sideways and up and down while looking at the circles. If the adjustment is correct, any given circle will not change its diameter, as the eye is thus moved. To be sure, the circles appear to move across the plates because their center is at the foot of the perpendicular dropped from the eye to the mirror  $D$ , but their apparent diameters are independent of the lateral motion of the eye. For this reason it is advisable to use the circular fringes whenever possible.

"To find the fringes in white light, adjust so that the monochromatic fringes are arcs of circles. Move the carriage rapidly by intervals of a quarter turn or so of the worm wheel. When the region of the white-light fringes has been passed, the curvature of the fringes will have changed sign, *i.e.*, if the fringes were convex toward the right, they will now be convex toward the left. Having thus located within rather narrow limits the position of the mirror  $D$ , which corresponds to zero difference of path, it is only necessary to replace the sodium light by a source of white light, and move the mirror  $D$  by means of the worm slowly through this region until the fringes appear.

“These white-light fringes are strongly colored with the colors of Newton’s rings. The central fringe—the one which indicates exactly the position of zero difference of path—is, as in the case of Newton’s rings, black. This black fringe will be entirely free from color, *i.e.* perfectly achromatic, if the plates *A* and *B* are of the same piece of glass, are equally thick, and are strictly parallel. If they are matched plates, *i.e.* if they are made of the same piece of glass and have the same thickness, their parallelism should be adjusted, until the central fringe of the system is perfectly achromatic. When this is correctly done, the colors of the bands on either side of the central one will be symmetrically arranged with respect to the central black fringe.”

If the instrument is illuminated with sodium light it will be found that the fringes become invisible periodically as the mirror is moved, for reasons which have been given in the chapter on Interference. It will be found instructive to illuminate the instrument with a lithium flame containing a little sodium, and note the shortness of the periods of indistinctness. In using the instrument to measure the refractive index or dispersion of a gas, the tube containing the gas can be closed with plates of thin plate-glass, which, if of good quality, does not much affect the appearance of the fringes. The tube is highly exhausted and the gas then slowly admitted, the shift in the fringe system being determined by counting the number of bands which cross the hair in the telescope, which can be used to view them.

The interesting investigation by Johonnott (*Phil. Mag.*, 47, page 501, 1899), on the thickness of the “black spot” on soap films, is an example of the many interesting applications of the interferometer. If we know the thickness of a transparent plate we can measure its refractive index by inserting it in one of the optical paths of the instrument and measuring the fringe displacement. The white system must be used of course in conjunction with the sodium or other monochromatic system, as the central fringe is the only one that can be identified. The abnormal displacement of the central band referred to in the chapter on Interference must also be remembered.

It is evident now that if the refractive index of a film is known the thickness can be determined. Johonnott found that, by employing a battery of 54 soap films mounted on frames, it was possible to get a measurable shift of the fringes even when the films were so thin that they refused to reflect light, *i.e.* showed the Newton black.

The thickness was found to vary between  $\cdot 00006$  mm. and  $\cdot 0004$  mm.

**Light-Waves as Standards of Length.**—Probably the most important use to which the interferometer has been put was the determination of the length of the standard meter in wave-lengths of the monochromatic radiations from cadmium. The invariableness of the wave-length of the radiation sent out from the atoms of a metal, brought to a state of luminescence by electrical discharges in a high vacuum, suggests their adoption as a standard of length. This proposition was first made by Lamont in 1823, and subsequently by Dr. Gould about thirty years ago. At that time the interferometer in its present form was unknown, and the method proposed involved the use of the diffraction grating, the measurement of its width, and the determination of angles, all of which measurements would have entailed no

very inconsiderable errors. Michelson suggested the use of his interferometer, and through the efforts of Dr. Gould, who represented the United States in the International Committee of Weights and Measures, was asked to carry out the experiments at the International Bureau at Sèvres in collaboration with Benoit. A very complete description of the method will be found in Prof. Michelson's book, *Light Waves and their Uses* (Chicago University Press, 1903).

The general principle of the method can be briefly outlined as follows:

The problem is to measure the distance between the two marks on the standard meter bar in terms of the wave-length of light, or, in other words, find out how many light-waves there are in a beam a meter long.

A bronze bar 10 cms. in length, of the form shown in Fig. 175, was

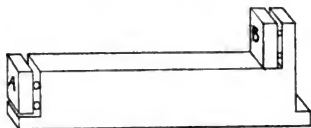


FIG. 175.

prepared, on the ends of which two silvered-glass mirrors were mounted which could be made accurately parallel by observing the interference fringes, formed in the manner to be described presently. The principle consisted in finding the number of light-waves in a beam whose

length was equal to the distance between the planes of the two mirrors, and then to find how many times this distance was contained in the meter. In a length of 10 cms. there are, however, roughly 300,000 light-waves, and the direct determination of this number by actual count would have involved too much labor and too great a risk of accidental mistakes. Nine other standards similar to the above were therefore prepared, each half as long as its predecessor, i.e. of lengths 10, 5, 2.5, 1.25, etc., cms.; the smallest unit had mirrors with reflecting planes only .39 mms. apart. The number of light-waves in this distance was first determined for the red, green, and blue radiations from a vacuum tube containing cadmium vapor. This was accomplished by putting the bar with its two mirrors in the place of one of the mirrors of the interferometer; the other mirror was then brought into such a position that the central fringe (white light) appeared in the field of, we will say, the lower mirror. By moving the mirror back the center of the system could be made to appear in the upper mirror, and by counting the number of fringes which passed during this operation the number of wave-lengths in the distance through which the mirror moved could be determined.

This first "*etalon*," as it was called, was next compared with the second by mounting the two side by side, in place of the movable mirror of the interferometer. The field of view now consisted of four square areas corresponding to the four mirrors of the *etalons*. The longer of the two (No. II.) was fixed in position, while the shorter (No. I.) could be moved by turning the screw of the instrument. The reference plane (image of the interferometer mirror seen in the plate) was then brought into coincidence with the front surface of the lower mirrors of the two *etalons*, the plane *R* (Fig. 176, *a*), by moving the

interferometer mirror until the colored fringes appeared. This mirror, which is usually fixed, in the present type of instrument could be moved along parallel ways. It was then moved back until the reference plane coincided with the upper mirror  $D$  of etalon I., the plane  $R'$ . The fringes passing during this motion of the mirror were counted, the number of course corresponding with the number previously determined. Etalon I. was now moved back until  $C$  came into coincidence with the reference plane  $R'$  (Fig. 176, *b*). The reference plane was now moved to  $R''$ , until it coincided with  $D$  in its new position, and was within a few wave-lengths of the plane of  $B$ , the number being found by turning the compensating plate. The second etalon was then compared with the third, and so on, until finally the number of wave-lengths in the 10 cm. etalon had been determined. A mark on this etalon was then brought into coincidence with one of the end marks on the meter bar under the microscope, and the etalon was then progressively advanced, its front mirror being brought into coincidence with the plane previously occupied by the rear mirror, the reference plane then moved back and the process repeated. In this way the

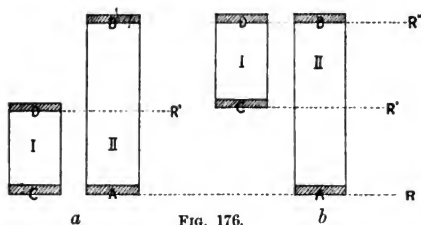


FIG. 176.

total number of waves in a length equal to the standard meter was determined. The final results were as follows, for  $15^{\circ}\text{C}$ . and 760 mm. pressure:

Red line 1 m. =  $1553163.5\lambda$ , i.e.  $\lambda = 6438.4722\text{AE}$ ,

Green line 1 m. =  $1900249.7\lambda$ , i.e.  $\lambda = 5085.8240\text{AE}$ ,

Blue line 1 m. =  $2083372.1\lambda$ , i.e.  $\lambda = 4799.9107\text{AE}$ .

The values given by Rowland for these same lines are

6438.680, 5086.001, and 4800.097.

An idea of the accuracy of the work can be obtained by comparing three independent observations, the first two by Michelson, the third by Benoit: 1553162.7, 1553164.3, 1553163.6.

In addition to recording the length of the standard meter in terms of an invariable unit, this remarkable piece of work has given absolute determinations of three standard lines, which will doubtless stand for a long time, if not for ever, as the standards from which all other lines will be measured.

It may be well to point out here that it has recently been shown by Michelson, and proven experimentally by Kayser, that Rowland's

coincidence method is not accurate. As a result of small errors of ruling, the second order ultra violet line of wave-length 2 may not fall exactly upon a first order line of wave-length 4. The use of the grating is thus restricted to obtaining the wave-lengths of lines between fixed standard lines, by interpolation, at least if the greatest accuracy is required. Revision of the standard wave-lengths is in progress at the present time by interferometer methods.

**The Visibility Curves.**—As we saw in the chapter on Interference, the fringe system formed with Newton's combination of a lens and flat plate, illuminated with sodium light, is not continuous. There are periodic regions of invisibility as we proceed outward from the center, due to the fact that when the maxima of  $D_2$  coincide with the minima of  $D_1$ , uniform illumination results. If now  $D_1$  and  $D_2$  were infinitely narrow lines and single, the fringes would be equally distinct when "in step," regardless of the path-difference. If, however, this is not the case, the visibility will vary at the different points of maximum distinctness. Suppose, for example, that each line is a close double; with a sufficiently large path-difference, the two components of  $D_1$  will get out of step, and we shall have uniform illumination and invisibility entirely independent of the light from  $D_2$ . Fizeau and Foucault, who may be regarded as the founders of interference spectroscopy, only recorded the successive recurrences of the fringes as the path-difference increased. Michelson went a step further, and measured the distinctness of the fringes at each reappearance. From these observations he was able to compute the nature of the lines, *i.e.* whether they were single or double, broad or narrow, etc. If  $J_1$  denotes the maximum brightness of a fringe, and  $J_2$  the intensity of the dark region between, Michelson calls

$$\frac{J_1 - J_2}{J_1 + J_2} = V,$$

the "Visibility," a quantity which represents the distinctness with which the fringes appear to the eye.

If we know the nature of the distribution of the light in the source, *i.e.* whether the lines are single or double, accompanied or not by fainter companions, etc., it is possible to construct a visibility curve in which the values of  $V$  are plotted as ordinates and the path-differences as abscissae.

Michelson commenced by calculating the visibility curves which would result from various types of single, double, and multiple lines. Examples of such curves are shown in Fig. 177, the intensity curves of the spectrum lines being shown to the left of each. The curves shown are resultant curves formed by the superposition of wave-trains such as would emanate from sources having a distribution of intensity as figured. The visibility curves are obviously the envelopes of the above curves. Michelson next took up the subject of the construction of an intensity curve from a visibility curve, a much more difficult problem. His work in this line was much aided by the invention of his harmonic analyzer, a machine which separates out of a complex curve the simple harmonic curves of which it is formed; in other words, makes a Fourier analysis of it.

As Lord Rayleigh has shown (*Phil. Mag*, 34, page 407, 1892), the rigorous solution of the problem is not possible, for, except in cases where there is symmetry in the group of lines, we may have a large number of different distributions of intensity, all of which give the same visibility curve. It is impossible, moreover, to decide from the visibility curve on which side of the principal line a fainter component lies. Michelson's predictions regarding the structure of many lines have been subsequently verified, however, and he is to be regarded as the pioneer in the field of investigations devoted to the minute study of spectrum lines.

The method has not been used to any great extent by other observers, partly from the great difficulty of estimating "visibilities" of the fringes, and partly from the difficulty in interpreting the results. Michelson's results were due to his great skill in this respect, which

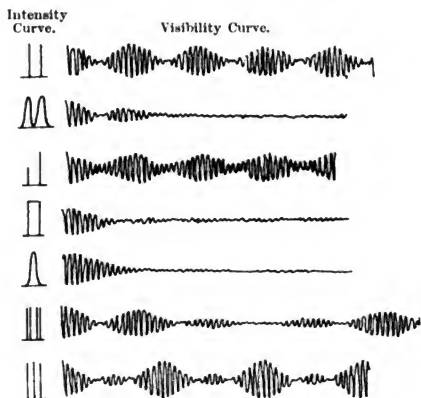


FIG. 177.

resulted from long experience and familiarity with his instrument. The more modern interferometers show objectively what before could only be guessed at, that is, they actually separate the line into its components just as the prism and grating separate the originally composite light into a spectrum of lines.

Michelson's genius gave us the next instrument in the series which we are considering, and we will now take up the subject of one of the most curious and interesting optical instruments ever devised, the echelon grating.

**The Echelon Grating.**—A remarkable kind of grating was constructed by Michelson.<sup>1</sup>

As we have seen in the chapter on Diffraction the resolving power of a grating is represented by  $mn$ , the product of the order of the spectrum

<sup>1</sup> "The Echelon Spectroscope," *Astrophys. J.*, 8, p. 36, 1898.

and the number

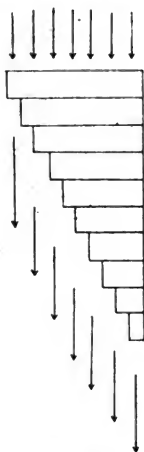


FIG. 178.

of lines. High resolving power had been secured previously by ruling a very large number of lines. Michelson attacked the problem in a new direction and constructed a grating for which  $m$  instead of  $d$  had a large value. The order of the spectrum is measured by the number of wave-lengths in the path-difference of disturbances coming from adjacent elements. If the path-difference can be made 1000 wave-lengths, we have a spectrum of 1000th order. Michelson accomplished this by building up a flight of steps of glass plates, all of exactly the same thickness and plane-parallel to within  $\frac{1}{20}$ th of a wave length of sodium light. The plates were cut from a single disc, which was figured with the greatest care by Mr. Petitdidier, and mounted as shown in Fig. 178. If light is sent through the series of plates in the direction indicated by the arrows, it is obvious that the streams emerging from the steps are retarded on each other by amounts depending on the thickness traversed, and the refractive index of the glass. Now the retardation by a plate 2 cms. in thickness is considerably over 20,000 wave-lengths; consequently we are dealing with a spectrum of the 20,000th order, if the plates have this thickness. The number of the plates

cannot be increased above 30 to advantage, owing to the loss of light by absorption and reflection from the surfaces. Our resolving power is thus about  $30 \times 20,000$  or roughly 600,000, or the grating should separate lines only  $\frac{1}{600,000}$ th of the distance between the  $D$  lines apart.

The echelon throws all its light into one, or at most two, spectra; consequently it is well adapted for the minute structure of faint spectrum lines. Its great disadvantage is the difficulty of interpreting the results obtained with it, and the impossibility of seeing more than a single line at a time. Even if sodium light is used nothing can be seen which can be interpreted. With certain thickness of plates the  $D_1$  and  $D_2$  spectrum lines may coincide, one being seen in, say, the 2000th order, and the other in the 2030th, owing to the difference of retardation. With plates of a different thickness the  $D_1$  spectra may fall midway between those due to  $D_2$ . As the spectra of succeeding orders are very close together, it is obvious that, except when employing extremely homogeneous radiation, we shall have a confused jumble of lines.



FIG. 179.

Only three different orders can be seen at one time, but by turning the echelon slightly others may be brought into view. We can set the echelon so as to have two adjacent orders of equal intensity, as in the first diagram of Fig. 179, or so as to have one bright line bordered by two faint ones. The latter condition is usually preferable.

The light must undergo previous prismatic analysis before it enters the collimator slit of the echelon spectroscope, or we may illuminate the slit with the heterogeneous light and place a prism between the echelon and the telescope. The instrument is especially well adapted for the exhibition of the Zeeman effect, as it is compact, and extremely saving of light, and requires practically no adjustment if the plates are properly mounted in a metal case. The writer has had no difficulty in showing the Zeeman effect with an improvised echelon made by fastening four interferometer plates on the table of a small spectrocope. A five-element grating is secured in this way, since a stream of unretarded light can be passed by the edge of the first plate. The width of the steps should not exceed one or two mms. and a cardboard screen should be so arranged as to cut off all the light except that which comes through the steps, a clear space of equal width to one side of the first plate, and a strip of the same width at the edge of the last plate, *i.e.* the top step. In other words, when looking at the echelon from the direction of the telescope the screen should hide everything except five vertical elements of equal width, four of them glass and one air. A direct vision prism can be put between the plates and the telescope to separate the echelon spectra of the different lines in the spectrum under investigation. A mercury vacuum tube between the conical poles of a powerful electromagnet is a suitable source of light to work with, the green line splitting up as soon as the current is turned on.

**Theory of Echelon.**—The theory as worked out by Michelson is as follows:

Let  $s$  be the width and  $t$  the height of each step, and  $\theta$  the angle of diffraction (Fig. 180). If the order of the spectrum is designated by  $m$ , and the ref. index of the glass by  $\mu$ , we have for the path-difference between the diffracted rays indicated by the arrows,  $m\lambda = \mu t - ac$ .

Now  $ac = t \cos \theta - s \sin \theta$ ;

$$\therefore m\lambda = \mu t - t \cos \theta + s \sin \theta.$$

$$(1) \quad m\lambda = (\mu - 1)t + s\theta,$$

since  $\theta$  is very small,  $\sin \theta = \theta$ ,  $\cos \theta = 1$ .

To find the dispersion which is represented by  $\frac{d\theta}{d\lambda}$ , we differentiate  $\theta$  with respect to  $\lambda$  in the above equation, remembering that  $\mu$  is also a function of  $\lambda$ ,

$$\frac{d\theta}{d\lambda} = \frac{m}{s} - \frac{t}{s} \frac{d\mu}{d\lambda}.$$

Substituting for  $m$  its approximate value  $(\mu - 1)\frac{t}{\lambda}$ , we obtain

$$(2) \quad \lambda \frac{d\theta}{d\lambda} = \frac{t}{s} \left[ (\mu - 1) - \lambda \frac{d\mu}{d\lambda} \right] = b \frac{t}{s},$$

if we represent the bracketed term by  $b$ .

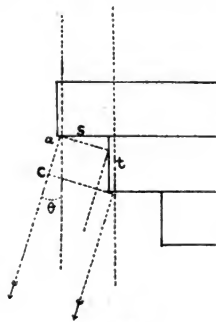


FIG. 180.

The resolving power, which we define by  $\frac{d\lambda}{\lambda}$ , is by equation (2)  $s \frac{d\theta}{b\lambda}$

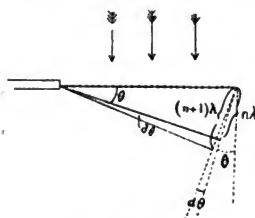


FIG. 181.

If  $d\theta$  is the increment in the angle of diffraction necessary for resolution (Fig. 181), i.e. the angle between the principal maximum and the first minimum (see page 177), we can write

$d\theta = \frac{\lambda}{ns}$ , as is made clear by the figure, in which we have

$$\frac{n\lambda}{ns} = \sin \theta, \quad \frac{(n+1)\lambda}{ns} = \sin(\theta + d\theta).$$

Owing to the small value of  $d\theta$  we get by subtraction of the first equation from the second

$$\sin d\theta = d\theta = \frac{\lambda}{ns},$$

in which  $ns$  is the total width of the grating.

This gives us for the resolving power,

$$(3) \quad \frac{d\lambda}{\lambda} = \frac{\lambda}{bnt}.$$

To find the distance between the spectra of succeeding orders, we differentiate equation (1) with respect to  $m$ ,

$$\frac{d\theta}{dm} = \frac{\lambda}{s},$$

or if we put  $dm = 1$  to obtain the change in  $\theta$ , in passing from order  $m$  to order  $m \pm 1$ ,

$$(4) \quad d\theta_1 = \frac{\lambda}{s},$$

in which  $d\theta_1$  is the angle of diffraction between adjacent spectra.

We will now determine the change in the wave-length which gives  $d\theta$  the same value as  $d\theta_1$ , that is, we will derive an expression which will enable us to compare the distance between the components of a double line, with the distance between the spectra. This we can easily do by substituting (4) in (2), which gives

$$(5) \quad \frac{d\lambda}{\lambda} = \frac{\lambda}{bt},$$

in which  $d\lambda$  is the increment of wave-length necessary to produce the increment  $d\theta_1$ , that is the components of a double line, with a wave-length difference  $d\lambda$  (as defined by (5)), will be separated by a distance exactly equal to the distance between the spectra of succeeding orders.

Comparison of (3) with (5) shows us that the limit of resolution is  $\frac{1}{n}$  of the distance between the spectra. "This," says Professor Michelson, "is a rather serious objection to this form of spectroscopy. Thus in

observing the effect of increasing density on the breadth of the sodium lines, if the broadening be of the order  $\frac{\lambda}{bt}$  the two contiguous spectra of the same line will overlap. As a particular case let us take  $t = 7$  mm., then  $\frac{\lambda}{bt} = \frac{1}{17000}$ , and it will be impossible to examine lines whose breadth is greater than  $\frac{1}{17}$  of the distance between the  $D$  lines. It is evidently advantageous *on this account* to make  $t$  as small as possible."

Obviously  $t$  has its smallest possible value (zero) in the case of the ordinary diffraction grating ruled on glass.

To get high resolving power, however, we must make either  $n$  or  $t$  large. The former quantity cannot be increased to advantage above 25 or 30, for reasons already specified, consequently  $t$  must be made large.

Michelson constructed three echelons, with plates 7, 18, and 30 mms. in thickness, having resolving powers equal to 210,000, 540,000, and 900,000 respectively, the smallest value surpassing that of the largest gratings ruled on speculum metal.

The distribution of intensity is deduced from the formula

$$A = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos px \, dx, \text{ where } p = \frac{2\pi}{\lambda} \theta.$$

Hence

$$I = A^2 = \frac{\sin^2 \pi \frac{s}{\lambda} \theta}{\left(\pi \frac{s}{\lambda} \theta\right)^2},$$

which vanishes for  $\theta = \pm \frac{\lambda}{s} = d\theta$ , the distance between the spectra.

Two spectra will thus in general be visible, of unequal intensity, as shown in Fig. 182, but by inclining the echelon a trifle one of them can be brought into the position  $\theta = 0$ , when it reaches its maximum intensity, and the two adjacent spectra, falling at  $\pi$ , practically disappear. Or we can so adjust the echelon as to have two spectra of equal intensity symmetrically placed. These two positions have been named positions of single and double order.

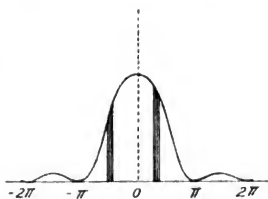


FIG. 182.

We can perhaps arrive at a better conception of the action of the echelon if we consider a type in which  $t$ , the thickness of the plates, is made very small. Such a grating can be constructed of mica films, .05 mm. or so in thickness. We can in this way obtain a type intermediate between the ordinary grating and the thick plate echelon of Michelson. With a grating of this description, the distance between two fairly close lines will not be greater than the distance between the spectra.

An echelon grating of 8 or 10 elements can be made of mica, which illustrates the principle, though of no use as a piece of optical apparatus. It is useful as coming midway between the ordinary grating and the echelon as usually constructed. By its aid lines can be separated which, with an ordinary grating of the same number of elements, would appear single. It shows spectra of the same general appearance as in the more powerful instruments, can be set for single and double order, and though useless as a tool for research, is almost as satisfactory for purposes of demonstration as the costly batteries of thick plates.

The very best quality of mica must be used, a sheet about .05 mm. thick being split off. This must be examined with an interferometer, and a portion picked out of uniform thickness, that is, an area must be found, across which the fringes run straight and unbroken. Possibly a simple examination of the film with a sodium flame would answer as well. The following description of a mica echelon constructed by the author and described in the *Philosophical Magazine* for June 1901, may be helpful to those who wish to undertake the task of constructing a similar piece of demonstration apparatus.

A number of thin sheets of mica were examined with the interferometer, and one selected over a considerable portion of which the fringes appeared straight and unbroken. This area was roughly outlined with a pin-scratch, and cut up into a dozen small rectangles with a print-trimmer. The retardation of one of these was measured with the interferometer, and found to be fifty wave-lengths for sodium light. The grating would therefore yield spectra of about the 50th order. A grating-space of 0.5 mm. was determined upon, and a strip of glass was accordingly ruled with this spacing on a dividing-engine. On this scale the echelon was built up, the plates being put in position under the microscope, and cemented at the edges by means of small bits of sealing-wax and a hot wire.

Considerable difficulty was found in attaching each plate without disturbing the spacing of the others. The first two or three gratings that were made were not very satisfactory; but some experience having been obtained by practice, an excellent one was finally obtained. Only nine plates were used owing to the opacity of the mica in thicker layers. The battery was mounted on a square of cardboard over a rectangular opening of the same size, a clear space 0.5 mm. wide being left to serve as the first grating-line of zero retardation. The whole number of lines was therefore ten.

The resolving power, represented by the product of the number of lines and the order of the spectrum, would accordingly be about 500. Obviously the sodium lines, requiring a product of at least 1000 for resolution, were beyond the power of the instrument; but the two yellow mercury lines, separated by 2.5 times the distance between the Na lines and requiring a product of only 280, seemed suitable.

The light from an "end on" mercury-tube, after passing through a collimating-lens and prism, was focused on the collimator of a spectrometer, the green (monochromatic) image of the tube being brought on the slit (Fig. 183). On placing the echelon on the table of the instrument the spectra showed clear and sharp, and by turning the grating a little could be brought into either single or double order. Faint

secondary maxima appeared between the principal maxima, owing to the small number of grating elements.

But slightly shifting the position of the lens, the yellow light from the tube was now focused on the slit, when the principal maxima immediately doubled in a most beautiful manner and the faint secondary maxima disappeared owing to overlapping. The distance between the components was about one third of the distance between the spectra. For the sake of comparison, a grating of the same spacing and number of lines was ruled on a piece of smoked glass. (To prevent the film from tearing it should first be moistened with alcohol and dried.)

The slit was illuminated with white light, and a cyanine film placed before it. This cut off all but the extreme red and blue; and it was

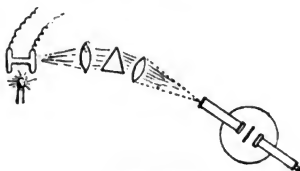


FIG. 183.

found that in the first order the grating was unable to separate the extreme red and blue of the spectrum, while the echelon easily resolved the *Hg* lines, showing the effect of the introduction of retardation.

The constants of the grating were as follows :

$t$  the thickness of plates, 0.05 mm.

Width of space, 0.5 mm.

Retardation of each plate, 50 waves.

Calculating the separation of the *Hg* lines by Michelson's formula, we have

$$\frac{\delta\theta}{\delta\theta_1} = \frac{t}{\lambda} \cdot \frac{\delta\lambda}{\lambda}, \text{ where } t = 0.05 \text{ mm,}$$

$$\frac{t}{\lambda} = 94, \quad \frac{\delta\lambda}{\lambda} = 280, \quad \delta\theta = \delta\theta_1 \frac{94}{280} = \frac{1}{3};$$

or the distance between the *Hg* lines is  $\frac{1}{3}$  of the distance between the spectra.

It will be found instructive to illuminate the slit-plate of the spectrometer with a focused continuous spectrum, and observe the way in which the different orders of echelon spectra file by, when the continuous spectrum is moved across the slit, showing the dependence of "order" on wave-length.

If with the slit illuminated with white light a continuous spectrum is formed in the telescope by means of a prism, this spectrum will be found to be crossed by heavy dark bands when the echelon is placed in front of the prism. The explanation of these bands makes a good problem for advanced students. A clue may be found by repeating

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the experiment with the slit-plate illuminated with the continuous spectrum instead of white light, and moving this spectrum very slowly.

**The Interferometer of Fabry and Perot.**—In the Michelson instrument, as we have seen, the fringes due to a double line disappear completely when out of step. It is obvious that if the ratio between the widths of the bright and dark bands could be decreased this disappearance would not occur, the bright fringes due to one line falling midway between those due to the other. An interferometer has been devised by Fabry and Perot in which the width of the bright band has been made many times smaller than the width of the dark band. To understand how this has been accomplished we must go back to the theory of the diffraction grating. It will be remembered that the intensity curve, in the system of fringes formed by light passing through two parallel slits, is similar to the curve which we have with the Michelson interferometer. Increasing the number of slits increases the steepness of the curve, i.e. the maxima become narrower and the minima wider. Regarding the slits as similar sources of light, we see that an increase in the number of the sources results in a narrowing of the bright fringes.

In the case of thin films or the Michelson interferometer, we are dealing essentially with light coming from two virtual sources, one immediately behind the other. If we can by any device increase the number of these sources, we shall in consequence decrease the width of the bright maxima without altering their distance apart. This can be done by availing ourselves of multiple reflections. The Fabry and Perot instrument is based upon this principle. It consists of two plates of plane-parallel glass, one stationary, the other movable, in a direction perpendicular to its surface by a device similar to that employed by Michelson. The general arrangement of the apparatus is shown in Fig. 184, the source of the light being located at *S* and the eye at *O*. The opposed surfaces of the glass plates are half silvered, and the distance between them can be varied from zero to 30 or 40 cm. Consider now a ray of light leaving the source and passing through the first silver film. A portion of it will be transmitted through the second film and a portion reflected back upon the first, from which it will be returned, the same sequence of events occurring again. Owing to these multiple reflections we shall have a number of beams of light of decreasing intensity emerging parallel to one another from the second silvered surface, the beams coming from a number of virtual sources situated in a line behind the actual source. The distance between the sources will be double the distance between the silvered films, which will consequently be the path-difference between two streams of light coming from adjacent sources.

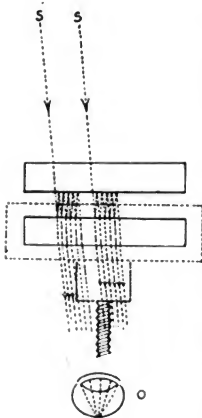


FIG. 184.

The case may be regarded as analogous in some respects to a number of slits situated in a line behind one another. Starting with the plates close together and accurately parallel, we shall observe in the case of sodium light a system of circular fringes similar to those seen with Newton's glasses, except that the maxima are extremely narrow circles of light with broad dark regions between them. On increasing the distance between the mirrors the rings due to the two components of the sodium light will gradually get out of step; but instead of disappearing, as they do in the Michelson instrument, they merely become double in the present case. In other words, we can observe the separation of the  $D_1$  fringes from the  $D_2$  ones, the number of circles of light doubling. On further increasing the path the fringes will again coincide.

It is clear that in the present case we can observe *directly*, that which we were obliged to infer from the visibility curves in the case of the Michelson interferometer. Faint components lying close to a bright line can be observed directly with this instrument by making the distance between the plates sufficiently great. Fabry and Perot have studied the minute structure of a large number of spectrum lines, and have obtained results which the Michelson instrument is incapable of yielding.



FIG. 185.

Two photographs by Dr. Barnes of the fringe system of the green line of mercury as seen in the Fabry and Perot instrument are shown in Fig. 185. The fainter companion lines are distinctly shown in the right-hand picture, which is a narrow strip of the ring system. The narrowness of the bright rings is to be noticed. Compare these photographs with the rings seen with Newton's arrangement of a plate and lens.

**The Interferometer of Lummer and Gehrcke.**—An extremely ingenious device for obtaining narrow interference fringes under conditions of large retardation of path, was originated by O. Lummer and E. Gehrcke.<sup>1</sup>

It resembles in principle the echelon spectroscope, but the multiple interfering beams are obtained in a much simpler way. The spectrum of the mercury vacuum tube is focused upon a slit  $S$  1 mm. in width (Fig. 186). The narrow monochromatic beam obtained in this way is passed into a slab of plane-parallel optical glass, at such an angle that it

<sup>1</sup> *Annalen der Physik*, 10, page 457, 1903.

meets the second surface at very nearly the critical angle. Only a small fraction therefore escapes along the path 1. After two internal reflections a second portion emerges along path 2. The path-difference between these two streams is evidently very large, for it is represented by  $\mu(AB + BC) - AD$ . Owing to the small amount of energy which escapes at each reflection a large number of parallel beams, 1, 2, 3, 4, 5, etc., are obtained with path-differences increasing in arithmetical progression. The condition of this multiple beam is precisely analogous to that of a beam which has traversed an echelon, and if we place our eye in it we shall see a large number of images of the slit side by side, which correspond to the elements of the echelon. To permit the introduction of the light at the proper angle, unaccompanied by the

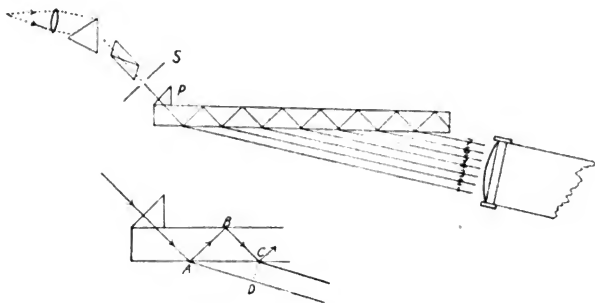


FIG. 186.

large loss by reflection at the first surface, the small prism  $P$  is cemented to the surface of the plate with Canada balsam.

By means of the Nicol prism the incident light can be polarized in such a plane that it is most copiously reflected at the inner surfaces. Lummer and Gehrecke studied the structure of the mercury lines with this instrument, obtaining results which have given rise to a good deal of discussion. The yellow line ( $\lambda = 579$ ) we found to consist of 12 components, the other yellow line ( $\lambda = 577$ ) of 11. The green line,  $\lambda = 546$ , had 21, and the blue line,  $\lambda = 492$ , 3 components.

These results have been criticised by Fabry and Perot,<sup>1</sup> on the ground that they do not agree with the results obtained with other instruments of equal power. Though Lummer and Gehrecke do not give the wavelengths of the component lines, Fabry and Perot estimate that the three brightest components of the green line, as figured by Lummer, ought to be easily visible with a small grating. They appear to be of nearly equal intensity, while other observers, working with instruments of different types, all agree upon one bright line with two faint companions. Lummer and Gehrecke state that the resolving power of their instrument is 400,000. The distance between

<sup>1</sup> *Journ. de Phys.*, Jan. 1904.

the extreme bright lines in the set is about twenty times the distance between the closest pair, from which Fabry and Perot estimate that for them  $\frac{\delta\lambda}{\lambda}$  must be  $\frac{1}{200000}$ , or in other words, that if they had a real existence, the distance between them would be only  $\frac{1}{20}$  of the distance between the  $D$  lines.

The multiple lines observed by Lummer they regard as optical "ghosts," due to slight imperfections in the glass plates. It is possible that the apparent confusion results from the presence of other "orders" of spectra, as with the echelon.

Lummer has devised another form of interferometer which is a modification of the one of Fabry and Perot; one side of a glass plate is heavily silvered, the other half silvered. A beam of light is admitted through a narrow slit in the silver surface and is reflected over paths as indicated in Fig. 187. The distinctness of the fringes was materially lessened when the last emergent streams of light were screened off, from which Lummer inferred that interference still took place, even with a path-difference of over two million wave-lengths.

Fabry and Perot in the same paper criticise this result, on the same grounds. With their interferometer, the fringes vanished when the path-difference was greater than 790,000 wave-lengths.

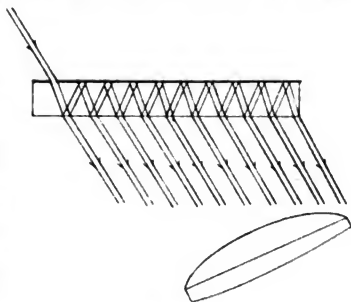


FIG. 187.

## CHAPTER IX.

### POLARIZATION OF LIGHT.

IN the preceding chapters we have considered rays of light as symmetrical around their direction of propagation; this amounts to saying that the rotation of the ray about its line of propagation is wholly without influence upon the optical phenomena exhibited by it. Rays of light exist, however, which possess a one-sidedness and behave differently when differently orientated. For example, it is possible to obtain light which a glass or water surface refuses to reflect at a certain angle of incidence. Such light is said to be polarized, and is distinguished from ordinary light in that its vibrations are of a fixed type, that is, the ether particle travels in a fixed orbit. If the motion is back and forth along a line, the light is said to be plane-polarized, and it is with this type of polarization that we are concerned in the present chapter. Elliptically polarized light, when the ether particle moves in an elliptical orbit of fixed eccentricity and orientation in space, and circularly polarized light, where the orbit is a circle, will be subsequently dealt with.

The fact that light can be obtained having a lack of symmetry around the direction of propagation is one of the most direct and convincing proofs which we have of the transverse nature of the waves, for we cannot very well conceive of a pressural or longitudinal wave, having different properties in the different directions perpendicular to the line of propagation.

**Discovery of Polarization.**—The polarization of light was discovered by Huygens in 1690, while experimenting with Iceland spar. He found that a ray of light was, by passage through the crystal, divided into two separate rays of equal intensity, except when the light traversed the crystal in a direction parallel to the crystallographic axis. He found furthermore that if one of these emergent rays was passed through a second crystal, it was divided into two rays of equal or unequal intensity, or not divided at all according to the orientation of the crystal. Though this single experiment was sufficient to establish the existence of light which was asymmetrical around its line of propagation, and though many other crystals exist having similar properties, Huygens was ignorant of the nature of the phenomenon, and the discovery remained an isolated fact for more than a century. The polarization is in this case produced by double refraction, which we shall study in detail in a subsequent chapter.

**Polarization by Reflection.**—The discovery was made by Malus in 1810 that light, which had suffered reflection at a certain angle from a surface of water or glass, exhibited the same peculiarities, which had previously only been observed in the case of light after its passage through a crystal of Iceland spar.

The polarization of light by reflection can be exhibited by means of the easily improvised apparatus shown in Fig. 188. The reflectors *A* and *B* are made of ordinary plate-glass, the backs being coated with black paint or asphalt varnish. The polarizing plate *A* is mounted on an iron stand on a hinged support so that it can be set at the polarizing angle. The other reflector is mounted on the vertical axis of an ordinary turn table, in such a way that the light reflected

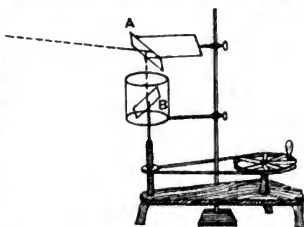


FIG. 188.

down from the polarizer meets the surface of the glass at an angle of  $57^\circ$  with the normal, i.e. the plate must make an angle of  $33^\circ$  with the axis. A cylindrical ring of parchment paper or other translucent medium surrounds the revolving plate, and receives the light reflected from it. The upper plate is so adjusted that its plane is parallel to the plane of the lower plate, in which position it will be found to reflect light capable of reflection from the latter: if, however, the lower plate is turned through an angle of  $90^\circ$ , it will be found that the light is no longer reflected from it, while in intermediate positions of the plate the reflection is partial. If the plate be set in rapid revolution a ring of light is seen on the translucent screen with two maxima and two minima, corresponding to the positions in which the light is most and least copiously reflected.

**Angle of Polarization. Brewster's Law.**—If the angle of the upper mirror is varied, it will be found that the reflected light is less completely polarized, and the maxima and minima obtained by revolving the lower plate are less marked. In general, as we increase the angle of incidence from normal to grazing, the polarization increases, passes through a maximum and then decreases. The angle at which the polarization is most complete varies with the nature of the substance, and is known as the polarizing angle. Jamin found that only a few substances with a refractive index of about 1.46 completely polarize the reflected light. For all other substances the polarizing angle is merely the angle at which the polarization is a maximum.

The relation between this angle and the refractive index of the substance was investigated by Brewster, who discovered the remarkable law that the index of refraction was the tangent of the angle of maximum polarization. When the light is incident at this angle the refracted ray makes an angle of  $90^\circ$  with the reflected ray, for

$$n = \tan i = \frac{\sin i}{\cos i} = \frac{\sin r}{\cos r};$$

$$\therefore \cos i = \sin r \text{ and } i + r = 90^\circ.$$

If this law is true the angle of maximum polarization will be different for the different colors owing to dispersion. In the case of most transparent media the dispersion is too small to greatly affect the angle, as can be shown by examining the image of the sun reflected in a glass plate through a Nicol prism so oriented as to cut off most completely the reflected light. The image of the sun appears uncolored, which would not be the case if the angle of polarization was very different for different parts of the spectrum. The Nicol prism, which will be presently described, takes the place of the second reflector, having the property of completely cutting off light polarized in a certain plane, and transmitting with greater or less facility light polarized in all other planes. In the case of substances having very high dispersion, the variation of the angle with change of wave-length becomes very marked.

The organic compound nitroso-dimethyl aniline, which has been found by the author to have the highest dispersion, in the brighter parts of the visible spectrum, of any known substance, is admirably adapted for the exhibition of what may be termed the dispersion of the angle of polarization. A little of the substance is fused on a glass or metal plate, or better in a small brass cell heated by steam (Fig. 188a).<sup>1</sup>

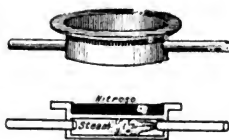


FIG. 188a.

A brilliant source of light of small dimensions—nothing is better than a Nernst lamp—is arranged so that its light is reflected from the liquid surface at an angle which can be varied.

On examining the light reflected at a fairly large angle with a Nicol prism, it will be found to vary from light blue to deep violet and purple, as the angle of incidence is increased, the Nicol being held in such a position as to refuse transmission to the light polarized by reflection. If a small direct-vision spectroscope is placed behind the Nicol, a dark band will be seen crossing the spectrum, which shifts its position as the incidence angle varies. The center of this dark band evidently marks the wave-length, for which the angle of incidence happens to be the angle of maximum polarization, or in other words the refractive index of the substance for this wave-length is the tangent of the angle of incidence. In the case of glass and substances of low dispersion, the different colors are polarized at nearly the same angle, i.e. very little color effect is observed when the reflected light is examined with the Nicol. In these cases the dark band is so broad as to occupy practically the entire visible spectrum. In the case of a substance with as high a dispersion as that of the nitroso, the angle of maximum polarization is quite different for the different colors, consequently the Nicol prism only extinguishes a portion of the spectrum for a given angle of incidence. This gives us a reflection method of determining the refractive index of a substance, for by

<sup>1</sup> Nitroso-benzyl aethyl aniline, which can be obtained from the Berlin Aniline Co., is better than the nitroso-dimethyl compound, as after fusion it remains liquid for some hours at ordinary temperatures.

determining the angle of incidence for which the center of the dark band is at a given point in the spectrum, we have only to look up the tangent of the angle in order to get the refractive index for the wavelength in question. The band will be found to be very sharp and quite narrow when it occupies a position in the green and greenish blue, but on attempting to drive it into the red, we shall find that it broadens and becomes much less sharply defined. This is of course due to the fact that the dispersion is much less in the red and orange portion of the spectrum. If the nitroso cannot be obtained selenium plates, made by pressing the molten substance between glass plates, which are to be separated by a blow from a hammer when cold, can be used for the exhibition of the dark band in the spectrum.

**Plane of Polarization.**—The plane of polarization is defined as the particular plane of incidence in which the polarized light is most copiously reflected. Referring to Figure 188 we will determine the plane of polarization of a ray reflected from the upper mirror. The ray meets the lower mirror at the polarizing angle, but as the mirror turns the plane of incidence changes, and the particular plane of incidence which we have when the light on the translucent screen is a maximum, is the plane of polarization of the ray. Since the mirrors are parallel (or turned through  $180^\circ$  from the parallel position) when this occurs, the plane of polarization of light polarized by reflection is obviously the plane of reflection. In the case of light polarized by some other method, we can determine its plane of polarization by reflecting it at the polarizing angle from a glass plate so oriented as to give maximum reflection. The plane of polarization of the ray is then the plane of incidence.

This definition of the plane of polarization is rather unfortunate, for, as we shall see later on, the vibrations of plane-polarized light are in a direction at right angles to this plane, and the plane of vibration is the one in which we are chiefly interested, for it is the one in which something is taking place. It would have been preferable if what we now believe to be the plane of vibration had been called the plane of polarization, but the definition was given before any very definite ideas were held regarding the direction of the vibration.

**Nörrenberg's Reflecting Polariscopes** —The reflecting polariscopes of Nörrenberg is a convenient and very easily constructed piece of apparatus for the study of polarized light. It consists of a wooden base with two vertical supports which carry the hinged polarizing mirror  $A_2$ , made of plate glass, and the two circular collars  $C$  and  $R$ , the latter fitted with a glass plate, upon which the object to be examined is laid. The upper collar carries the analyzing mirror, which is mounted on a revolving collar fitting concentrically into the other. This mirror is made of plate-glass, backed with black varnish, and is hinged like the polarizer. If the upper collar is roughly graduated, the utility of the apparatus is increased, for it may then be used for measuring the rotation of the plane of polarization, which occurs when light passes through certain substances. On the base of the instrument, between the two vertical supports, a small circular mirror of silvered glass is cemented. The entire apparatus can be made of wood, if facilities for metal turning are not available.

A beam of sunlight, coming in a direction  $a'a$  such that it makes an angle of  $66^\circ$  with the vertical, is reflected from the mirror at an incidence angle of  $57^\circ$ , the mirror being adjusted at an angle of  $33^\circ$  with the vertical, so that the reflected ray is thrown down normally upon the silvered reflector. It is then reflected back, and for the most part

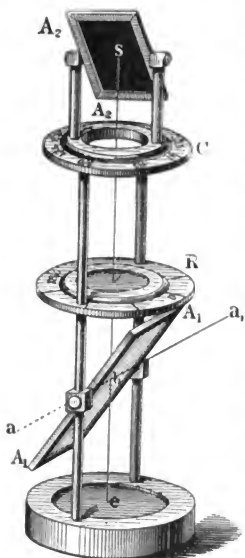


FIG. 189.

passes through the mirror  $A_2$  and the circular glass table, falling upon the upper mirror  $A_1$ , which is also set at an angle of  $33^\circ$  with the vertical. The upper mirror we shall call the analyzer, since it is used for the study of the light which comes from the polarizer. On looking down into this mirror at an angle of  $66^\circ$ , we see the reflected image of the sun or other source of light, the intensity depending on the position of the revolving collar, being greatest when the mirrors are parallel or turned through  $180^\circ$  with respect to this position, and practically zero when the planes of reflection are perpendicular to each other.

The proper angles can be found by making small adjustments of the mirrors when they are in the latter position, until the reflected image disappears entirely. The instrument will be found useful in the study of the colors of thin crystalline plates, and the phenomena of circular and elliptical polarization, which we shall take up in subsequent chapters.

The polarized light from the reflector passes through the object under examination once or twice, according as it is laid on the glass table or the silver mirror;

placing the object in the latter position is equivalent to doubling its thickness. On this account the instrument is sometimes called the Nörrenberg doubler.

**Polarization by Refraction.**—If we examine the light transmitted through a plate of glass placed at the polarizing angle, we shall find that the light is partially polarized; *i.e.* its intensity varies slightly when examined by means of an analyzer. Arago discovered that the reflected and refracted portions of the light contained equal quantities of polarized light, and that the planes of polarization were at right angles.

The greater intensity of the transmitted light is responsible for the incompleteness of the polarization. If the light transmitted through a plate placed at the polarizing angle is received upon a second plate, the unpolarized portion suffers a further resolution into two polarized components, one of which is reflected out through the upper plate and the other transmitted. By increasing the number of plates we can increase the intensity of the reflected polarized light, and consequently the completeness of the polarization of the trans-

mitted light, seven or eight being sufficient to give us nearly complete polarization in the transmitted, as well as in the reflected beams.

A simple polariscope can be constructed on this principle, which has an advantage over the reflecting instrument in that it can be directed towards the source of light, and requires no adjustment. The glass plates used should be as thin as possible, in order to avoid loss of light by absorption. The large sized rectangular cover glasses used for microscopical preparations are best for the purpose, though the thin glass employed for lantern slides is almost as good. The plates should be carefully cleaned and freed from dust, and mounted in two piles, of eight plates each, in tubes of wood or pasteboard at an angle of about  $33^\circ$  with the axis of the tube. It is best to determine the angle experimentally, as it varies slightly with the nature of the glass. The two piles of plates should almost completely cut off light when the planes of incidence are at right angles. If this is not the case a few more plates can be added. Bundles of plates thus mounted form very fair substitutes for the more expensive Nicol prisms, and are well adapted to lantern experiments. It may appear at first sight as if each successive plate in the pile would diminish the intensity of the polarized portion coming through the one next to it by reflection, but it must be remembered that *transmitted* polarized light is polarized in a plane at right angles to the plane of the reflected portion, and consequently is incapable of reflection. After passage through a sufficient number of plates the light is completely polarized, and the addition of more plates does not decrease the intensity except by absorption. This is of importance in connection with the calculation of the loss of light by reflection from the surfaces of prism-trains in spectroscopes. Transmission through the oblique surfaces polarizes the light, and we must take this into account in calculating the loss at each surface, the amount reflected decreasing as the polarization becomes more complete. After passage through five prisms there is practically no further loss by surface reflection, and the spectrum is almost completely polarized.

**Law of Malus.**—The law which governs the decrease in the intensity of the light, as the upper mirror of the Nörrenberg polariscope is revolved, was formulated by Malus. It may be stated thus: "When a beam of light, polarized by reflection at one plane surface, is allowed to fall upon a second, at the polarizing angle, the intensity of the twice reflected beam varies as the square of the cosine of the angle between the two planes of reflection. The assumption was made that the incident vibration, polarized in a plane making, say, an angle  $\theta$  with the plane of incidence, was resolved into two components, one perpendicular, the other parallel to the plane of incidence, the former being partially reflected, and the latter wholly transmitted. This will make the reflected amplitude  $a \cos \theta$ , if  $a$  is the reflected amplitude when  $\theta = 0$ , and the intensity will be  $a^2 \cos^2 \theta$ , or the maximum reflected intensity multiplied by the  $\cos^2$  of the angle between the plane of polarization and the plane of incidence.

The law of Malus is therefore simply a statement of the resolution of a vibration into two rectangular components, the direction of the vibration being considered perpendicular to the plane of polarization.

It is important to distinguish between the behavior of vibrations parallel to the plane of incidence, and vibrations perpendicular to the plane, when they meet a reflecting surface at the polarizing angle. If

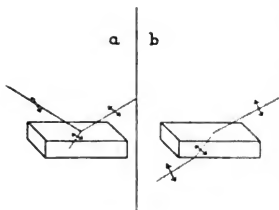


FIG. 190.

the light is so polarized that the vibrations are perpendicular to the incidence plane, *i.e.* parallel to the reflecting surface, a portion of the energy will be reflected, and a much larger part transmitted, the directions of the vibrations remaining parallel to the incident vibration (Fig. 190, *a*). If, on the other hand, the direction of vibration is parallel to the incidence plane, practically no energy is reflected, the light being refracted with-

out loss of intensity (Fig. 190, *b*). If now the vibration takes place in a direction making, say, an angle of  $45^\circ$  with the plane of incidence, it will be resolved into two components, one parallel to the reflecting surface and the other parallel to the plane of incidence.

Let  $ABCD$  (Fig. 191) represent a portion of the wave-front of the incident beam, which is coming towards us, the direction of vibration being  $AC$ . We have resolution into the components  $AB$  parallel to the incidence plane, and  $AD$  parallel to the reflecting surface. The former is wholly transmitted ( $A'B'$ ), the latter part reflected and part transmitted ( $A'D'$ ). The reflected light is therefore polarized with its vibration parallel to the surface, since only this component is reflected; the refracted light is made up of the completely transmitted component  $A'B'$ , and the partially transmitted component  $A'D'$ , their resultant being  $A'C'$ , a polarized vibration, rotated counter clock-wise with respect to the incident vibration  $AC$ . If now the resultant  $A'C'$  be received on a second reflecting surface, the same resolution will take place, and there will be a further rotation of the plane. The effect of a pile of plates will therefore be to bring the plane of vibration of the transmitted light into coincidence with the plane of incidence, since  $A'B'$  is transmitted each time without loss, while  $A'D'$  is reduced in intensity by the partial reflection. This rotation of the plane of polarization is clearly the result of the reduction in the intensity of one of the rectangular components, and may be shown best by means of a pair of Nicol prisms, so oriented as to refuse transmission. If a glass plate, or better, a pile of three plates, be placed between the prisms at the polarizing angle and so oriented

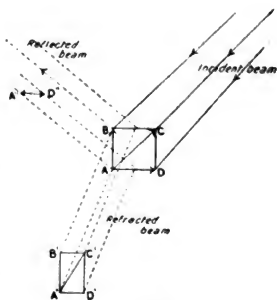


FIG. 191.

that the plane of incidence is inclined at  $45^\circ$  to the principal planes of the Nicols, there will be a partial restitution of light, and the analyzing Nicol will have to be turned counter clock-wise to produce complete extinction.

It is clear now why the transmitted light is only partially polarized, when a ray is refracted at a single surface, and completely polarized by refraction at a large number of parallel surfaces. We may consider ordinary light as consisting of vibrations polarized in all possible planes. Each vibration is therefore transmitted with a slight rotation towards the plane of incidence, and the light will not differ greatly in its properties from ordinary light. By every succeeding surface there is a further rotation, and eventually all are brought into the plane of incidence and the transmitted light is plane-polarized. If the vibrations of the incident light be represented

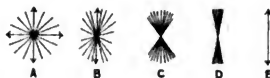


FIG. 192.

by *A* (Fig. 192), the effect of successive refractions may be represented by the succeeding diagrams *B*, *C*, *D*, *E*, the transmitted light being plane-polarized in the latter case. As we shall see in a subsequent chapter, the probable condition in ordinary light can be more nearly represented by considering it plane-polarized light, the plane of polarization changing with great rapidity. This conception will not alter the above representation when the condition present during a finite time is under consideration. The foregoing elementary treatment of polarization by reflection will suffice for the present. The theoretical treatment of the subject will be given in a subsequent chapter. We will now consider some of the other methods by which polarized vibrations can be obtained.

**Polarization by Double Refraction.**—The double refraction of light by crystals of Iceland spar was first noticed by Erasmus Bartholinus, and subsequently more completely investigated by Huygens. It occurs whenever light enters the crystal in a direction not parallel to the optic axis, and is due to the fact that the incident vibration is decomposed into two mutually perpendicular components which travel through the crystal with different velocities. One of the two rays obeys the ordinary laws of refraction and is called the ordinary ray, while the other behaves in a most peculiar manner and is called the extraordinary ray, for it is bent away from the normal *even at perpendicular incidence*. At first sight this appears impossible, for even if the two rays have different velocities there seems to be no reason why there should be a change of *direction* for normal incidence. As we shall see when we come to the chapter on Double Refraction, this is due to the fact that the wave-front of the extraordinary disturbance is not spherical but ellipsoidal. For the present, however, we are not concerned with this question, and merely consider the crystal as a means of resolving ordinary light into two polarized beams. One of these may be cut off by any suitable device, leaving us plane-polarized light. Crystals of tourmaline have the remarkable property of absorbing the ordinary ray and transmitting the extraordinary, consequently a thin section of a crystal transmits only polarized light, and may be used either as a

polarizer or analyzer. The sections are cut parallel to the optic axis, and when superposed with their axes parallel transmit light quite freely. If one is rotated through a right angle, the combination becomes opaque, since the polarized vibrations transmitted by one are absorbed by the other. The tourmaline polariscope is a very simple instrument, consisting of a pair of crystal sections mounted in a pair of wire tongs in such a way that one of them can be rotated in front of the other. The object to be examined, for example, a mica or selenite film, is placed between the two tourmalines, and the instrument directed towards a bright light; owing to the deep color of the tourmaline crystals, this form of polarizer and analyzer is very inferior to the Nicol prism, which only cuts down the intensity of the original light one half.

**The Nicol Prism.**—Iceland spar, on account of the large size in which the crystals occur, and their great transparency, is especially suitable for the construction of polarizing prisms. Since both the ordinary and extraordinary rays are transmitted with equal facility, it becomes necessary to eliminate the one or the other by some optical contrivance. The method employed by Nicol was to stop the ordinary ray within the crystal by total reflection. A rhomb of spar is cut in two along a plane, as indicated in Fig. 193, and the two halves cemented together with Canada balsam, the two oblique surfaces having first been polished. The refractive index of the balsam is intermediate between that of the spar for the ordinary and extraordinary rays, and the former meeting the balsam film at an angle greater than the critical angle is totally reflected to one side and absorbed by a coating of black paint.



FIG. 193.

The prism therefore only transmits the extraordinary ray, which is plane-polarized, the direction of the vibration being parallel to the short diagonal of the prism, *i.e.* perpendicular to the balsam surface. In some prisms it is impossible to tell by mere inspection in which direction the emergent light vibrates, owing to the way in which the crystal has been cut. It is always possible, however, to determine in a moment the direction in question. We have only to reflect ordinary light from a glass surface in the neighborhood of the polarizing angle and examine it through our prism, holding it in such a position that the transmitted light has its maximum intensity. The vibrations of the light are parallel to the glass surface, consequently the diagonal of the prism which is parallel to this surface is the direction in which the transmitted light vibrates. If any difficulty is experienced in fixing in the mind the direction of vibration in the case of reflected and transmitted light, the following analogy may prove useful. If a cylindrical wooden rod is thrown in an oblique direction and with great force upon the surface of water, it will bounce off if parallel to the surface, the rod being supposed to move in a direction perpendicular to its length. If, however, the rod is perpendicular to the surface, the lower end will enter the water first and the rod "cut down" into the fluid, without suffering reflection. Consider our polarized vibrations as parallel to

the rod, and we have the optical analogy, which is only of use, however, in enabling us to remember the direction of the vibration in the reflected and refracted components.

The Nicol prism is sometimes made with end surfaces perpendicular to the axis. This necessitates a more oblique section, and consequently a greater length in proportion to the width, the ratio being nearly 4 : 1. The Foucault prism is similar to the Nicol, except that the balsam film is replaced with an air film. This device reduces the ratio of length to width to 1.5 : 1, but the prism is less efficient than the Nicol owing to multiple reflections in the air film.

**Angular Aperture of Polarizing Prisms.**—Since many experiments require the passage of a convergent or divergent beam through the prism, it is of some interest to consider the maximum angular aperture of the prism, or maximum divergence which a cone of rays may have, and still be completely polarized by passage through a prism of given type. If the divergence exceeds a certain amount, it is obvious that some of the ordinary rays will not suffer total reflection. The type of prism originally designed by Nicol, was made by grinding down and polishing the ends of the rhomb, by an amount sufficient to reduce the angle between the end surfaces and the sides from  $72^\circ$  to  $68^\circ$ , and make the section in a plane perpendicular to the end surfaces and the plane containing the optic axis and long axis of the crystal. The angular aperture of a prism of this type is about  $30^\circ$ , while that of the Foucault prism is only  $8^\circ$ . A prism was devised by S. P. Thompson<sup>1</sup> in which the optic axis was perpendicular to the long axis of the prism, which had an aperture of  $39^\circ$ . Glazebrook<sup>2</sup> constructed one along similar lines, but with end surfaces perpendicular to the long axis.

The question of the best construction for a prism with large aperture and end surfaces perpendicular to the long axis was investigated by Feussner,<sup>3</sup> who found that the optic axis should be perpendicular to the section, and the refractive index of the cementing film the same as that of the crystal for the extraordinary ray. Such a prism has an aperture of  $42^\circ$  and a ratio of length to width of 4 : 1.

In another and quite different type of polarizing prism, the doubly refracting substance acts as the rarer medium, the extraordinary ray being totally reflected from a thin plate of Iceland spar immersed in a liquid of higher refractive index. The first prism constructed on this principle was made by Jamin, who immersed a thin plate of spar in a glass trough filled with carbon bisulphide. Zenker improved the device by substituting prisms of flint glass for the liquid, while Feussner suggested the use of a plate of sodium saltpetre instead of Iceland spar, on account of the greater difference between the ordinary and extraordinary refractive index. Such a prism would have an aperture of  $56^\circ$ .

**Detection of Polarized Light.**—If the amount of polarized light present in a beam is too small to be detected by the slight changes in

<sup>1</sup> *Phil. Mag.*, 5, 12, p. 349, 1881.

<sup>2</sup> *Phil. Mag.*, 5, 15, p. 352, 1883.

<sup>3</sup> *Zeitsch. für Instr.*, 4, p. 41, 1884.

intensity produced by passing it through a slowly revolving Nicol, some more delicate method must be adopted.

The Nicol used alone will not give evidence of the presence of less than about 20 % of polarization, consequently in cases where the polarization is not considerable (the solar corona for example), we cannot rely upon its testimony. By the use of the so-called "bi-quartz," which is described in the chapter on Rotatory Polarization, the presence of 5 or 10 % of polarization may be detected by the slight coloration of the two segments of the plate when it is placed in front of a Nicol and directed towards the light.<sup>1</sup>

Savart's plate and Babinet's compensator are still more sensitive detectors of small amounts of polarization, and are generally employed in the study of the polarization of the sun's corona during total eclipses. The presence of polarized light is shown by a system of colored fringes which appear when the plate is used in the same manner as the bi-quartz. Savart's plate is made by cutting a plane-parallel plate from a quartz crystal at an angle of  $45^\circ$  with the optic axis. The plate is then cut into two halves, which are mounted the one above the other, but rotated through  $90^\circ$  with respect to each other. The sensitiveness is at a maximum, *i.e.* the fringes are most distinct when the direction of the polarized vibration is perpendicular to the fringes, the instrument then being capable of showing the presence of 1 % of polarization.

**Determination of the Percentage of Polarized Light.**—In the case of light partially polarized it is often of importance to determine the percentage polarized. This may be accomplished in a number of ways. A method frequently used in studying the polarization of the corona consists in compensating the polarization by means of one or more inclined glass plates. This method has been referred to in the last section (Naval Obs. Report). A method devised by Cornu is, however, better adapted to accurate work in the laboratory.

A screen perforated with a rectangular opening measuring about  $2 \times 4$  mms. is placed in front of the source of light, and a double image prism oriented in such an azimuth that one of its planes of vibration is parallel to the polarized vibration which is to be measured. It frequently happens that we know the position of this plane beforehand; for example, if we are studying light reflected from transparent media, we know that the plane of vibration of the polarized portion of the reflected light is parallel to the reflecting surface. The screen and prism are to be placed at such a distance apart that the two images of the aperture just touch one another. One of these images will be found to be brighter than the other, since *all* of the polarized light is present in it, while the unpolarized light is divided equally between the two. We now compensate the intensities by means of a Nicol prism mounted on a graduated circle.

In Fig. 194, let  $AB$  be the plane of vibration of the polarized portion of the light. Then the right-hand image of the aperture will be the

<sup>1</sup> An excellent account of various methods of detecting polarized light and measuring the percentage will be found in the "Report of the U.S. Naval Obs. of the Total Eclipse of July 29, 1878."

brighter, if the planes of vibration of the double image are as indicated by the arrows  $a$  and  $b$ . Let  $a$  and  $b$  equal the amplitudes of the vibrations,  $a^2$  and  $b^2$  the intensities. The proportion of polarized light will then be given by

$$p = \frac{a^2 - b^2}{a^2 + b^2}.$$

Call  $\omega$  the angle between the transmission plane (short-diagonal) of the Nicol and the plane of the vibration  $a$ . Then the intensities of the two images seen through the Nicol will, by the law of Malus, be  $a^2 \cos^2 \omega$  and  $b^2 \sin^2 \omega$ . If we orient the Nicol so that equality is established, we can equate these two quantities

$$a^2 \cos^2 \omega = b^2 \sin^2 \omega \quad \text{or} \quad \frac{a^2}{b^2} = \frac{\sin^2 \omega}{\cos^2 \omega}.$$

The proportion of polarization is given by

$$p = \frac{a^2 - b^2}{a^2 + b^2} = \frac{\sin^2 \omega - \cos^2 \omega}{\sin^2 \omega + \cos^2 \omega} = -\frac{\cos^2 \omega - \sin^2 \omega}{1} = -\cos 2\omega.$$

It will be at once seen that  $2\omega$  will be over  $90^\circ$ , since  $\omega = 45^\circ$  when  $a$  and  $b$  are originally equal, i.e. when the light contains no traces of polarization. If the plane of partial polarization is not known, we may take a reading with the double image prism in any position, and then rotate the whole instrument through  $90^\circ$  and take a second reading. The proportion of polarized light will then be given by

$$p = \sin(\omega_2 - \omega_1).$$

**The Eye as an Analyzer: Haidinger's Brush.**—The polarization of light can sometimes be detected by the eye alone. If we look through a Nicol prism at a white cloud, and slowly revolve the prism, a faint blue and yellow double brush appears at the point upon which the eye is fixed, which revolves with the prism. It is not easy to see it at the first attempt, but once noticed, it is easily recognized on subsequent occasions. It consists of four quadrants, colored blue and yellow alternately, and is usually very faint. Various explanations of the phenomena have been given, most of them based on the laminary structure of the lens of the eye. If this were the case, however, the center of the brush ought to appear a little to one side of the point observed, since the point on the retina which receives the image of the point upon which the eye is fixed lies to one side of the axis of the eye. This fact led Helmholtz to look for the cause of the phenomenon in the structure of the "yellow spot" of the retina, which is the point of the retina alluded to above. He found that the brush was due to the fact that the yellow elements of the spot were doubly refracting, and that the extraordinary rays of blue color were more strongly absorbed than the ordinary rays. For a more complete treatment of the subject the reader is referred to Helmholtz's *Physiological Optics*.

**Polarization by Oblique Emission.**—Arago found that the light emitted in an oblique direction from the white hot surface of a solid

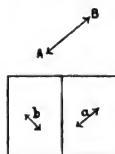


FIG. 194.

or liquid, was partially polarized in a plane perpendicular to the plane of emission, i.e. it resembled light refracted at an oblique surface. From this we may infer that the emitted light comes not only from the surface molecules, but also from those lying below the surface, and that the polarization is due to the refraction of this light when it emerges into the air. The light from an incandescent gas is never polarized, owing doubtless to the low refractive index.

The state of polarization of the light coming from an oblique surface of a body, gives us a clue as to the origin of the light and the state of the body. The light of the moon, for example, is partially polarized in a plane passing through the sun, moon, and earth, which shows us that the moon shines by reflected light.

If it shone by emission the light would be polarized slightly in the opposite plane. The absence of polarization in the light coming from near the edge of the sun, led Arago to infer that it was emitted by an incandescent gas.

**Stationary Polarized Waves.**—Wiener found in repeating his experiments (page 146) with plane polarized light, that if the light was incident at an angle of  $45^\circ$  the effects of stationary waves were only obtained when the plane of polarization was parallel to the plane of incidence. Stationary waves can only be formed when the effective vector in the reflected disturbance is parallel to the vector of the incident light, from which we infer that the vector which is effective in producing photographic action is perpendicular to the plane of polarization. The same was found to hold true for fluorescence action. Wiener having already determined that a node of the stationary wave system occurred at the reflecting surface, the inference to be drawn from the experiments with oblique light was that the electric vector was the one concerned in photographic and fluorescent action, and that it was perpendicular to the plane of polarization. This will be made clearer in the chapter on the Theory of Reflection.

**Landolt's Fringe.**—If a brilliant source of light is viewed through a pair of Nicol prisms, so oriented that their principal planes are at

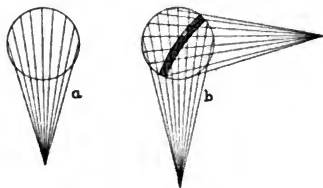


FIG. 195.

right angles, it will be seen that the whole field does not become absolutely dark when exact adjustment is reached, but that the darkened field is crossed by a back fringe which changes its position if either prism is rotated ever so slightly. This fringe, which was first noticed by Landolt, was explained by Lippich<sup>1</sup> who showed that it

<sup>1</sup> F. Lippich, *Wien. Akad. Ber.*, III., Bd. lxxxv., p. 268, 1882.

was due to the fact that the directions of vibration in the different parts of the field were not strictly parallel, a circumstance resulting from the varying obliquity of the rays. Lippich showed further that in the case of polarizing prisms with end faces perpendicular to the prism axis, the direction of vibration was represented by a system of converging lines which met in a point outside of the prism, as shown in Fig. 195*a*. Complete darkness will occur with crossed Nicols only in those parts of the field where the directions of vibration in the two prisms are perpendicular. If we draw two fields similar to the one represented by *a*, at right angles to each other, we shall find that the small areas formed by the intersection of the converging lines are in general diamond shaped, but that they are approximately rectangular along a curved line represented by the dark band in Fig. 195*b*. This is the region where the directions of vibration are strictly perpendicular, and it in consequence appears black. A slight rotation of either field will obviously change the position of this locus, the squares becoming diamond shaped, and the adjacent diamonds straightening out into squares. Rotation of one of the prisms through  $180^\circ$  will be found to give a locus of squares perpendicular to the one shown in the figure, and since the fringe moves broadside across the field, the directions of motion in the two cases are perpendicular.

Lippich made use of the fringe in the construction of a polarimeter, in which the position of the fringe was determined with reference to a pair of cross-hairs. It was possible in this way to set the analyzing Nicol with an error not exceeding two or three seconds of arc.

## CHAPTER X.

### DOUBLE REFRACTION.

WE have seen in the chapter on Polarization, that crystals of Iceland spar have the property of dividing a ray of light into two plane polarized rays, the directions of polarization being at right angles to each other. In the present chapter we shall study in detail the propagation of light in crystalline media, and the laws which govern it.

The division of a ray of light by a crystal of Iceland spar, or double refraction, was observed in 1669 by Erasmus Bartholinus, and the polarization of the two rays was subsequently discovered by Huygens, though he was unable to explain the phenomenon, since at the time light was supposed to consist of waves in which the displacement was parallel to the direction of propagation. It was not until Young and Fresnel introduced the idea of transverse waves that the true nature of polarization was understood. Double refraction can be easily observed by laying a crystal of Iceland spar over an ink dot on a sheet of paper. Two images are seen which can be quenched in succession by the rotation of a pile of glass plates held at the polarizing angle. On revolving the crystal of spar, one of the images is seen to remain stationary, while the other revolves around it. The distance between

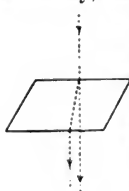


Fig. 196.

the two images is independent of the position of the eye, showing that the rays of light, after refraction through the crystal, emerge parallel to one another, as shown in Fig. 196. At first sight it may appear strange that rays of light, incident normally upon the refracting surface, should be deviated away from the normal, since we are sometimes accustomed to associate the bending due to refraction with oblique incidence, the light-waves turning through an angle as they enter the denser medium. We shall see presently, however, that the phenomenon is easily explained by Huygens's principle, when applied to the peculiar type of waves which we have in doubly refracting media.

Experiments have shown that in crystals belonging to the hexagonal and tetragonal system one of the rays obeys the ordinary laws of refraction, *i.e.* the refracted ray lies in the plane of incidence, and the sine of the angle of incidence bears a constant ratio to the sine of the angle of refraction. This ray is called the ordinary ray;

the other ray in general conforms to neither of these two laws, though in certain cases it may conform to one or to both of them. This ray is called the extraordinary ray. In the case of all other crystals except those of the cubic system, neither of the two refracted rays conforms of necessity to the ordinary laws of refraction. Crystals belonging to the cubic system do not exhibit the phenomenon of double refraction, the light being propagated as in isotropic media.

**Physical Explanation of Double Refraction.**—On the elastic solid theory we can explain double refraction in crystalline media by assuming that there are three directions called axes of elasticity, which have the distinctive properties of the two planes of vibration of Blackburn's pendulum, which consists of a weight suspended by strings as shown in Fig. 197*a*. If the weight is displaced either in, or perpendicular to, the plane of the paper it will oscillate in a straight line, the period being greater for vibrations perpendicular to this plane than for those parallel to it. If, however, it is displaced in an oblique direction, the force acting upon it will no longer be directed towards the position of equilibrium, and the weight will move in a curved orbit. In the case of crystals a particle displaced parallel to any one of the axes of elasticity will be acted upon by a force directed towards the equilibrium position, and the vibration will be plane-polarized. If displaced in any other direction and released, it does not return to its original position, but moves in a curved path in a manner analogous to that of the pendulum. We require an explanation of the splitting of a beam of light into two polarized components, and for their unequal velocities of propagation. The vibrations of a cylindrical rod form a useful analogy. In this case the elasticity is the same in all directions, and traverse vibrations in all planes are transmitted with the same velocity. Suppose the rod to be struck in a very brief time in every possible direction, then each particle will move in an orbit which is the resultant of all these impulses. The waves transmitted along the rod in this case are analogous to the waves of light in isotropic media. Consider now that the rod has an elliptical cross section (Fig. 197*b*). The elasticity is now not the same in all directions, being greatest in the plane of the major axis, and least in the plane of the minor. Wave-motion will traverse it with greater velocity if the direction of vibration is parallel to the longer diameter, than if the direction is perpendicular. If an attempt is made to transmit vibrations making an angle with the axes of the elliptical cross section, by striking the end of the rod in a direction other than that parallel or perpendicular to the major axis, the vibration will be decomposed into two components which travel along the rod with different velocities. The rod in other words is incapable of transmitting vibrations which make an angle with the axes.

We have in doubly refracting media a somewhat similar condition, the elasticity being different in different directions. Luminous vibrations will be decomposed into two polarized components which traverse the crystal with different velocities.

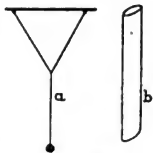


FIG. 197.

**Wave-Surface in Uniaxial Crystals.**—The fact was established by Huygens that, in isotropic media, the form of the wave-surface was spherical, and as one of the rays in Iceland spar was found to obey the ordinary laws of refraction, he assumed that the corresponding wave was a sphere. In the case of the extraordinary ray, which does not obey such simple laws, he made the assumption that the wave-surface was a spheroid, *i.e.* an ellipsoid of revolution. The velocity of the extraordinary ray in any direction is therefore given by the following construction: "Let an ellipsoid of revolution be described around the optic axis, having its centre at the point of incidence, and let the greater axis of the generating ellipse be to the lesser, in the ratio of the greater to the lesser index of refraction. Then the velocity of any ray will be represented by the radius vector of the ellipsoid which coincides with it in direction."

The optic axis may be defined as the direction in the crystal in which a ray of light may be propagated without double refraction. The law just given was found to apply to Iceland spar and many other crystals, but in all of these there was but a single optic axis. Brewster, however, discovered that in many crystals there were two directions in which light could be propagated without double refraction. Such crystals are termed bi-axial, and the law of Huygens was found not to apply in these cases. Fresnel then established a theory which not only conformed to all of the known facts, but made possible predictions which were afterwards verified by experiment. This theory we shall take up a little later.

According to the theory of Huygens the wave-surface in uniaxial crystals consists of two sheets, one a sphere, the other a spheroid, which touch each other at two points. The direction of the line joining these points of contact is called the optic axis of the crystal. This conception applies, however, only to uniaxial crystals. In the case of Iceland spar and all so-called negative crystals, the sphere lies within the spheroid. In such crystals the angle of refraction of the extraordinary is greater than that of the ordinary ray. In the case of quartz and other positive crystals, the spheroid lies within the sphere, and the angle of refraction of the extraordinary ray is less than that of the ordinary. This will be clearer when we come to the construction of the refracted ray.

**Huygens's Construction.**—Suppose a luminous disturbance to start within a uniaxial crystal. The wave will spread out in two sheets, a sphere and a spheroid, which touch each other at two points. In the direction of the line joining these two points both waves travel with the same velocity. If we apply Huygens's construction to crystalline refraction, giving to the secondary wavelets, which originate on the refracting surface, the forms of spheres and ellipsoids, we can account for, and calculate the position of the two refracted rays. In all other directions the velocities will be unequal and we shall have a division of the ray, as may be seen by Huygens's construction. Consider a wave-front *AB* incident in an oblique direction upon the surface of a uniaxial crystal (Fig. 198). The direction of the optic axis is represented by the dotted line. The point *A* becomes the center of two secondary wavelets which are propagated with different velocities. Making use of

the same construction which we applied in the case of isotropic media (see Chapter IV.), we draw tangent planes from the point  $C$  to the two wave-surfaces; the directions of the refracted rays are given by joining the point  $A$  with the points of tangency. In the case of the ordinary wave the refracted ray lies in the plane of incidence. This is also the case with the extraordinary ray, *provided the optic axis lies in the plane of the paper*. If, however, the optic axis is not in the plane of the paper, the point of tangency for the extraordinary wave will lie above or below the plane of the paper, and the refracted ray will no longer be in the plane of incidence. In the latter case neither of the ordinary laws of refraction is obeyed, for the sine relation only holds when the section of the secondary wave is circular. If the optic axis is perpendicular to the plane of incidence, the section of the spheroid is equatorial and therefore circular, the extraordinary refracted ray in this case lying in the plane of incidence and obeying the sine law. The ratio of the sines of the angles of incidence and refraction in this case, is termed the extraordinary index of refraction.

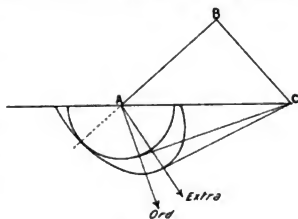


FIG. 198.

We will next consider the case in which rays of light are incident in a normal direction upon the crystal. As we have seen, double refraction occurs in this case, one of the rays passing straight through, while the other is deflected away from the normal. It is obvious that we cannot apply in this case the simple explanation of refraction which assumes successive portions of the wave-front retarded upon entrance into the refracting medium. For the wave-fronts originally parallel to the surface must remain so after refraction. What we have actually, if our original waves be plane, are two plane-waves travelling through the crystal with unequal velocities but parallel always to the surface. The deflection of the extraordinary ray is obvious if we apply Huygens's

construction to the present case. Assume that the points on the surface of the crystal become simultaneously the centers of ellipsoidal wavelets as indicated in Fig. 199. If the incident wave-front is limited to the region  $AB$ , the refracted wave-front will be the tangent plane of the ellipsoidal wavelets, and the refracted rays will be the lines  $AA'$ ,  $BB'$ . What actually happened may be described as follows: The refracted wave-front travels in the

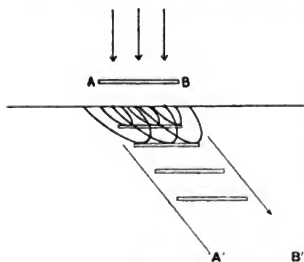


FIG. 199.

medium in a direction normal to its surface, but any limited portion of it bears away constantly to one side, and the ray is defined as the direction in which a limited portion of the wave-front

travels. We see in this case that the ray is not perpendicular to the wave-front, which is, in general, the case in doubly refracting media.

**Verification of Huygens's Construction.**—The assumptions made by Huygens regarding the form of the wave-surfaces in uniaxial crystals were speedily verified by experiment. That the ordinary wave front is a sphere, was shown by constructing a prism formed of pieces cut in all possible directions from a crystal of Iceland spar and cemented together. The spectra formed by the extraordinary rays were deviated by different amounts, whereas a single spectrum only was formed by the ordinary rays which traversed the different elements of the prism. To verify the construction of the extraordinary wave-front we will consider several cases.

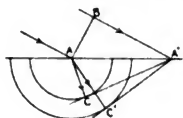


FIG. 200.

(1) **The refracting face is parallel to the optic axis, and the plane of incidence perpendicular to it** (see Fig. 200).—In this case the axis is perpendicular to the plane of the paper.

The sections of the two wave-surfaces will in this case be circles, as we have seen. The tangent planes touch the sphere and spheroid at C and C'. Let the velocity in air be 1, then the velocity of the ordinary and the extraordinary rays will be proportional to b and a the radii, and the refractive index of the extraordinary ray will be

$$\frac{\sin i}{\sin r} = \frac{1}{a} = \mu_e.$$

By cutting a prism of Iceland spar with its refracting edge parallel to the optic axis we obtain two spectra, and by measuring the deviation with a spectrometer we can calculate in the usual manner the refractive indices  $\mu_o$  and  $\mu_e$  for the ordinary and extraordinary rays. It can easily be shown that both rays are propagated through the prism according to the same law which holds in the case of a glass prism. This indicates that the section of the wave-surface is a circle for both rays, the radius for the extraordinary ray being  $\frac{1}{\mu_e}$  and for the ordinary  $\frac{1}{\mu_o}$ .

The extraordinary wave is therefore a surface of revolution around the optic axis, and to determine the form of the generating curve we shall consider the refraction which takes place under different conditions.

(2) **Optic axis parallel to the surface of the crystal and to the plane of incidence.**—The sections of the wave-surfaces in this case are shown in Fig. 201. Assume the extraordinary wave section to be an ellipse, the minor axis of which lies in the surface. The section of the sphere will be a circle touching the ellipse at the extremities of the minor axis. Drawing tangent planes from A', as before, to the two wave surfaces, and joining the points of tangency with A, we obtain the refracted rays. A line joining the two points of contact and produced, will cut the minor axis at a right angle, since the polar of any point in the

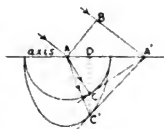


FIG. 201.

chord of contact of a circle and ellipse having double contact, is the same with regard to both curves. We have then

$$\frac{\tan r}{\tan r'} = \frac{AD}{CD} \times \frac{C'D}{AD} = \frac{C'D}{CD} = \frac{a}{b} = \frac{\mu_o}{\mu_e},$$

or the ratio of the tangents of the angles of refraction is equal to the ratio of the two indices of refraction.

This relation, which was deduced on the assumption that the wave-section was an ellipse, was verified by Malus in the following manner: Two scales  $AC$  and  $BC$  (Fig. 202) were engraved on a plate of polished

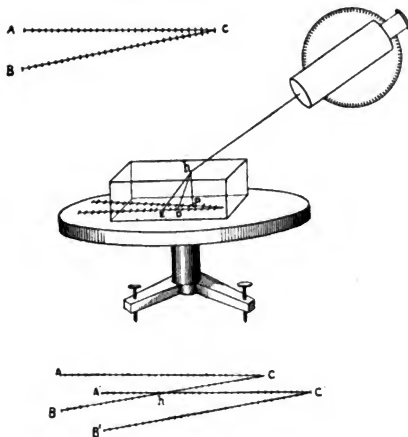


FIG. 202.

steel, and a thick plate of crystal with its faces parallel to the optic axis was laid on the scale and viewed through a telescope mounted on a graduated vertical circle. The crystal was brought into the horizontal position by means of levelling screws, the correct position being that in which the image of a distant point of light was not changed by rotation of the platform. Two images of the scale were seen in the telescope, and if we denote these by  $AC$ ,  $A'C$ ,  $BC$ ,  $B'C$ , there will be some point of  $BC$  coinciding with some point of  $A'C$ . We call this point  $h$ . If the axis of the telescope is directed towards this point it will cut the surface of the crystal at  $h$ , the position of which can be determined with reference to the scales. The divisions at  $E$  and  $D$  which appear to coincide can be read off and the distance  $ED$  determined by actual measurement. If  $e$  is the thickness of the crystal, we have

$$ED = EP - DP = e(\tan r' - \tan r),$$

in which  $\tan r$  is known, for the angle of incidence (considering the ray reversed) is equal to the angle which the axis of the telescope makes with the vertical. Moreover,  $\sin i = \mu_o \sin r$  (since the ordinary ray

obeys both laws of refraction for all conditions), therefore  $r$  is known, and  $r'$  may be determined by the above formula. If the value of  $r'$  thus found agrees with the value determined by the formula  $\frac{\tan r'}{\tan r} = \frac{\mu_e}{\mu_0}$ , the experiment will have proved that the section of the extraordinary wave is, in the present case, an ellipse. Inasmuch as we have already proven that the wave-front is a surface of revolution, the experiment will prove that it is a spheroid of axes  $A$  and  $B$ . The experiments made by Malus completely verified this theory.

**Fresnel's Theory of Double Refraction.**—In the foregoing discussion we have considered only uniaxial crystals, making certain assumptions regarding the form of the wave-surfaces, and showing that certain relations deduced from them were verified by experiment.

We will now consider the phenomenon of double refraction in its more general aspect, following the treatment of Fresnel.

As we have seen, the velocity of a transverse wave in an isotropic medium is proportional to  $\sqrt{\frac{e}{d}}$ , in which  $e$  is the elasticity of the medium.

In doubly refracting media  $e$  is assumed to vary with the direction of the displacement, and there will be two directions in every possible plane for which  $e$  has its maximum and minimum values. The corresponding velocities of propagations  $\sqrt{\frac{e_1}{d}}$ ,  $\sqrt{\frac{e_2}{d}}$  are for vibrations parallel to these two directions. If the displacement is in any other direction, the wave is not propagated with an intermediate velocity, as might at first be supposed, but is decomposed into two waves, which travel with the above velocities, the directions of their vibrations being perpendicular to each other. If we are dealing with trains of waves, as is always the case, the actual motions of the vibrating particles will not be along straight lines, for they are the resultants of the two sets of disturbances which are travelling with different velocities. Until the rays become completely separated by the double refraction, we must regard the vibration as changing its type from point to point, changing from plane to elliptical and circular, and then back again to plane, as the relative phases of the two perpendicularly polarized disturbances alter.

If the direction of displacement coincides with one or the other of the two directions of maximum or minimum elasticity, a single plane-polarized wave will be propagated in the medium. From this it is clear that in the case of the changing type of vibration alluded to above, the vibration along a line will never occur in either one of these directions, for if it did, it would be propagated from that point on, as a plane-polarized vibration without further change.

Fresnel arrived at a conception of the wave-surface by considering it as the envelope of an infinite number of plane-waves, which have passed simultaneously in all possible directions, through a given point in the doubly refracting medium.

Consider now the following construction. Through the point in question imagine an infinite number of planes, in all possible orientations, and draw through the point, on each plane, two lines at right

angles to each other, and coinciding with the directions of maximum and minimum elasticity, and of lengths proportional to velocities of propagation of disturbances vibrating parallel to the lines in question.

If the two lines are made to bisect each other at the point, the terminal points of the lines for all the planes taken collectively will lie upon an ellipsoid. This fact can be deduced theoretically, by making certain specifications regarding the medium, but as the deduction will not help us much in understanding the phenomena, we will simply consider it as representing experimental facts. Having the ellipsoid given, it is possible to find the direction of vibration and the velocities of propagation of a plane-wave, by drawing a central section of the ellipsoid parallel to the plane wave-front.

The ellipsoid is called the **ELLIPSOID OF ELASTICITY**.

Let its equation be  $a^2x^2 + b^2y^2 + c^2z = V^2$ , in which  $V$  is the velocity of light in vacuo.

The constants  $a$ ,  $b$ , and  $c$  are related to the elastic properties of the medium, and represent the velocities of waves vibrating parallel to the axes of elasticity, which may be defined as the three directions at any point, along which we can displace the ether, and have the force of restitution parallel to the displacement. In any given *plane* there are but two such directions, in space, however, there are three.

If we take as our unit of time the time occupied by a wave in travelling unit distance in vacuo, then  $V = 1$ . If we put  $x = 0$  in our equation, we obtain the equation of the intersection of the ellipsoid with the  $yz$  plane, which is an ellipse having  $\frac{1}{b}$  and  $\frac{1}{c}$  as semi-axes, and a plane-polarized wave will be propagated along the  $x$  axis with a velocity  $b$ , if the direction of vibration is parallel to  $y$ , or with a velocity  $c$  if it is parallel to  $z$ .

The reciprocals  $\frac{1}{a}$ ,  $\frac{1}{b}$ ,  $\frac{1}{c}$  correspond to refractive indices, and are called the principal refractive indices. If we designate them by  $\mu_1$ ,  $\mu_2$ ,  $\mu_3$ , we can write the equation of the ellipsoid in the form

$$\frac{x^2}{\mu_1^2} + \frac{y^2}{\mu_2^2} + \frac{z^2}{\mu_3^2} = 1.$$

The deduction of the equation of the ellipsoid from a consideration of the elastic properties of the medium is generally accomplished by considering the potential of medium. The following simple method is taken from Schuster's *Optics*:

"Fresnel's method of treating double refraction which led him to the discovery of the laws of wave-propagation in crystalline media, though not free from objection, is very instructive, and deserves consideration as presenting in a simple manner some of the essential features of a more complete investigation. Consider a particle  $P$  attracted to a centre  $O$  with a force  $a^2x$  when the particle lies along  $OX$ , and a force  $b^2y$  when it lies along  $OY$ . The time of oscillation, if the particle has unit mass, is  $2\pi/a$  or  $2\pi/b$  according as the oscillation takes place along the axis of  $X$  or along the axis of  $Y$ . When the displacement has

components both along  $OX$  and  $OY$ , the components of the force are  $a^2x$  and  $b^2y$ , and the resultant force is

$$R = \sqrt{a^4x^2 + b^4y^2}.$$

"The cosines of the angles which the resultant makes with the coordinate axes are  $a^2x/R$  and  $b^2y/R$ . The direction of the resultant force is not the same as that of the displacement, the direction cosines of which are  $x/r$  and  $y/r$ . The cosine of the angle included between the radius vector and the force is found in the usual way to be

$$\frac{a^2x^2 + b^2y^2}{Rr},$$

and the component of the force along the radius vector is

$$(a^2x^2 + b^2y^2)r.$$

"If we draw an ellipse  $a^2x^2 + b^2y^2 = k^2$  (Fig. 203), where  $k$  is a constant having the dimensions of a velocity, the normal to this ellipse at a point  $P$ , having coordinates  $x$  and  $y$ , forms angles with the axes, the cosines of which are in the ratio  $a^2x$  to  $b^2y$ , hence the force in the above problem acts in the direction of  $ON$  of the line drawn from  $O$  at right angles to the tangent at  $P$ . The component of the force along the radius vector is  $k^2/r$ , and the

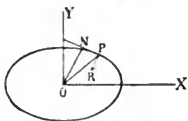


FIG. 203.

force per unit distance is  $k^2/r^2$ , so that if the particle were constrained to move on the radius vector  $OP$ , its period would be  $2\pi r/k$ . The ratio  $r/k$  depending only on the direction of  $OP$ , our result is independent of the particular value we attach to  $k$ .

"If we extend our investigation to three dimensions, the component of attraction along  $OZ$  being  $c^2z$ , we obtain the same result, and the component of force acting along any radius vector  $OP$  per unit length is  $k^2/r^2$ , where  $r$  is the radius drawn in the direction of  $OP$  to the ellipsoid

$$a^2x^2 + b^2y^2 + c^2z^2 = k^2."$$

For a plane-wave to be propagated without alteration it is essential that the effective force of restitution shall be parallel to the displacement.

Though in general this force does not even lie in the plane of the wave-front, we can always resolve it, however, into two components, one in, the other perpendicular to the front. Fresnel neglected the latter component, as it contributes nothing towards the propagation of a transverse wave. The longitudinal disturbance which, in the case of elastic solids, is produced by the normal component, is considered as non-existent in the case of light, owing to the incompressibility of the medium.

The direction of the component of force parallel to the wave-front is along the radius vector of the ellipsoid which is perpendicular to the section conjugate to the direction of the displacement. This will be made clearer by reference to Fig. 204. Let  $abcd$  be a plane-wave travelling within the crystal, the direction of the displacement being parallel to  $ab$ . The ellipsoid is assumed constructed around a point

lying on the wave front, which cuts it in the elliptical cross-section as indicated. The displacement is along  $AO$ , which we will assume to be the semi-major axis of the ellipse, while the direction of the force of restitution is along the radius  $ON$ , perpendicular to the plane  $BOC$ . If the projection of  $ON$  on the plane of the wave-front coincides with the direction of the displacement  $OA$ , the plane  $AON$  must be perpendicular to the wave-front, and since  $ON$  is perpendicular to  $OB$ ,  $OB$  must be perpendicular to  $OA$ ; in other words,  $OA$  and  $OB$  are the axes of the elliptical section. This is the condition which we assumed at the start. If the direction of the displacement is not along one

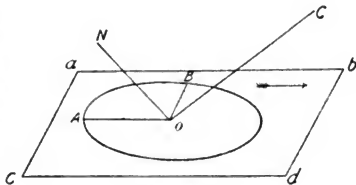


FIG. 204.

of the axes, the effective force of restitution will not be directed parallel to the displacement, and *two* plane-polarized waves will result as we have seen. Two sections of the ellipsoid will be circular, and plane-waves parallel to these sections will be propagated without alteration, whatever may be the direction of the displacement, though there may be a division of the ray, as we shall see presently. These circular sections of the ellipsoid of elasticity are perpendicular to the optic axes of the crystal. We may sum up as follows:

In any given direction in the crystal two systems of plane-waves can be propagated normally, the vibrations being along the axes of the elliptical cross-section, and the velocities of normal propagation inversely proportional to the lengths of the axes. Two directions exist, however, in which but a single wave-front is propagated, known as the axes of single wave-velocity or optic axes. In these directions the velocity of normal propagation of a plane-wave is independent of the direction of vibration, although the direction in which a limited portion of the wave-front travels (the ray direction) depends upon the nature of the vibration, for the ray is not necessarily perpendicular to the wave-front in crystalline media.

We will now investigate the form of the wave surface, which we can do by considering a geometrical construction known as the normal velocity-surface.

**The Normal Velocity Surface.**—Around any point  $O$  within a crystal construct the ellipsoid of elasticity, and consider a system of plane-waves passing simultaneously through  $O$  in all possible directions. We have seen that, in general, a crystal has the property of transmitting only vibrations polarized in a definite direction, and that all other types of vibration are resolved into two components which travel with unequal velocities. We shall thus have two systems of plane-waves passing through the point. To determine the velocities of these waves in different directions we proceed as follows. Let any one of the plane-waves passing through  $O$  cut the ellipsoid in the section  $AOB$  (Fig. 205), of which the axes are  $OA$  and  $OB$ . Draw a normal to the plane at  $O$  and measure off on it distances  $ON$  and  $ON'$ , inversely pro-

portional to the axes  $OA$  and  $OB$ . If now planes are drawn through  $N$  and  $N'$  parallel to the original plane of the section, they will represent the positions of the two waves which passed through the point  $O$  simultaneously, the one having its vibrations parallel to  $OA$  and the other parallel to  $OB$ . If we rotate the plane  $AOB$  around  $O$  in every possible direction, the points  $N$  and  $N'$ , as defined above, will trace out a surface consisting of two sheets termed the surface of normal velocities, any radius vector of which determines the normal velocity of the plane-wave propagated in that direction. Since for two positions of the plane  $AOB$  the section of the ellipsoid is circular, it is

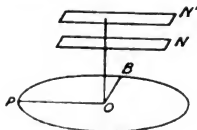


FIG. 205.

obvious that the points  $N$  and  $N'$  will coincide, when the waves are parallel to these sections. In other words, the inner sheet will touch the outer at four points.

This surface is not, however, identical with the wave-surface, which is the surface enveloped by the plane-waves which we have just considered. This family of planes is represented by the equation

$$lx + my + nz = v,$$

in which  $l, m, n$  are the direction cosines of the direction in which the wave travels with a velocity  $v$ , which is, however, a function of  $l, m$ , and  $n$ . We require a relation connecting these quantities. The following treatment is taken from Rayleigh's *Wave-Theory*:

"If  $v$  be the velocity of propagation in the direction  $l, m, n$ , the wave-surface is the envelope of planes  $lx + my + nz = v$ , where  $v$  is a function of  $l, m, n$ , the form of which is to be determined. If  $(\lambda\mu\nu)$  be the corresponding direction of the vibration, then

$$l\lambda + m\mu + n\nu = 0."$$

According to the principles laid down by Fresnel, we see at once that the force of restitution ( $a^2\lambda, b^2\mu, c^2\nu$ ) corresponding to a displacement unity is equivalent to a force  $v^2$  along  $(\lambda\mu\nu)$  together with some force ( $P$ ) along  $(lmv)$ .

Resolving parallel to the coordinate axes, we get

$$lP = a^2\lambda - v^2\lambda, \quad mP = b^2\mu - v^2\mu, \quad nP = c^2\nu - v^2\nu,$$

$$\text{or} \quad \lambda = \frac{lP}{a^2 - v^2}, \quad \mu = \frac{mP}{b^2 - v^2}, \quad \nu = \frac{nP}{c^2 - v^2}$$

Multiplying these by  $l, m, n$  respectively, and remembering the relation  $l\lambda + m\mu + n\nu = 0$ , we obtain

$$\frac{l^2}{a^2 - v^2} + \frac{m^2}{b^2 - v^2} + \frac{n^2}{c^2 - v^2} = 0,$$

an equation which we shall use presently.

**The Wave-Surface.**—If for every possible position of the section  $AOB$  in the construction which we have just considered, we construct planes through  $N$  and  $N'$  parallel to the section, these planes will envelop a surface which consists of two sheets, and resembles in its general appearance the normal velocity-surface which we have just

considered. The surface thus defined is the true wave-surface, representing the form of the wave which we should have if a luminous disturbance started within the body of the crystal.

The equation which represents the system of plane-waves which envelop the wave-surface is

$$lx + my + nz = v,$$

in which  $l$ ,  $m$ ,  $n$  and  $v$  are subject to the conditions

$$\frac{l^2}{v^2 - a^2} + \frac{m^2}{v^2 - b^2} + \frac{n^2}{v^2 - c^2} = 0 \quad \text{and} \quad l^2 + m^2 + n^2 = 1.$$

The equation of the wave-surface was found by Archibald Smith (*Phil. Mag.*, 1838, p. 335) in the following manner:

By differentiation of the three equations above regarding  $l$ ,  $m$ ,  $n$  as variables, we obtain

$$xdl + ydm + zdn = 0,$$

$$\frac{l dl}{v^2 - a^2} + \frac{m dm}{v^2 - b^2} + \frac{n dn}{v^2 - c^2} - \left\{ \frac{l^2}{(v^2 - a^2)^2} + \frac{m^2}{(v^2 - b^2)^2} + \frac{n^2}{(v^2 - c^2)^2} \right\} v dv = 0,$$

$$l dl + m dm + n dn = 0,$$

whence by indeterminate multipliers we obtain

$$(1) \quad x = Al + \frac{Bl}{v^2 - a^2}, \quad (2) \quad y = Am + \frac{Bm}{v^2 - b^2}, \quad (3) \quad z = An + \frac{Bn}{v^2 - c^2},$$

$$(4) \quad Bv \left\{ \frac{l^2}{(v^2 - a^2)^2} + \frac{m^2}{(v^2 - b^2)^2} + \frac{n^2}{(v^2 - c^2)^2} \right\} = 1.$$

Multiplying the first three of these equations by  $l$ ,  $m$ , and  $n$ , and adding, we obtain

$$(5) \quad v = A.$$

By transposing the third terms, squaring, and adding, we get, since

$$r^2 = x^2 + y^2 + z^2,$$

$$r^2 + 2Av + A^2 + B^2 \left\{ \frac{l^2}{(v^2 - a^2)^2} + \frac{m^2}{(v^2 - b^2)^2} + \frac{n^2}{(v^2 - c^2)^2} \right\} = 0,$$

which by (4) and (5) gives us

$$B = v(r^2 - v^2).$$

We now substitute these values of  $A$  and  $B$  in equation (1) and obtain

$$x = lv + lv \frac{r^2 - v^2}{v^2 - a^2} = lv \frac{r^2 - a^2}{v^2 - a^2}; \quad \therefore \quad l = \frac{v^2 - a^2}{r^2 - a^2} \cdot \frac{x}{v},$$

and similarly

$$m = \frac{v^2 - b^2}{r^2 - b^2} \cdot \frac{y}{v}, \quad n = \frac{v^2 - c^2}{r^2 - c^2} \cdot \frac{z}{v}.$$

Substitution of these values in  $lx + my + nz = v$ , the equation of the plane-wave system, gives us the equation of the wave-surface,

$$x^2 \frac{v^2 - a^2}{r^2 - a^2} + y^2 \frac{v^2 - b^2}{r^2 - b^2} + z^2 \frac{v^2 - c^2}{r^2 - c^2} = v^2 = \frac{v^2 x^2}{r^2} + \frac{v^2 y^2}{r^2} + \frac{v^2 z^2}{r^2},$$

whence 
$$\frac{x^2 a^2}{r^2 - a^2} + \frac{y^2 b^2}{r^2 - b^2} + \frac{z^2 c^2}{r^2 - c^2} = 0,$$

or  $(r^2 - b^2)(r^2 - c^2)a^2x^2 + (r^2 - a^2)(r^2 - c^2)b^2y^2 + (r^2 - a^2)(r^2 - b^2)c^2z^2 = 0.$

Multiplying out and dividing by  $r^2$ , we obtain

$$r^2(a^2x^2 + b^2y^2 + c^2z^2) - a^2(b^2 + c^2)x^2 - b^2(c^2 + a^2)y^2 - c^2(a^2 + b^2)z^2 + a^2b^2c^2 = 0.$$

We are now in a position to determine the general form of the wave-surface, which we can do by studying its sections with the planes  $xy$ ,  $xz$ ,  $yz$ . This we can do by making  $x=0$ ,  $y=0$ ,  $z=0$  in succession in the equation of the wave-surface, when we obtain the equations of the curves of section. Assume  $a > b > c$ .

If we make  $z=0$ , we get at once

$$(x^2 + y^2)(a^2x^2 + b^2y^2) - a^2(b^2 + c^2)x^2 - b^2(c^2 + a^2)y^2 + a^2b^2c^2 = 0,$$

or  $(x^2 + y^2 - c^2)(a^2x^2 + b^2y^2 - a^2b^2) = 0,$

which is separately satisfied by

$$x^2 + y^2 = c^2, \text{ a circle of radius } c,$$

and  $a^2x^2 + b^2y^2 = a^2b^2$ , an ellipse of semi-axes  $a$  and  $b$ .

The circle lies wholly within the ellipse, since we have assumed  $c$  less than either  $a$  or  $b$ . Making  $x=0$ , we find the section with the  $yz$  plane to be

$$y^2 + z^2 = a^2, \text{ a circle of radius } a,$$

and  $b^2y^2 + c^2z^2 = b^2c^2$ , an ellipse of semi-axes  $b$  and  $c$ .

In this case the ellipse lies within the circle.

For  $y=0$ , the section with the  $xz$  plane,

$$x^2 + z^2 = b^2, \text{ a circle of radius } b,$$

$$a^2x^2 + c^2z^2 = a^2c^2, \text{ an ellipse of semi-axes } a \text{ and } c.$$

In this case the circle meets the ellipse at four points. The three sections are shown in Fig. 206.

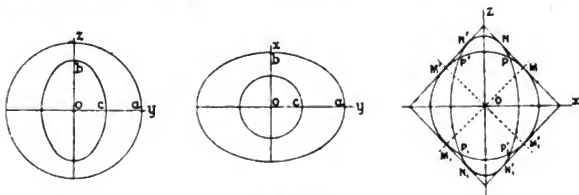


FIG. 206.

A model of the surface can be made by cutting the sections out of cardboard and fitting them together in three perpendicular planes, which can easily be done by cutting them up in a suitable manner, and fastening them together again by means of strips of gummed paper. Such a model is shown in Fig. 207. The inner surfaces should be blackened on both sides of the section as indicated. By a little exercise of the imagination it is easy to see the general form of the

inner and outer sheets, though a still better idea can be obtained from the wire or plaster models, which can be procured from dealers in physical apparatus. The outer sheet has the general form of an ellipsoid with four depressions or pits similar to the pit found on an apple around the point where the stem is inserted, only much shallower. At these four points the two sheets come in contact, and some very remarkable optical phenomena are associated with this peculiar condition, which we will now investigate.

**The Optic Axes or Axes of Single Wave-Velocity.**—Consider now the  $xz$  section of the wave-surface, in which the curves intersect at four points and have four common tangents, one of which is represented by  $MN$  (Fig. 208). Planes passing through these tangents and



FIG. 207.

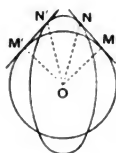


FIG. 208.

perpendicular to the plane of the section, are tangent planes to the wave-surface. They touch the surface, moreover, not at two points, as was imagined by Fresnel, but all around a circle of contact, a condition which can be represented by pressing a flat card against the dimple on an apple. This was first proved by Sir William Hamilton, who predicted from it the remarkable phenomena of internal conical refraction, which we shall consider presently. The lines  $OM$ ,  $OM'$  perpendicular to the tangent planes are the directions in which a single wave only is propagated, for the planes  $MN$  and  $M'N'$  touch both sheets. These directions are therefore the optic axes of the crystal.

**Internal Conical Refraction** —Huygens's construction may be applied to determine the direction of the refracted rays, the points on the surface of the crystal becoming centers of wave-surfaces of the form which we have just studied. If light is incident upon the crystal in such a direction that the refracted wave-front is parallel to  $MN$  or  $M'N'$  (Fig. 208) any line joining the center  $O$ , with the circle of contact of  $MN$  with the wave-surface, is a possible direction of the refracted ray. The direction of the refracted ray will depend on the direction of the vibration in the incident wave-front. The type of the vibration will not be altered by the crystal, since the wave-front is moving parallel to an optic axis, but the direction of the ray will depend on the plane of polarization. If the incident light is polarized in all possible plans, i.e. unpolarized, the ray upon entering the crystal will open out into a cone, each elementary ray of the cone being plane-

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polarized. This result was predicted from theory by Sir William Hamilton and verified by Lloyd (*Trans. Roy. Irish Ac.*, vol. xvii., p. 145, 1833) with a plate of aragonite cut so that its faces were equally inclined to the two optic axes.

A divergent cone of light from a screen  $AB$  (Fig. 209) perforated with a very small hole, upon which sunlight is concentrated by means of a lens, is intercepted by a second perforated screen  $CD$ . This screen can be moved about over the surface of the crystal, and serves to isolate a narrow pencil from the divergent cone. In general, if the transmitted light is received upon a screen at  $E$ , two spots of light appear, but by moving the screen  $CD$  about, it is possible to find a position such that the two spots run together into a ring of light, the diameter of which is independent of the distance of the screen  $E$  from the lower face of the crystal plate. This proves that the rays

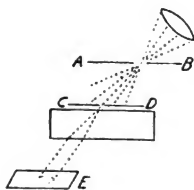


FIG. 209.

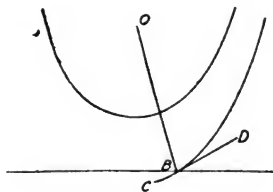


FIG. 210.

leave the plate in a parallel direction, notwithstanding their strong divergence within the crystal. The angle of the cone was found to be  $1^{\circ} 50'$ , while the value calculated was  $1^{\circ} 55'$ , a very close agreement between theory and experiment.

**Axes of Single Ray Velocity. External Conical Refraction.**—The directions determined by joining the point  $O$  (Fig. 208) with the points at which the two sheets of the wave-surface meet are termed the axes of single ray velocity. At each one of the conical points or pits an infinite number of tangent planes can be drawn to the surface, which collectively form a tangent cone. A rough model of such a cone can be made by cutting a paper circle along a radius and then pasting the edges together, making them overlap a little. This cone fits into the conical depressions of the wave-surface.

Suppose now that a ray is travelling within the crystal along the axis of single ray velocity, and emerges from the surface of the crystal. The direction of the ray after refraction out into the air is determined by the position of the plane tangent to the element of wave-surface corresponding to the ray. For example, suppose we are dealing with a simple spheroidal wave starting at  $O$  within the crystal (Fig. 210). We wish to determine the direction of the ray  $OB$  after emergence. This direction will be that traversed by a plane-wave  $CD$  tangent to the spheroid at  $B$ . In other words, the small element of the wave at  $B$  can be considered as a portion of the tangent plane. The direction of the refracted ray is thus seen to be determined by

the position of the plane tangent to the wave-front at the point where it intersects the surface.

Now a ray travelling along an axis of single ray velocity has an infinite number of tangent planes which envelope a cone, and the refracted ray may pursue a direction determined by any one of them.

It will therefore open out into a hollow cone, and if the light be received upon a screen we shall see a ring, which increases in diameter as the distance from the crystal face is increased. The phenomenon is exhibited by concentrating a pencil of rays upon the surface of the crystal. This converging system of rays contains the hollow cone of rays which we should have if we transmitted a ray up through the crystal along the axis of the single ray velocity. The cone is indicated by solid lines (Fig. 211), the superfluous rays which pursue other paths in the crystal being indicated by dotted lines. A screen perforated with a small hole limits the emergent light to the ray which has traversed the plate in the direction of the single ray axis, and if the beam which issues is received upon a white screen it is seen to have the form of a hollow cone.

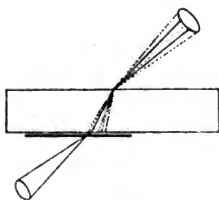


FIG. 211.

**Crystal Plates in Convergent or Divergent Polarized Light. Colors of Thin Crystal Plates.**—In the chapter on Elliptical Polarization we have seen that a beam of plane-polarized light falling upon a crystal plate cut perpendicular to the optic axis (for example, a film of mica or selenite) is in general doubly refracted, that is, the incident vibration is resolved into two mutually perpendicular vibrations, one polarized in, and the other at right angles to the principal plane. The two disturbances traverse the crystal with different velocities, and consequently emerge with a difference of phase depending upon the thickness of the crystal plate. The plane vibration on entering the medium becomes transformed into an elliptical vibration, owing to the different velocities of the two rectangular components. As the disturbance proceeds its type changes, becoming circular, elliptical, and plane in succession, each plane phase being turned through  $90^\circ$  with respect to the phase immediately preceding or following. It is obvious that if the plate is thick, and the two rays become separated by double refraction, each ray will be plane polarized, that is, we shall no longer have a circular and an elliptical type. If white light falls upon the plate the difference of phase at emergence of the two components will vary with the wave-length, certain colors, for example, emerging plane polarized parallel to the original plane of polarization; other colors polarized at right angles to it. Certain colors will, therefore, be absent when the emergent light is examined with Nicol prism held with its principal plane parallel to the principal plane of the polarizing Nicol. On rotating the Nicol through  $90^\circ$  each color changes to its complementary tint for obvious reasons. The state of polarization for waves of length intermediate between those specified will, in general, be of the circular or elliptical type. We will now derive an expression

for the intensity of the illumination as a function of the position of the polarizer and analyzer and the phase-difference between the two emerging streams. Let the principal plane of the polarizer be parallel to  $OD$  (Fig. 212), and the principal plane of the analyzer parallel to  $OA$ . Since the vibrations are parallel to the principal plane of the Nicol it is clear that the incident light will vibrate parallel to  $OD$ . On entering the plate it is resolved into two components vibrating at right angles along  $OX$  and  $OY$ , where  $OX$  and  $OY$  are the two directions in the crystal in which vibrations may occur without change

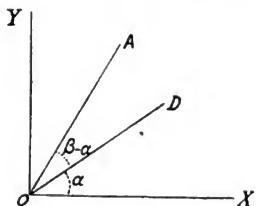


FIG. 212.

of type. Let the incident vibration be represented by  $Y = A \sin \omega t$ . The vibrations in the crystal will then be  $A \cos \alpha \sin \omega t$  and  $A \sin \alpha \sin \omega t$ , along  $OX$  and  $OY$ , where  $DOX = \alpha$ .

These two disturbances on emerging will have a difference of phase which we will represent by  $\delta$ . The vibrations, therefore, take the form  $A \cos \alpha \sin \omega t$  and  $A \sin \alpha \sin(\omega t + \delta)$ . The analyzing Nicol resolves these vibrations parallel to its principal plane  $OA$ , transmitting one component and suppressing the other. If  $AOX = \beta$  we have two vibrations parallel to the plane of the analyzer, one along  $OA$  represented by

$$A \cos \alpha \cos \beta \sin \omega t,$$

contributed by the  $OX$  component, and another also along  $OA$  represented by

$$A \sin \alpha \sin \beta \sin(\omega t + \delta).$$

These two combine into the resultant vibration

$$y = A \cos \alpha \cos \beta \sin \omega t + A \sin \alpha \sin \beta \sin(\omega t + \delta).$$

The intensity is represented by the square of this quantity, which reduces to

$$I = A^2 \left\{ \cos^2(\alpha - \beta) - \sin 2\alpha \sin 2\beta \sin^2 \frac{\delta}{2} \right\},$$

where  $\alpha - \beta$  is the angle between the principal planes of the polarizer and the analyzer.

If we are working with white light  $\delta$  will vary with the wave-length, and if  $A$  also varies with the wave-length the general expression for the intensity is

$$I = \cos^2(\alpha - \beta) \Sigma A^2 - \sin 2\alpha \sin 2\beta \Sigma A^2 \sin^2 \frac{\delta}{2}.$$

The first term is seen to be independent of  $\delta$  the phase-difference, and will therefore have no effect in producing color in the image; and the transmitted light will therefore consist of two parts, one of which is white, depending on the first term, and the other colored to a greater or less extent, depending on the second. If we rotate the plate around its normal, the Nicols remaining fixed, the colors will be affected in the same proportion, and the tint of the emerging light

therefore remain unaltered, except that it will be diluted to a greater or less extent with white light arising from the first term. The colors will be most intense when  $\alpha - \beta = 90^\circ$  and least intense when  $\alpha - \beta = 0$ , the former case corresponding to crossed Nicols, and the latter to parallel. In both cases the effects are most pronounced when  $\alpha = 45^\circ$ ; in other words, when the principal planes of the polarizer and analyzer bisect the angle between the principal planes of the plate.

**Colors of Crystal Plates in Convergent and Divergent Polarized Light.**—A remarkable series of phenomena are presented when we examine crystal plates, cut in various ways, in a beam of strongly convergent or divergent light. Colored fringes of varied forms appear crossed by dark crosses and brushes, the variety being almost as great as in the kaleidoscope. A complete investigation of the forms which occur under all possible conditions is hardly profitable, and we will examine a few typical cases only.

The simplest form of polariscope for viewing the rings and crosses in convergent light is the tourmaline tongs. When the crystal plate is placed between the tourmalines, and the eye brought close to the apparatus, which is directed to a brilliant light of large size, such as the sky, the rays which enter the pupil have traversed the crystal in the form of a cone of wide aperture, as shown in Fig. 213. Tourmaline crystals are, however, usually so strongly colored, that only an imperfect idea of the color distribution can be obtained in this way, though the general form of the fringes can be made out. It is therefore customary to use some such arrangement as that shown in the lower part of the figure.

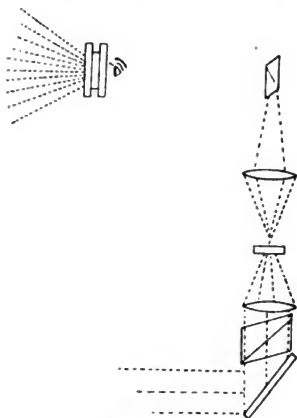


FIG. 213.

**Uniaxial Crystal Cut Perpendicular to Axis.**—Consider what happens when a cone of plane-polarized rays diverging from  $S$  (left hand Fig. 214) passes through a crystal plate, the central ray  $SO$  of the cone coinciding with the optic axis. Consider the source  $S$  in front of the plane of the paper, and let the vibrations be vertical. The ray incident at  $O$  passes through the plate in the direction of the optic axis, and its vibration plane remains unaltered. Other rays in general will suffer double refraction, and emerge with a phase-difference between the components of the vibration. This will not be true, however, for certain rays. Consider the ray  $SP$ . The direction of vibration is in the principal plane  $SOP$ , i.e. the plane containing the ray and the optic axis, and it will therefore be transmitted by the crystal without resolution. The same is true for the ray  $OP'$ , since in this case the vibration is perpendicular to the principal plane. Hence all

rays striking the crystal plate along  $OP$  or  $OP'$  or their prolongations, will not suffer double refraction, and will be wholly transmitted or completely stopped by a Nicol held behind the plate, according as its principal plane is vertical or horizontal. Consider now a ray incident at some other point, say  $Q$  (right-hand Fig.). The vibration  $a$  will be resolved into two components,  $b$  and  $c$ , one lying in the

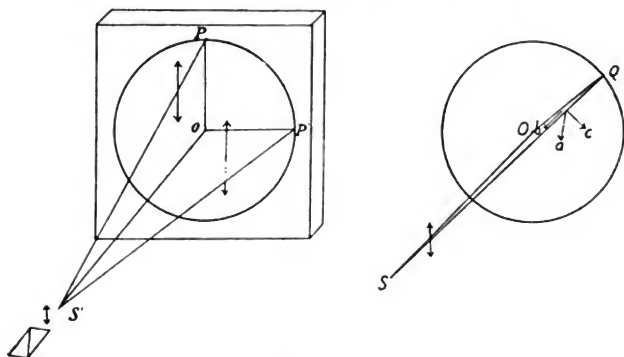


FIG. 214.

principal plane  $SOQ$  and the other perpendicular to it. The vibrations will traverse the crystal with different velocities, and emerge with a difference of phase, which will depend upon the thickness traversed, and also with the wave-length of the light. Now the thickness traversed will increase as we pass from  $O$  to  $Q$ , owing to the increasing obliquity of the rays. The phase-difference of the emergent components will therefore vary along the line  $OQ$ , and the emergent light at some points on this line will be polarized in the same plane as the incident light, at other points in a plane perpendicular to it. The analyzing Nicol will quench one or the other, according to its position. By symmetry the conditions of equal phase difference will occur along concentric circles with a common center at  $O$ . We shall accordingly see bright and dark circles surrounding  $O$  if the light is monochromatic, and colored fringes if it is white. These circles will, however, be interrupted

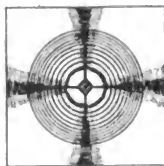


FIG. 215.

along the lines  $OP$  and  $OQ$  (for reasons above specified) by a cross which appears bright or dark according to the position of the analyzing Nicol (Fig. 215).

**Isochromatic Surfaces.**—The characteristic of a fringe is that the retardation  $\delta$  is constant along its length, and the locus of points in space for which  $\delta$  is constant is called an isochromatic surface. For every value of  $\delta$  there will be a corresponding surface, and if we describe these surfaces around  $S$  as an origin, with retardations

of 1, 2, 3, 4, etc., half-wave lengths, the intersections of these surfaces with the second surface of the crystal will determine the isochromatic lines or fringes. The form of the surface was worked out by Bertin<sup>1</sup> in the following way:

We may suppose the source from which the rays diverge located on the surface of the crystal.

Let  $O$  be the source: then the time occupied by the two disturbances in traversing  $OP$  will be  $\frac{OP}{v_o}$  and  $\frac{OP}{v_e}$  for the ordinary and extraordinary disturbances. The time retardation is therefore

$$t_o - t_e = OP \left( \frac{1}{v_o} - \frac{1}{v_e} \right),$$

and the phase retardation

$$\frac{2\pi}{T} (t_o - t_e) = \frac{2\pi}{T} \left( \frac{1}{v_o} - \frac{1}{v_e} \right) OP.$$



FIG. 216.

The wave-surface consists of a sphere of radius  $b$ , and a spheroid of which the generating curve is the ellipse

$$a^2 x^2 + b^2 y^2 = a^2 b^2.$$

If  $r$  be a radius vector of this curve, we have  $v_o$  proportional to  $b$  and  $v_e$  proportional to  $r$ , and the time retardation is, for a thickness  $\rho$ ,

$$\delta = \rho \left( \frac{1}{b} - \frac{1}{r} \right) = \rho \left( \mu_o - \frac{1}{r} \right).$$

If we write the equation of the ellipse in the form

$$\mu_o^2 x^2 + \mu_e^2 y^2 = 1,$$

we have

$$\frac{1}{r^2} = \mu_o^2 \cos^2 \theta + \mu_e^2 \sin^2 \theta,$$

which, if we combine with the equation for  $\delta$ , gives us

$$\frac{1}{r^2} = \left( \frac{\delta}{\rho} - \mu_o^2 \right),$$

$$\left( \frac{\delta}{\rho} - \mu_o^2 \right)^2 = \mu_o^2 \cos^2 \theta + \mu_e^2 \sin^2 \theta,$$

$$(\delta - \rho \mu_o)^2 = \mu_o^2 x^2 + \mu_e^2 y^2,$$

and since  $\rho^2 = x^2 + y^2$ ,

$$\{(\mu_e^2 - \mu_o^2)y^2 - \delta^2\}^2 = 4\mu_o^2 \delta^2 (x^2 + y^2),$$

which is the generating curve of the isochromatic surface, which we form by rotating the curve around the optic axis. Its general form is shown in Fig. 217. Its sections with the surface of a plate cut perpendicular to the axis are circles, with a plate parallel to the axis hyperbolae.

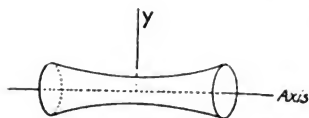


FIG. 217.

<sup>1</sup> *Ann. de Chim. et de Phys.*, lxiii., p. 57, 1861.

**Isochromatic Surfaces in Biaxial Crystals.**—The form of the surface in biaxial crystals is shown in Fig. 218. A section parallel to the plane

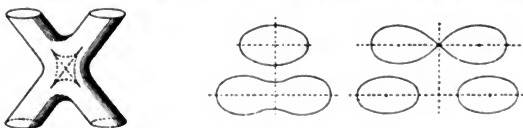


FIG. 218.

containing the axes gives us curves closely resembling hyperbolae. A section perpendicular to the bisector of the angle between the optic axes gives us a family of lemniscates. Sections in planes along  $a$ ,  $b$ ,  $c$ ,  $d$  give us fringes of the form shown on the right-hand side of the figure. These different curves correspond to successive values of  $\delta$ , and they may all be seen simultaneously, as in Fig. 219.

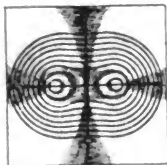


FIG. 219.

The region of constant illumination, which in the case of uniaxial crystals had the form of a cross, in the case of biaxial crystals presents the appearance of a double brush of hyperbola form. The subject of the various modifications which the fringes and brushes may undergo is a very large one, but its study teaches us very little regarding the phenomenon of double refraction, the problems being purely geometrical. We shall examine but one other case, the remarkable transformation of a biaxial into a uniaxial crystal resulting from an elevation of temperature.

**Position of Axes as a Function of Temperature.**—A remarkable phenomenon occurs when certain biaxial crystals are heated, for as the temperature rises the angle between the optic axes becomes decreased until the axes finally coalesce, the crystal becoming uniaxial. An oblique section of selenite is usually used for exhibiting the phenomenon. As the plate is warmed the lemniscates close in, the centers approaching, and presently meeting, at which stage the isochromatic fringes are circles crossed by a rectangular cross. A further elevation of temperature causes the axes to cross one another, so to speak, the crystal becoming again biaxial. The experiment makes one of the most beautiful lantern demonstrations ever devised.

**Phenomena exhibited by Twin Crystals.**—Calcite is sometimes found with one or more layers crystallized in opposite directions. Such crystals sometimes show the rings and crosses without either polarizer or analyzer, the front and back parts of the crystal taking their place, and the oppositely crystallized plane serving as the thin film. A slice from a nitre crystal frequently exhibits four systems of rings.

If a crystal possesses rotating power, still further complications result, notable among which are the beautiful spirals described by Airy and named after him. They appear when plates of right and left-handed quartz cut perpendicular to the axis are superposed and viewed in convergent light. Or a single plate may be made to exhibit them if

it is placed on the lower mirror of the Nörremberg polariscope, on account of the reversal of the rotation.

**Convergent Circular Light.**—If a quarter-wave plate is interposed between the first Nicol and the crystal plate in a converging polariscope, the appearances are completely altered. As we should expect, the black cross disappears almost completely, the arms being replaced by thin lines of nebulous grey, which rotate with the analyzer without changing in appearance. The rings in adjacent quadrants are dislocated as shown in Fig. 220, the light rings in one quadrant being opposite the dark ones in the next.

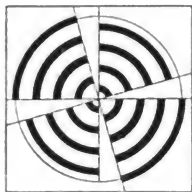


FIG. 220.

The explanation of this can easily be found by working out the resolution of the circular vibration in two opposite quadrants, taking care to distinguish between components parallel to the radii and those perpendicular to them.

**Double Refraction in Non-Crystal Media.**—Many of the phenomena of double refraction can be observed in isotropic substances subjected to strain, or to sudden differences of temperature. Glass plates squeezed in a vice (Fig. 221) and viewed between crossed Nicols exhibit most beautifully coloured fringes, the lines of strain being clearly indicated

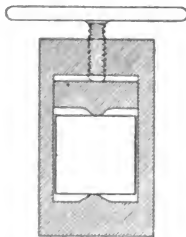


FIG. 221.

Tyndall found that a long strip of glass, thrown into sonorous vibration, restored the light when placed between crossed polarizing prisms. The experiment has since been modified in a beautiful manner, the transmitted light being examined in a revolving mirror and found to be restored periodically, the band appearing broken up into beads, showing that the double refraction was coincident with the vibration.

On inserting a selenite plate the band was found to vary in color.

A permanent strained condition can be established by heating a block of glass nearly to a red heat and cooling it suddenly. Polarized light is an extremely sensitive test for imperfect annealing. Prince Rupert drops make excellent objects. They can be easily prepared by melting the end of a glass rod in a powerful blast-lamp, and allowing the drop to fall into a bowl of water with some filter paper on the bottom. Four out of five fly to pieces, but with a little practice a number can be prepared in a short time. They are best viewed by immersing them in a small rectangular cell of glass filled with a mixture formed by dissolving about ten parts of chloral hydrate in one part of hot glycerine. This mixture has the same refractive index as the glass.

## CHAPTER XI.

### CIRCULAR AND ELLIPTICAL POLARIZATION.

IN the case of plane-polarized light the vibration of the ether is linear, as we have seen. We will now consider another type of polarization, in which the ether particle moves in a circular or elliptical orbit. Such a vibration results when two rectangular vibrations, of the same period but differing in phase, are simultaneously impressed upon a point.

If the amplitudes are the same and the phase-difference an odd number of quarter periods, we shall have a circular vibration which is right or left-handed according to the circumstances. This can be easily shown by means of the circular pendulum: suspend a weight by a string and strike the weight a blow in any direction: a linear vibration results. Strike a second blow, at right angles to the direction of the first, and a quarter period later, *i.e.* when the weight has reached its position of greatest displacement, and the linear vibration will be replaced by a circular one. If we delay the second blow until a half period has elapsed, the resultant motion will be linear, but in a direction making an angle of  $45^\circ$  with the original direction, while if we wait until three quarters of a period have elapsed, we again get the circular vibration, but in an opposite direction.

We have cases precisely similar to the above when plane-polarized light is transmitted through a thin crystalline plate which is doubly refracting. The incident vibration is in general decomposed into two rectangular vibrations which traverse the plate with different velocities, and consequently emerge with a phase-difference depending on the thickness of the plate. If the plate is very thick, the two components are completely separated and emerge plane polarized, but in the case of very thin plates the components emerge without appreciable separation, and compound into a vibration which may be circular, elliptical, or linear according

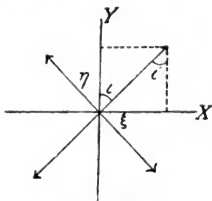


FIG. 222.

to the path-difference within the plate, and the amplitudes of the two components. The circular vibration results only when the amplitudes are equal and the path-difference is  $(2n + 1)\frac{\lambda}{4}$ .

Let the  $x$  and  $y$  axes (Fig. 222) represent the directions of the vibrations of the extraordinary and ordinary ray in the crystal plate, and let the incident vibration of amplitude  $a$  be represented by the arrow making an angle  $i$  with the  $y$  axis. The incident vibration is represented by the equation  $\sigma = a \sin 2\pi \frac{t}{T}$  and the projections of  $\sigma$ , the displacement along the  $x$  and  $y$  axes, by

$$\xi = a \sin i \sin 2\pi \frac{t}{T},$$

$$\eta = a \cos i \sin 2\pi \frac{t}{T}.$$

The projections on  $x$  and  $y$  after passage through the plate are given by

$$\xi = a \sin i \sin 2\pi \left( \frac{t}{T} - \frac{E}{\lambda} \right),$$

$$\eta = a \cos i \sin 2\pi \left( \frac{t}{T} - \frac{O}{\lambda} \right),$$

in which  $E$  and  $O$  are the reduced paths, *i.e.* the thicknesses of the two air films which would be traversed in the same times by the extraordinary and ordinary rays, as the times occupied by the rays in traversing the crystal plate.

These equations can be written in the form

$$\xi = a \sin i \sin 2\pi \left( \frac{t}{T} - \frac{O}{\lambda} + \frac{O-E}{\lambda} \right),$$

$$\eta = a \cos i \sin 2\pi \left( \frac{t}{T} - \frac{O}{\lambda} \right).$$

**Plane Polarization of the Emergent Light.**—The light on entering the plate is decomposed into the components parallel to  $x$  and  $y$ . For the resultant to be plane polarized  $\frac{\eta}{\xi}$  must be a constant, *i.e.* independent of the time. This occurs for any thickness of plate when  $i=0$  and when  $i=90$ , the disturbance being propagated in these two cases without change. For all other values of  $i$  we have the condition for plane-polarized emergent light given by the equation

$$\frac{\sin 2\pi \left( \frac{t}{T} - \frac{O}{\lambda} + \frac{O-E}{\lambda} \right)}{\sin 2\pi \left( \frac{t}{T} - \frac{O}{\lambda} \right)} = K$$

$$\begin{aligned} \text{or} \quad \sin 2\pi \left( \frac{t}{T} - \frac{O}{\lambda} \right) \cos 2\pi \frac{O-E}{\lambda} + \cos 2\pi \left( \frac{t}{T} - \frac{O}{\lambda} \right) \sin 2\pi \frac{O-E}{\lambda} \\ = K \sin 2\pi \left( \frac{t}{T} - \frac{O}{\lambda} \right). \end{aligned}$$

This equation is satisfied for all values of  $t$  only when

$$\sin 2\pi \frac{O-E}{\lambda} = 0 \quad \text{or} \quad O-E = n \frac{\lambda}{2},$$

the path-difference being a whole number of half wave-lengths.

If  $n$  is even  $\frac{n}{\xi} = \cot i$ , and the emergent light is polarized in a plane parallel to the original plane of vibration.

If  $n$  is odd  $\frac{n}{\xi} = -\cot i$ , and the emergent light is plane polarized in azimuth  $2i$ , the vibration being represented by the dotted arrow.

**Circular Polarization of the Emergent Light.**—This occurs when  $i = 45^\circ$  and  $O-E = (2n+1)\frac{\lambda}{4}$ .

This makes

$$\sin i = \cos i = \frac{1}{2}\sqrt{2}, \quad \cos 2\pi \left( \frac{O-E}{\lambda} \right) = 0, \quad \text{and} \quad \sin 2\pi \left( \frac{O-E}{\lambda} \right) = 1.$$

Substituting, we have

$$\xi = \pm \frac{a}{2} \sqrt{2} \cos 2\pi \left( \frac{t}{T} - \frac{O}{\lambda} \right),$$

$$\eta = \frac{a}{2} \sqrt{2} \sin 2\pi \left( \frac{t}{T} - \frac{O}{\lambda} \right).$$

Squaring and adding, these two equations give us  $\xi^2 + \eta^2 = \frac{1}{2}a^2$ , the equation of a circle of radius  $\sqrt{\frac{a^2}{2}}$  = projection of original amplitude on  $x$  and  $y$ .

**Intensity of Circularly Polarized Light.**—The intensity of plane-polarized light is as the square of the amplitude. We will now find an expression for the intensity of circularly polarized light. As we shall see presently, when plane-polarized light is transformed into circular light the intensity remains unaltered. This means that the intensity is measured by twice the square of the radius of the circle as defined above, or by twice the square of the amplitude of one of the plane-polarized components. When therefore we add two rectangular disturbances together to produce circular light we get double illumination, exactly as when we add the effects of two independent sources of light.

**Production and Properties of Circular Light.**—The easiest method of producing circularly polarized light, is by means of a plate of mica of such thickness that the path-difference between the ordinary and extraordinary rays is a quarter of a wave-length, the proper thickness for yellow light being .032 mm. Such a plate is called a **quarter-wave plate**, which we shall hereafter speak of as a  $\frac{\lambda}{4}$  plate. These plates can

be prepared without difficulty by splitting a good quality of mica by means of a needle into the thinnest possible sheets, and selecting such as completely restore the light when they are placed in the proper azimuth between a pair of crossed Nicols.

The thickness of the plates should be measured with a micrometer caliper or spherometer, as otherwise the mistake may be made of getting the plate three times too thick, the optical effects produced (with sodium light) by a  $\frac{3}{4}\lambda$  plate being similar in appearance. The two directions on the plate parallel to vibrations which are propagated without change should be marked. They can be easily found by holding the plate between two crossed Nicols, in such an azimuth that the field appears dark. The directions in question will then be parallel and perpendicular to the longer diagonal of the field of the analyzing Nicol. It is also important to know which of the two directions corresponds to the greater retardation. Singularly enough this point has been very generally neglected by text books. In fact I have failed to find any mention of it anywhere.

If the plate is mounted with its principal directions vertical and horizontal respectively, in front of one of the silvered mirrors of a Michelson interferometer and the fringes found with white light, it is not difficult to determine the direction corresponding to the faster propagation. The central black fringe is brought upon the cross hair of the telescope in which the fringes are viewed, and the light passed through a Nicol before it reaches the instrument. It will be found that a shift of  $\frac{1}{2}$  a fringe width occurs when the vibration plane is changed from horizontal to vertical. If this shift is in the same direction as the shift originally produced by the introduction of the mica plate, it means that the retardation has been increased by changing the direction of the vibration from horizontal to vertical, consequently the vertical direction in the plate is the direction in which the slower component vibrates. This direction should be marked "Slow," the other "Fast."

As this method involves some trouble, the following, based on observations made with a plate previously tested as above, will be found simpler:

A Nicol prism is mounted in front of a sodium flame with its short diagonal turned in the direction in which the hands of a clock move, through an angle of  $45^\circ$  from the vertical. The light polarized in azimuth  $45^\circ$  is then reflected from a polished metal surface, *e.g.* silver or speculum metal, at an angle of about  $60^\circ$ , which introduces a phase-difference between the components of nearly a quarter of a period (the component perpendicular to the plane of incidence being retarded).

If the light is then passed through the quarter-wave plate and an analyzing Nicol, it will be found that it can be extinguished by the latter, that is the quarter-wave plate reduces the nearly circular vibration into a plane vibration. The plane of this vibration, which is given by the *long diagonal* of the analyzing Nicol when set for complete darkness, makes an angle of  $45^\circ$  with the two directions of vibration which we have marked

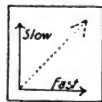


FIG. 223.

on the  $\frac{\lambda}{4}$  plate; this direction is indicated by the dotted arrow in Fig.

223, the directions of vibration of the fast and slow disturbances being as shown. We have then merely to fit this diagram to our  $\frac{\lambda}{4}$  plate,

making the dotted arrow coincide with the direction of vibration of the plane-polarized emergent ray.

For the present we will not concern ourselves with the explanation of why this method enables us to distinguish between the ordinary and extraordinary rays. The reason will become clear after the study of the direction of revolution of circularly polarized light. If plane-polarized light is passed through the  $\frac{\lambda}{4}$  plate, with its plane of vibration making an angle of  $45^\circ$  with the two principal directions, it will be found to suffer very little change in intensity when examined with a slowly rotating analyzer. In this respect it resembles ordinary unpolarized light. It may be distinguished from the latter, however, by passing it through a second  $\frac{\lambda}{4}$  plate, which, by bringing the retardation between the components up to  $\frac{\lambda}{2}$ , converts it into plane-polarized light, which can be extinguished with a Nicol. It also shows brilliant colors in the designs made up of thin flakes of selenite, when the latter are viewed through a Nicol or other analyzer.

**Circular Polarization by Total Reflection. Fresnel's Rhomb.**—When light polarized in a plane making an angle of  $45^\circ$  with the plane of incidence is totally reflected at an angle of  $54^\circ$ , the two reflected components have a phase-difference of one-eighth of a period (for glass-air reflection). Two such reflections give the required  $\frac{\lambda}{4}$  difference, and produce circular polarization. In the case of total internal reflection, the phase of the component of vibration parallel to the plane of incidence is retarded  $135^\circ$ , or a total retardation of  $270^\circ$  for two reflections. This is virtually the equivalent of an *acceleration* of  $90^\circ$ , and we can so consider it in all experimental work. (See Lord Kelvin's *Baltimore Lectures*, page 401.)



FIG. 224.

This phenomenon will be more completely discussed in the article on the theory of reflection, and for the present we shall merely assume the fact to be true. Fresnel constructed a rhomb of glass to verify his calculations of the effect of total reflection upon plane-polarized light, and found that after two internal reflections at an angle of  $54^\circ$ , as shown in Fig. 224, the light emerged circularly polarized.

A rhomb of this description can be easily made out of a rectangular piece of thick plate-glass, the dimensions of which should be in about the proportion 1:2:3. The plate-glass employed should be as thick as possible. It is usually possible to get strips of glass an inch or two in width and an inch thick, which have been trimmed from large plates. These make excellent rhombs, though equally good results on a smaller scale can be obtained with pieces cut from quarter-inch plate. The ends of the block are to be ground down on a grindstone

to an angle of  $54^\circ$ , as shown by the dotted lines in Fig. 224. With quarter-inch plate this can be done in a short time, but if the very thick plate is employed it is better to saw off the ends with a mineralogist's saw, as the slow grinding is very tedious. Small pieces of thin plate-glass, cemented to the rough ground ends of the rhomb with "boiled-down" Canada balsam, make an excellent substitute for polished faces, and save several hours of labor.

If the light entering one of the oblique faces of the rhomb is polarized in a plane making an angle of  $45^\circ$  with the plane of incidence, the emergent light will be freely transmitted by a Nicol in every azimuth. If, however, thin mica or selenite films are interposed between the rhomb and the analyzer, they will show brilliant colors, which is not the case when ordinary light is used.

The Fresnel rhomb has an advantage over the  $\frac{\lambda}{4}$  plate, for the phase-difference between the rectangular components is nearly independent of the wave-length, which is not so in the case of the mica film.

**Elliptical Polarization of the Emergent Light.**—Suppose  $i$  to have some value between  $0$  and  $45^\circ$ , and  $O - E = (2n + 1)\frac{\lambda}{4}$ . This is the case of the quarter-wave plate, with the light polarized in such a plane as to give neither plane nor circularly polarized light.

The components of displacement along the  $x$  and  $y$  axes are then

$$\xi = \pm a \sin i \cos 2\pi \left( \frac{t}{T} - \frac{O}{\lambda} \right),$$

$$\eta = a \cos i \sin 2\pi \left( \frac{t}{T} - \frac{O}{\lambda} \right),$$

which give by squaring and adding

$$\frac{\xi^2}{\sin^2 i} + \frac{\eta^2}{\cos^2 i} = a^2,$$

the general equation, the two which we have already discussed being special cases.

This equation shows us that the vibration is an elliptical one, the axes of the ellipse being parallel to  $x$  and  $y$ , and proportional to  $\sin i$  and  $\cos i$  respectively.

Let  $i = 30^\circ$ , then  $\sin i = \frac{1}{2}$  and  $\cos i = \frac{1}{2}\sqrt{3}$ , and we have

$$4\xi^2 + \frac{4}{3}\eta^2 = a^2.$$

The major and minor axes are then  $\sqrt{\frac{3}{4}}a$  and  $\sqrt{\frac{a^2}{4}}$  respectively, and since the sum of their squares is equal to  $a^2$ , and the intensity of the plane-polarized light is not changed by converting it into elliptical light, we have the *intensity* represented by the sum of the squares of the major and minor axes. The more general equation we obtain by

considering our plate of any thickness, in which case we have the components

$$\xi = a \sin i \sin 2\pi \left( \frac{t}{T} - \frac{O}{\lambda} \right) \cos 2\pi \frac{O-E}{\lambda} + a \sin i \cos 2\pi \left( \frac{t}{T} - \frac{O}{\lambda} \right) \sin 2\pi \frac{O-E}{\lambda},$$

$$\eta = a \cos i \sin 2\pi \left( \frac{t}{T} - \frac{O}{\lambda} \right).$$

Eliminating  $t$  from this equation gives us

$$\eta^2 \frac{1}{a^2 \cos^2 i} + \frac{\left( \xi \cos i - \eta \sin i \cos 2\pi \frac{O-E}{\lambda} \right)^2}{a^2 \cos^2 i \sin^2 i \sin^2 2\pi \frac{O-E}{\lambda}} = 1,$$

or

$$\xi^2 \cos^2 i + \eta^2 \sin^2 i - 2\xi\eta \sin i \cos i \cos 2\pi \frac{O-E}{\lambda} = a^2 \sin^2 i \cos^2 i \sin^2 2\pi \frac{O-E}{\lambda},$$

the equation of an ellipse of which the axes are parallel and perpendicular to the principal section *only when*  $O-E = (2n+1)\frac{\lambda}{4}$ . The use of a mica plate of some other thickness gives us an elliptical vibration, the axes of which are inclined to the original direction of vibration.

**Production and Properties of Elliptical Light**—Elliptically polarized light can be produced in a number of ways: by the transmission of plane-polarized light through a quarter-wave plate, the plane of vibration making an angle of say  $20^\circ$  with the principal plane of the plate: by decreasing this angle the ellipse becomes more eccentric, degenerating into a line when  $i=0$ . By increasing  $i$  the ellipse becomes less eccentric, and passes through the circular condition when  $i=45^\circ$ . If we use a plate of some other thickness, we obtain an ellipse with its axis inclined to the original direction of vibration.

Elliptically polarized light, when examined through a Nicol, shows fluctuations in brilliancy as the prism is rotated, the change in intensity becoming more marked as the eccentricity of the ellipse is increased. It thus resembles partially polarized light, but can be distinguished from it by introducing a retardation of a quarter of a period by means of a  $\frac{\lambda}{4}$  plate, which converts it into plane polarized light. The directions of the axes can be determined by the  $\frac{\lambda}{4}$  plate, for they are parallel and perpendicular to the principal section of the plate when it is so oriented as to give plane-polarized light.

The ratio of the axes can be determined by observing the angle between the principal plane of the analyzing Nicol when it extinguishes the light, and the principal plane of the  $\frac{\lambda}{4}$  plate. The tangent of this angle is the ratio of the axes of the ellipse, for when two rectangular vibrations compound into a linear vibration, the tangent of the angle

which the resultant makes with one of the components is the ratio of the components.

When the  $\frac{\lambda}{4}$  plate and the Nicol are in such positions as to extinguish the light, we have the arrangement shown in Fig. 225, in which the elliptical disturbance  $A$  (with components  $a$  and  $b$ ) approaches the observer, passing through the  $\frac{\lambda}{4}$  plate

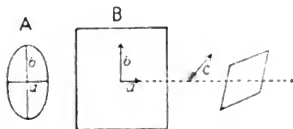


FIG. 225.

$B$ , which decomposes it into its components. On emerging, the resultant linear vibration  $c$  is extinguished by a Nicol oriented as shown, the tangent of the angle  $A$  giving us the ratio  $\frac{b}{a}$ . The constants of elliptical polarization cannot be very accurately determined with the  $\frac{\lambda}{4}$  plate, owing to the difficulty

of making accurate settings of the mica plate and Nicol. A better contrivance is Babinet's compensator, which has been adapted by Jamin to the study of elliptically polarized light. It consists of two acute prisms of quartz, which, when placed in contact, form a plate the thickness of which can be varied by sliding the prisms. The optic axes are parallel to the surfaces of the plate, but perpendicular to each other, as shown in Fig. 226. If plane-polarized light falls normally on the face of the compensator, the plane of vibration not coinciding with either of the principal planes, it will be broken up into two components

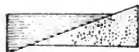


FIG. 226.

parallel and perpendicular to the optic axis. When these vibrations enter the second prism, their directions will remain unaltered, but they will exchange velocities; i.e. the ordinary ray in one becomes the extraordinary ray in the other. If  $\mu_e$  and  $\mu_o$  be the refractive indices for the two polarized disturbances, and if a ray traverses a thickness  $\epsilon$  in one prism, the relative retardation of the two disturbances is  $\epsilon(\mu_e - \mu_o)$ , and for a thickness  $\epsilon'$  in the second prism it is  $-\epsilon'(\mu_e - \mu_o)$ , for the disturbance which is the faster in the first prism, is the slower in the second. The retardation produced by the plate as a whole at the point in question is obviously

$$(\epsilon - \epsilon')(\mu_e - \mu_o).$$

The retardation is zero for the central ray, for at this point  $\epsilon = \epsilon'$ , and the light emerges polarized in the original plane. On either side of this point we shall have points at which the relative retardation is  $\pi$ ,  $2\pi$ ,  $3\pi$ , etc.; the plane of polarization of the light emergent at the points at which the retardation is an even multiple of  $\pi$  is parallel to the original plane of the incident light. At intermediate points, where the retardation is an odd multiple of  $\pi$ , the transmitted light will be polarized in a plane inclined to the original plane by an amount  $2a$ , where  $a$  is the angle between the plane of the original vibration and the plane of vibration of the retarded component. There will thus be

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a system of lines across the face of the compensator along which the light is polarized in the original plane, and another system midway between them where the light is polarized at angle  $2a$  with the original plane.

If the incident vibration makes an angle of  $45^\circ$  with the principal planes of vibration of the compensator, the plane of vibration along this second set of lines will be at right angles to the vibration along the first system. At points between the lines the light will be elliptically or circularly polarized, the condition over the surface of the compensator being roughly represented in Fig. 227.

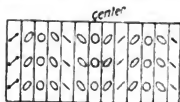


FIG. 227.

If the surface of the compensator be viewed through a Nicol with its planes of vibration parallel to the planes of the linear vibrations along the surface, the light will be extinguished along one set of lines, and the field will appear traversed by equidistant dark bands.

The bands of circular polarization may be detected by bringing a quarter-wave plate between the compensator and the analyzing Nicol; this gives us plane polarization along the lines which were previously circularly polarized, and a new system of dark bands results. The mica plate should be so oriented as not to affect the appearance of the systems of plane-polarized lines.

The dark bands previously alluded to are of course most distinct when  $a = 45^\circ$ .

**Determination of the Constants of Elliptical Polarization.**—When the elliptical polarization is produced by a quarter-wave plate we can calculate the position and ratio of the axes, but in cases where the ellipticity is the result of reflection, it becomes necessary to determine the constants experimentally. These determinations are of importance in connection with the theory of reflection, as we shall see in a subsequent chapter.

The compensator in its original form was provided with a fine cross wire moved by a micrometer screw, by means of which the distance between the bands could be measured, and the displacement of the bands determined. As modified by Jamin for the study of elliptically polarized light, the instrument has a fixed cross wire, one of the quartz wedges being moved by the screw. The relative retardation  $e - e'$  is increased or diminished at a given point according to the direction of the motion of the wedge, consequently the dark bands are displaced by a corresponding amount.

The wedge must, however, be moved through double the distance moved by the wire in the old form of instrument, in order to displace the system by the width of a band, since in this case  $e$  varies while  $e'$  remains constant, while in the case of the movable wire both  $e$  and  $e'$  vary, the one increasing and the other diminishing; the difference between  $e$  and  $e'$  consequently increases twice as rapidly with a moving wire as it does with a fixed wire and moving wedge. Let  $2a$  be the distance between two dark bands as measured by the wire, and  $2b$  the distance through which the wedge is moved in order to produce the same shift—from the foregoing  $b = 2a$ .

The retardation  $\delta$  at distance  $x$  (measured by moving wedge) from the central band is  $\delta = \frac{x}{b} \frac{\lambda}{2}$ , since moving the wedge a distance  $b$  changes the retardation by  $\frac{\lambda}{2}$ . We have now the necessary data for the study of an elliptical vibration, and will first determine the phase-difference between the two components. The components into which the incident vibration is resolved at the quartz surface will differ in phase by an amount  $\alpha - \beta$ , if we represent them by

$$x = a \cos(\omega t + \alpha), \quad y = b \cos(\omega t + \beta).$$

Transmission through the plate alters this phase-difference by an amount  $\delta = \frac{2\pi}{\lambda}(e - e')(\mu_e - \mu_o)$ , and there will be a system of lines along which the total phase-difference  $\alpha - \beta + \delta$  will be multiples of  $\pi$ , and the transmitted light plane polarized.

We first adjust the wedges so that with plane-polarized light to start with, the central dark band is bisected by the cross wire. The phase-difference at this point is zero. Substituting elliptically polarized light we find the central band shifted to a point, so situated that the phase-difference between the components of the elliptical vibration is compensated exactly by  $\delta$ , the phase-difference resulting from transmission through the plate. The quartz wedge is now to be moved by the micrometer screw until the central band is again bisected by the wire. If this distance is denoted by  $x$ , we have

$$\frac{\alpha - \beta}{\pi} = \frac{x}{b} \text{ or } \alpha - \beta = \pi \frac{x}{b},$$

$b$  having been previously determined.

**Position of the Axes.**—The phase-difference of the component vibrations along the axes is  $90^\circ$ . We set the compensator as before, so that with plane-polarized light the central band falls under the wire, and then move the wedge a distance  $\frac{1}{2}b$ . There is now a phase-difference of  $90^\circ$  along the line under the wire. Substituting the elliptical light we rotate the compensator until the central band is again bisected by the wire. The axes of the elliptical vibration are now parallel to the axes of the quartz wedges.

**Ratio of the Axes.**—If the compensator is set so that its axes are parallel to the axes of the elliptical vibration, the tangent of the angle between one of its principal planes and the principal plane of the analyzer is the measure of the ratio of the axes. The compensator acts in this case in the same way as the  $\frac{\lambda}{4}$  plate, the use of which in the determination of the ratio of the axes has already been given.

**Elliptical Polarization by Reflection.**—We have seen that when plane-polarized light is twice internally reflected at an angle of  $54^\circ$ , it emerges as circularly polarized light if the original plane of polarization made an angle of  $45^\circ$  with the plane of incidence; each reflection in this case introduces a phase-difference of  $\frac{1}{2}\pi$  between the

reflected components, consequently a single internal reflection from glass will give us elliptically polarized light. This can be shown with an ordinary right-angle prism. In general, when plane-polarized light is reflected at an azimuth of  $45^\circ$ , *i.e.* with its plane of vibration inclined at  $45^\circ$  to the plane of incidence, the reflected light will be, to a greater or less extent, elliptically polarized. In the case of glass and other transparent media the eccentricity of the ellipse is very great; in other words, the reflected light is very nearly plane polarized, but in the case of metals the elliptical polarization is very marked. If plane polarized light is reflected from a silvered mirror it will be found to be quite freely transmitted by a Nicol prism in all positions, if the plane of polarization originally made an angle of  $45^\circ$  with the plane of incidence. These cases will be more fully discussed in the chapter on the Theory of Reflection.

**Direction of Revolution in Circularly Polarized Light.**—The direction of revolution of the circular vibration depends on the thick-

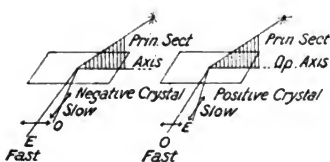


FIG. 228.

ness of the crystalline plate, its orientation, and its nature, *i.e.* whether it is a positive or negative crystal. The positions of the ordinary and extraordinary ray, and the directions of vibration in each are shown for positive and negative crystals in Fig. 228. In the former the extraordinary component travels slower than, and consequently

lags behind, the ordinary; in the latter the reverse is true. We will

now take the case of a  $\frac{\lambda}{4}$  plate of mica, which is a negative crystal, and

determine the direction of revolution for two different orientations.

First suppose the direction of vibration of the incident light to make an angle of  $+45^\circ$  with the principal section (optic axis in Fig. 229). It is decomposed into the components  $O$  and  $E$ , the former lagging behind the latter by one quarter of a wavelength. The  $E$  component consequently carries the ether particle to

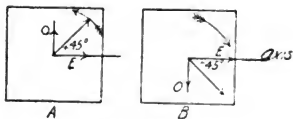


FIG. 229.

the right, and when it is at its point of greatest displacement, the  $O$  component acts in a vertical direction, consequently the direction of rotation is from right to left, as shown by the arrow. If we now rotate the plane of vibration of the incident light through  $180^\circ$ , making the angle between it and the principal section  $-45^\circ$ , we have the condition shown in *B*, and applying the same reasoning we find that the direction of rotation is now from left to right. In the case of positive crystals we apply the same construction, considering, however, that the  $O$  component acts first, since it is in advance of the other. The directions of revolution will be found to be the reverse of those in the former cases.

We can determine experimentally the direction of revolution with the quarter-wave plate. Suppose the light to be coming towards us, and the direction of revolution clock-wise. It can be decomposed into two rectangular components  $A$  and  $B$ ,  $B$  being a quarter of a period behind  $A$ . We will now suppose it trans-

mitted through the  $\frac{\lambda}{4}$  plate (placed as shown in Fig. 230)

and examined with an analyzer. The component  $A$ , which is a quarter period ahead, will traverse the plate at the slower velocity and be brought into the same phase as the component  $B$ , the resultant plane vibration having the direction  $CD$ . If the direction of revolution be reversed, component  $A$  will be a quarter period behind  $B$ , and will experience a further quarter period

relative retardation in traversing the  $\frac{\lambda}{4}$  plate, the result-

ant having the direction  $EF$ . The direction of revolution is thus determined by observing whether the plane vibration makes an angle of  $+$  or  $-45^\circ$  with the direction designated "fast."

If we have a Nicol prism and quarter-wave plate so oriented as to give us a right-handed circular vibration, by turning the Nicol through  $180^\circ$  we reverse the direction of rotation. This can be readily understood by constructing two diagrams representing the two conditions.

Let us now return to the method which we use for determining the

"fast" and "slow" directions in our  $\frac{\lambda}{4}$  plate (page 268). With the arrangement of the Nicol and metallic reflector which we employed, we obtained a clock-wise circular vibration. Obviously, if the direction of rotation is known, the fast and slow directions of the  $\frac{\lambda}{4}$  plate can be determined by observing the direction in which the plane-polarized disturbance vibrates on leaving the plate.

**Direction of Revolution in the Case of Fresnel's Rhomb.**—As has been stated before in the case of total reflection, the component perpendicular to the plane of incidence virtually lags behind the other. If the rhomb is placed in a vertical position, and the incident light polarized in a plane turned clock-wise  $45^\circ$  from the vertical, the direction of revolution will be clock-wise. A convenient way of determining the direction of revolution when the rhomb is set for circular polarization, is to notice the direction in which it must be turned in order to bring the plane of incidence into coincidence with the plane of vibration. This is the direction of revolution of the circularly polarized light.

**Natural and Partially Polarized Light.**—By natural light we mean ordinary unpolarized light, which is characterized by showing no change of intensity when passed through a  $\frac{\lambda}{4}$  plate and Nicol prism, no matter how oriented, and by being doubly refracted by certain crystals, the intensities of the two refracted rays being independent of the orientation of the crystal. Partially polarized light, such as we obtain by reflection from a glass surface at some other angle than the

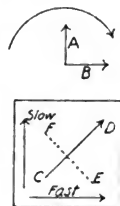


FIG. 230.

polarizing angle, is characterized by showing fluctuations of intensity when it is examined through a rotating Nicol, never being completely extinguished however. In this respect it resembles elliptically polarized light, but the difference between the two can be shown with the  $\frac{\lambda}{4}$

plate, which converts the latter into plane-polarized light. Partially polarized light is doubly refracted by crystals, the relative intensities of the two rays varying with the orientation.

We have now to consider the simplest forms of vibration which are consistent with the above results, and we will begin with a discussion of natural or unpolarized light. Brewster explained natural light by assuming it to be made up of two plane-polarized disturbances, perpendicular to each other, and independently propagated. A disturbance of this nature, if it could exist, would undoubtedly have the properties of natural light, but there are mechanical objections to the conception of a disturbance in which it is necessary to assume that the adjacent ether particles on the wave-front move in totally different directions. Fresnel accordingly, in 1821, advanced another hypothesis, namely that natural light was in reality plane-polarized light, the azimuth of which changed with exceeding rapidity. Fresnel considered that a ray which came from a single center of disturbance was plane polarized, but that the azimuth varied rapidly. If we could isolate such a ray and experiment with it, we should find that it was alternately transmitted and cut off by a Nicol prism in a fixed azimuth. The isolation of a ray coming from a single center of disturbance is, however, impossible, and even if it could be done there would be no way of verifying the hypothesis experimentally unless the changes took place so slowly that the fluctuations in the intensity of the light, after passage through the Nicol, could be followed by the eye. In every source of light we have a vast number of independent centers of disturbance, and the joint effect of all at a given moment, on a given point in the ether, will be a movement in a definite

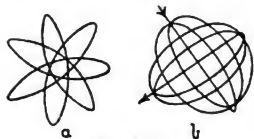


FIG. 231.

direction which will, however, change from moment to moment. Fresnel probably included the ellipse and circle in his conception of the motion of ordinary light, though he does not mention them specifically. The elliptical vibration is the most general form, and we may, on Fresnel's hypothesis, consider natural light as an elliptical

vibration, the form and orientation of which changes with great rapidity, passing through the circle and straight line as special forms.

The change in the orientation of the ellipse cannot, for mechanical considerations, be conceived as taking place suddenly. If it takes place gradually, the curve ceases to be an ellipse and takes the complicated form shown in Fig. 231a. If the ratio of the axes changes as well as the orientation, we have a curve of the form shown in Fig. 231b, except that the change from the straight line to the circle is much more gradual, and the whole curve must be conceived of as constantly changing its orientation.

Curves of this nature occur in acoustics as the resultants of harmonic disturbances of different periods, and as early as 1831 Airy raised the objection that unless the ellipse changed its form and orientation suddenly, the light could not be strictly monochromatic, for a curve such as  $b$  can be considered as the resultant of two perpendicularly plane-polarized components of different periods. Dove, in 1847, showed that light which had traversed a Nicol prism rotating at high speed showed all the properties of natural light, the emergent beam being plane polarized, the plane turning with the same speed as the prism. If a mica plate was added and made to revolve with the prism, the revolving elliptical vibration was found to have the properties of circularly polarized light. Airy pointed out, however, that the revolving plane-polarized vibration could be considered as the resultant of two oppositely polarized circular components of different periods. This case will be more fully dealt with presently.

Lippich<sup>1</sup> came to the conclusion that unpolarized vibrations are only possible with non-homogeneous light, and that only polarized vibrations are possible with strictly monochromatic light. His objections to the conception of unpolarized monochromatic vibrations were the same as those raised by Airy, but they are not serious, if we assume that the change in the orbit of the ether particle takes place very slowly in comparison to the time of revolution, *i.e.* if it executes several thousand revolutions in practically the same orbit, the departure from strict homogeneity of the light will be too slight to be detected.

Interference experiments under the condition of large difference of path, point out that the form of the vibration remains constant for many thousand periods, which makes the above assumption seem very probable. Michelson has obtained with unpolarized light, interference fringes, with a path-difference of 540,000 waves, which indicates that the light executes at least 540,000 vibrations before changing its state of polarization. A million vibrations take place in  $2 \cdot 10^{-9}$  sec., and it is therefore impossible for the eye to detect traces of polarization in natural light even if it remains polarized for many million complete periods.

The lower limit for the duration of a constant condition of polarization is given by interference experiments, and is probably somewhere in the neighborhood of  $1 \cdot 10^{-9}$  sec. The length of the path-difference in this case is about 32.4 cms.

The only conceivable way in which an upper limit for the constancy of the nature of the vibration might be obtained is by the employment of flashes of light of very brief duration. By a suitable arrangement of apparatus it is possible to obtain electric sparks, the duration of which is as brief as  $2 \cdot 10^{-9}$  sec.; the length of the wave-train from such a spark would be about 72 cms.

If no very great change of form of the vibration in one of these brief flashes took place, we might be able to discover traces of polarization in the light providing all the radiating centers of luminous energy in the spark vibrated in practically the same plane. In this case the light of the spark would vary with the position of a Nicol prism through which

<sup>1</sup> *Sitzungsber. d. Wien. Akad.*, Bd. xlviii., Abth. ii., page 146, 1863.

it was viewed. It is quite conceivable that an electric discharge might start vibrations of similar form and orientation, but all experiments have failed to show evidences of polarization in the light of sparks and electric discharges in vacuum tubes, the few positive results that have been obtained having been subsequently shown to be due to the polarization of the light by oblique transmission through the glass wall of the tube.

If the individual luminous centers in the spark were vibrating in different planes, the light would not appear polarized even if no change occurred in the states of vibration, for the smallest area of the spark which the eye could recognize, even under the microscope, would emit light from thousands of independent centers. The light from some would be cut off by the Nicol, while that from others would be transmitted; but the appearance would be the same as with natural light.

**Light-Beats.**—Airy's conception that revolving plane polarized light was merely the resultant of two circularly polarized disturbances, of different periods and opposite directions of rotation, was tested experimentally by Righi.<sup>1</sup>

In the case of the interference between two sounds of nearly the same pitch we have the familiar phenomenon of beats. At a given point the intensity of the disturbance is a function of the time, the waves alternately reinforcing and destroying one another. The optical analogy would be a moving system of interference fringes, the illumination at a given point varying with the time.

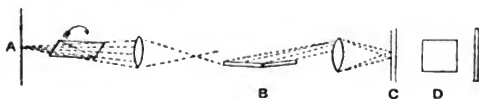


FIG. 232.

Righi arranged an optical system which effected interference between the two circularly-polarized components of different periods, which were first transformed in plane-polarized vibrations by means of a  $\frac{\lambda}{4}$  plate.

Light from a vertical slit *A* was passed through a Nicol prism which made *n* revolutions per second, and then brought to a focus by means of a lens. The revolving plane-polarized light was then reflected from a pair of Fresnel mirrors *B*, and again brought to a double focus by a second lens, the two images resulting from reflection from the inclined mirrors. By this device two similar sources are obtained side by side, the light from each being plane-polarized, the plane rotating with the same speed as the Nicol. Behind each image a  $\frac{\lambda}{4}$  plate *C* was placed, so oriented that the optic axes were mutually perpendicular, and at an angle of  $45^\circ$  with the vertical. We have seen that right and left-handed circular vibrations are transformed by a  $\frac{\lambda}{4}$  plate into plane

<sup>1</sup> *Journal de Physique* (2), 2, p. 437, 1883.

vibrations which make angles of  $+45^\circ$  and  $-45^\circ$  with the axes of the plate.

According to Airy's conception, if the light makes  $N$  vibrations per second and the plane of polarization  $n$  revolutions per second, it can be considered as the resultant of two circular vibrations, of periods  $N+n$  and  $N-n$  respectively. The two circular components from one source are resolved by the  $\frac{\lambda}{4}$  into a vertical disturbance with a period  $N+n$  and a horizontal disturbance with a period  $N-n$ . The light from the other source, since the  $\frac{\lambda}{4}$  plate behind it is differently oriented, is resolved into vertical vibrations of period  $N-n$  and horizontal ones of period  $N+n$ . The vertical vibrations of period  $N+n$  from one source interfere with the vertical vibrations of period  $N-n$  which come from the other, and since the number of beats per second is equal to the difference between the frequencies of the interfering disturbances, a given point will receive maximum illumination  $2n$  times per second. The fringe system formed by the interference of the vertical vibrations was separated from that formed by the horizontal components by means of a doubly refracting prism  $D$ ; and the fringes were found to be moving in opposite directions, passing a given point at the rate of  $2n$  per second.

## CHAPTER XII.

### THEORY OF REFLECTION AND REFRACTION.

A THEORY of reflection was worked out by Fresnel, based upon the elastic-solid hypothesis, and equations were obtained which represented the relations between the intensities of the reflected and refracted components, their states of polarization, etc.

This treatment, however, is only of historical interest, since it has been supplanted by one based on the electro-magnetic theory, and we shall in the present chapter trace the derivation of the fundamental equations of the more modern theory of luminous disturbances, the foundations of which were laid down by Maxwell.

The luminous vibrations will be regarded as rapidly alternating displacement currents in the ether or in matter, as the case may be, these currents giving rise to magnetic forces similar to those brought into existence by currents flowing in conductors. On this theory not only are the optical and electrical properties of matter being rapidly harmonized, but predictions are being constantly made which are subsequently verified by experiment. In certain cases, however, the discussion from the elastic-solid standpoint is more intelligible, and we shall therefore make use of it from time to time, regarding the older theory more as a convenience, however, than as a true representation of what is actually going on. We will begin by the derivation of the fundamental equations of Maxwell.

**Derivation of Maxwell's Equations.**—The current may be defined either in electrostatic or electromagnetic units, and will be designated by  $i$  or  $i'$  accordingly. As we shall have occasion to pass from one system to the other frequently, it is well to fix firmly in the mind at the start that the accent is used to distinguish quantities measured in *electromagnetic* units from those measured in *electrostatic*.

The current  $i$  which flows through cross section  $q$  is defined as the number of electrostatic units which traverse  $q$  in unit time, so that if the quantity of electricity  $de$  flows through  $q$  in the element of time  $dt$  we have

$$i = \frac{de}{dt}, \dots\dots\dots(1)$$

and if  $q$  is equal to unit cross section,  $i$  is equal to  $j$ , the current density. The components of  $j$  along the  $x, y, z$  axes we will designate

$j_x, j_y, j_z$ . We will now derive an expression for the current in electromagnetic measure. The current is surrounded by a magnetic field, the lines of force being circles in the case of a current flowing along a cylindrical wire. An isolated magnetic pole will follow these lines of force, travelling around the wire as long as the current continues to flow.

If we carry the magnetic pole around the wire in the opposite direction, we are obliged to do a certain amount of work on it; and if it is allowed to move under the influence of the magnetic force, the current does work on it, developing a certain amount of kinetic energy. We shall define the current  $i'$  measured in electromagnetic units thus.

The work done will be proportional to the strength of the current, and for convenience we make use of the proportionality factor  $4\pi$ . If  $A$  represents the work done by the current on unit magnetic pole in driving it around one complete turn, we write  $A = 4\pi i'$ .

Now the work is represented by the force multiplied by the distance through which it acts. Assume that we have a rectangle  $dx, dy$ , which is traversed normally by a current  $i' = j_z dx dy$ ,  $j_z$  being the  $z$  component measured in electromagnetic units. If the current flows towards the observer (Fig. 233), a plus magnetic pole will be carried around  $dx, dy$ , in the direction indicated by the arrows. The total work done by the current in moving unit pole around the rectangle will be

$$A = \alpha dx + \beta' dy - \alpha' dx - \beta dy, \quad (2)$$

in which  $\alpha$  and  $\beta$  are the components of magnetic force along  $AB$  and  $AD$ , and  $\alpha'$  and  $\beta'$  are the components along  $DC$  and  $BC$ .  $\alpha$  may not be constant along  $dx$ , but if we regard it as variable, for example having the value  $\alpha$  at  $A$  and  $\alpha + \partial\alpha$  at  $B$ , the average value will be  $\alpha + \frac{\partial\alpha}{2}$  which, when multiplied by  $dx$ , gives us  $\alpha dx$  + an infinitesimal of the second order. The minus signs occur for the obvious reason that the forces along  $DC$  and  $CB$  are oppositely directed from the forces along  $AB$  and  $AD$ .  $\alpha'$  differs from  $\alpha$  since it works along a line, the  $y$  coordinate of which is greater by an amount  $dy$  than that of  $AB$ . Under certain conditions of course  $\alpha$  would be equal to  $\alpha'$ .

If  $dy$  be taken sufficiently small  $\frac{(\alpha' - \alpha)}{dy}$  may be regarded as the partial differential coefficient  $\frac{\partial\alpha}{\partial y}$ , and we have

$$\alpha' = \alpha + \frac{\partial\alpha}{\partial y} dy \quad \text{and} \quad \beta' = \beta + \frac{\partial\beta}{\partial x} dx.$$

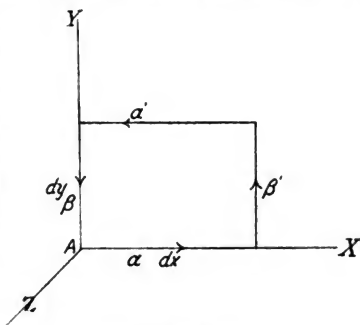


FIG. 233.

We now have for the work, substituting these values in (2) and cancelling,

$$A = \left( \frac{\partial \beta}{\partial x} - \frac{\partial \alpha}{\partial y} \right) dx dy,$$

and since

$$A = 4\pi i' = 4\pi j_z' dx dy,$$

$$4\pi j_z' = \frac{\partial \beta}{\partial x} - \frac{\partial \alpha}{\partial y}, \text{ and similarly, } 4\pi j_x' = \frac{\partial \gamma}{\partial y} - \frac{\partial \beta}{\partial z}, \quad 4\pi j_y' = \frac{\partial \alpha}{\partial z} - \frac{\partial \gamma}{\partial x}, \quad (3)$$

Maxwell's differential equations of the magnetic field.

If  $c$  represents the ratio of the two systems of units, *i.e.*  $\frac{i}{i'} = c$  and  $j_x' = c$ , we can introduce  $j$  (defined electrostatically) into the equations, which now become

$$\frac{4\pi}{c} j_x = \frac{\partial \gamma}{\partial y} - \frac{\partial \beta}{\partial z}, \quad \frac{4\pi}{c} j_y = \frac{\partial \alpha}{\partial z} - \frac{\partial \gamma}{\partial x}, \quad \frac{4\pi}{c} j_z = \frac{\partial \beta}{\partial x} - \frac{\partial \alpha}{\partial y}. \quad \dots \dots (4)$$

These equations hold for all media, for it can be shown that the work done in carrying the magnetic pole around the circuit is independent of the nature of the medium. The quantity  $c$  has the dimensions of velocity, and can be determined by observing the magnetic effect of discharging a quantity of electricity measured electrostatically through a circuit.

While the above equations, which connect the current with the magnetic force, hold for all media, we shall presently develop expressions connecting the current with the electric force, and these expressions take particular forms, depending upon the nature of the medium.

They will suffice for the study of reflection, absorption and dispersion, but when we come to consider the behavior of media when brought into a magnetic field, we shall require another set of similar equations, which connect a magnetic current with the lines of electric force which accompany it.

The magnetic current or magnetic flux occurs when the strength of a magnetic field changes, and the lines of flow will be surrounded by lines of electric force just as the electric current is surrounded by lines of magnetic force. By determining the work done by the magnetic current in drawing unit charge once around the circuit, expressions are obtained which connect the strength of the flux with the accompanying electric field. The equations are similar to those which we have already deduced, and like them hold for all media :

$$\frac{4\pi}{c} s_x = \frac{\partial Y}{\partial z} - \frac{\partial Z}{\partial y}, \quad \frac{4\pi}{c} s_y = \frac{\partial Z}{\partial x} - \frac{\partial X}{\partial z}, \quad \frac{4\pi}{c} s_z = \frac{\partial X}{\partial y} - \frac{\partial Y}{\partial x}. \quad \dots \dots (5)$$

**Displacement Currents in Free Ether.**—A displacement current will occur in the ether whenever the density of the lines of electric force changes, and the strength of the current will be proportional to the rate at which the change takes place. It is not easy to form a physical conception of the displacement current. We may perhaps think of it as a lateral shift of the ether, which takes place parallel to the lines of force; in this case our picture of the wave will be not unlike the

conception of a wave in an elastic solid. A different way of looking at the matter is that adopted by J. J. Thomson in his *Recent Researches in Electricity and Magnetism*, the first chapter of which will be found of great assistance in forming a concrete picture of what may be taking place in the ether when it is traversed by waves. Thomson represents the phenomena of the electromagnetic field in terms of Faraday tubes (lines of electric force). The motion of one of these tubes gives rise to a magnetic force perpendicular to the direction of its motion, and an electromotive intensity (which we have spoken of as the electric force), which is perpendicular to both of the specified directions. The displacement current takes place in the direction of the electromotive intensity, which is not constant, unless the density of the moving tubes is constant.

Thomson showed that the equations which we have already examined, and those which we are about to consider, could be derived from the consideration of the motion of the Faraday tubes.

We will now derive expressions which connect the displacement current with the electromotive intensity (electric force). Since a charge  $e$  sends out  $4\pi$  lines of force, the product of the current density and  $4\pi$  will be the change in the number of lines of force in unit time. It is obvious that in the case of steady currents there will be no change in the number of the lines, but in the case of displacement currents, where the current strength is changing with the time, the density of the lines of force changes. We can now write

$$4\pi j_x = \frac{\partial N_x}{\partial t}, \quad 4\pi j_y = \frac{\partial N_y}{\partial t}, \quad 4\pi j_z = \frac{\partial N_z}{\partial t}, \dots\dots\dots (6)$$

in which the expressions  $N_x$ ,  $N_y$ ,  $N_z$  represent the components of the density of the lines of electric force (polarization in free ether) parallel to the three axes. Similarly for the magnetic current we have

$$4\pi s_x = \frac{\partial M_x}{\partial t}, \quad 4\pi s_y = \frac{\partial M_y}{\partial t}, \quad 4\pi s_z = \frac{\partial M_z}{\partial t}. \dots\dots\dots (6')$$

We can form an idea of a magnetic current such as we have in the case of light waves in the following way. Suppose that we have an iron wire with a coil of insulated wire around one end of it, which is traversed by an alternating current. The density of the magnetic lines of force in the iron wire varies periodically, rising from zero to a maximum, and then falling to zero during the first half period, and then rising again to a maximum, with a reversal in the direction of the force however. The wire is thus traversed by a periodic magnetic current, which is surrounded by circular lines of electric force, which set up alternating induced currents in conductors which are brought into the field.

In the free ether the electric force is considered as numerically equal to the density of the lines of force, so that we may substitute for  $N_x$ ,  $N_y$ ,  $N_z$  their equivalents  $X$ ,  $Y$ ,  $Z$ . Our equations now become

$$\left. \begin{aligned} 4\pi j_x &= \frac{\partial X}{\partial t}, & 4\pi j_y &= \frac{\partial Y}{\partial t}, & 4\pi j_z &= \frac{\partial Z}{\partial t}, \\ 4\pi s_x &= \frac{\partial \alpha}{\partial t}, & 4\pi s_y &= \frac{\partial \beta}{\partial t}, & 4\pi s_z &= \frac{\partial \gamma}{\partial t} \end{aligned} \right\} \dots\dots\dots (7)$$

Substituting these values for the current in equations (4) gives us expressions which connect the variation of the electric force with the magnetic field which results from the displacement current.

$$\left. \begin{aligned} \frac{1}{c} \frac{\partial X}{\partial t} &= \frac{\partial \gamma}{\partial y} - \frac{\partial \beta}{\partial z}, & \frac{1}{c} \frac{\partial Y}{\partial t} &= \frac{\partial \alpha}{\partial z} - \frac{\partial \gamma}{\partial x}, & \frac{1}{c} \frac{\partial Z}{\partial t} &= \frac{\partial \beta}{\partial x} - \frac{\partial \alpha}{\partial y} \\ \frac{1}{c} \frac{\partial \alpha}{\partial t} &= \frac{\partial Y}{\partial z} - \frac{\partial Z}{\partial y}, & \frac{1}{c} \frac{\partial \beta}{\partial t} &= \frac{\partial Z}{\partial x} - \frac{\partial X}{\partial z}, & \frac{1}{c} \frac{\partial \gamma}{\partial t} &= \frac{\partial X}{\partial y} - \frac{\partial Y}{\partial x} \end{aligned} \right\} \dots\dots(8)$$

**Isotropic Dielectrics.**—The equations which we have just derived do not hold for dielectrics, for in media the force exerted by two charges  $e' e''$  at distance  $r$  is less than the force which would be exerted in the free ether, being represented by  $\frac{e' e''}{\epsilon r^2}$ , in which  $\epsilon$  is the dielectric constant of the medium. The dielectric constant is greater than unity for all media, and we shall see that the velocity with which the wave is propagated in the medium becomes less as the dielectric constant increases.

In the same way the force between two magnetic poles in a medium is represented by  $\frac{m' m''}{\mu r^2}$ ,  $\mu$  being the magnetic permeability, a quantity which differs only slightly from unity except in the case of iron, and we shall see later on that we are justified in writing  $\mu = 1$  in practically all optical problems. The change in the law of the force which occurs in ponderable media, makes it necessary to modify our last equations, since with the same change in the current intensity the electric force is weaker in the proportion  $\frac{1}{\epsilon}$ , the current in dielectrics being represented by  $4\pi j_z = \epsilon \frac{\partial X}{\partial t}$ , etc.,  $4\pi s_x = \mu \frac{\partial \alpha}{\partial t}$ , etc.

Equations (7) now become

$$\left. \begin{aligned} \epsilon \frac{\partial X}{\partial t} &= \frac{\partial \gamma}{\partial y} - \frac{\partial \beta}{\partial z}, & \epsilon \frac{\partial Y}{\partial t} &= \frac{\partial \alpha}{\partial z} - \frac{\partial \gamma}{\partial x}, & \epsilon \frac{\partial Z}{\partial t} &= \frac{\partial \beta}{\partial x} - \frac{\partial \alpha}{\partial y} \\ \frac{1}{c} \frac{\partial \alpha}{\partial t} &= \frac{\partial Y}{\partial z} - \frac{\partial Z}{\partial y}, & \frac{1}{c} \frac{\partial \beta}{\partial t} &= \frac{\partial Z}{\partial x} - \frac{\partial X}{\partial z}, & \frac{1}{c} \frac{\partial \gamma}{\partial t} &= \frac{\partial X}{\partial y} - \frac{\partial Y}{\partial x} \end{aligned} \right\} \dots\dots(9)$$

which expressions completely determine all properties of the magnetic field in an isotropic dielectric.

A comparison of equations (6) with the equations preceding (9) shows us that, if we consider the number of lines of force issuing from given charge independent of the surrounding medium, we have the condition within a medium of dielectric constant  $\epsilon$ , and permeability  $\mu$ ,

$$N_x = \epsilon X, \quad N_y = \epsilon Y, \quad N_z = \epsilon Z,$$

$$M_x = \mu \alpha, \quad M_y = \mu \beta, \quad M_z = \mu \gamma;$$

in other words, the densities of the lines of magnetic and electric force are equal to the forces only in a vacuum, for which  $\epsilon$  and  $\mu$  both equal unity.

If a charge  $e$  is contained in the cube  $dx, dy, dz$ ,  $4\pi e$  lines of force issue from its surface. We can also reckon the number of lines issuing from the cube as the sum of the lines issuing from the six surfaces.

The two squares perpendicular to  $x$  contribute the part

$$-(N_x)_1 dy dz + (N_x)_2 dy dz.$$

By Taylor's Theorem  $(N_x)_2 = (N_x)_1 + \frac{\partial N_x}{\partial x} dx$ ;

the two squares therefore contribute

$$\left( (N_x)_1 + \frac{\partial N_x}{\partial x} dx \right) dy dz - (N_x)_1 dy dz = \frac{\partial N_x}{\partial x} dx \cdot dy dz.$$

The total contribution of all six faces is

$$\left( \frac{\partial N_x}{\partial x} + \frac{\partial N_y}{\partial y} + \frac{\partial N_z}{\partial z} \right) dx dy dz.$$

Equating this to  $4\pi e$ , and bearing in mind the expression for  $N_x, M_x$ , etc., gives us, if we write  $\frac{e}{dx dy dz} = \rho$  the charge in unit volume,

$$4\pi\rho = \frac{\partial \epsilon X}{\partial x} + \frac{\partial \epsilon Y}{\partial y} + \frac{\partial \epsilon Z}{\partial z}, \dots\dots\dots (9a)$$

an equation which we shall not make use of for the present.

**Boundary Conditions.**—Since in optical problems we are continually dealing with cases where the waves pass across the boundary which separates two media of different optical properties, it will be necessary to determine what changes, if any, occur in the components of the electric and magnetic forces at the surface of separation.

We will begin by considering that the transition is abrupt, i.e. that the dielectric constant changes suddenly in crossing a mathematical plane, which we will take parallel to the  $xy$  plane of our co-ordinate system. Let the dielectric constant of the upper median be  $\epsilon_1$ , and that of the lower  $\epsilon_2$ , and let  $N_1$  equal the density of the lines of force in the upper medium, that is the number which pass in a normal direction through a plane of unit area. These lines of force are incident on the boundary at an angle  $\theta_1$  with the normal. We will first assume that the lines pass through into the second medium without change of direction. The electric force in the upper medium

is  $R_1 = \frac{4\pi N_1}{\epsilon_1}$ , that in the lower medium  $R_2 = \frac{4\pi N_2}{\epsilon_2}$ , the force due to a

given density of the lines of force decreasing as the dielectric constant increases. If  $\epsilon_2 > \epsilon_1$ , it is obvious that the electric force is less below the boundary, assuming as we have done that  $N_1 = N_2$ . Since  $R_1$  is less than  $R_2$  and the direction the same, it is obvious that the components of  $R_2 - X_2, Y_2, Z_2$ —will all be less in the second medium. We shall see that this condition is contrary to the principle of the conservation of energy, and that our original assumption that the lines passed through the boundary without change of direction was incorrect. If  $X_2$  is less than  $X_1$ , and the boundary infinitely thin, we can derive an unlimited amount of work by carrying a charged particle along the boundary in

the lower medium against the electric force, and then, carrying it across the boundary (which requires no work unless the force is infinite), allow it to move back in the upper medium through the same distance. It will obviously yield more work than has been spent upon it in the lower medium, since it is moving under the influence of a stronger force. We are thus forced to the conclusion that the  $x$  component has the same value on both sides of the boundary, otherwise a perpetual motion would be possible. The same is true for the  $y$  component, and we consequently have for the boundary condition of these two components  $X_1 = X_2$  and  $Y_1 = Y_2$ . Let us now see how we can reconcile this condition with the smaller value of  $R_2$  in the second medium. It is obvious that if we consider that if the lines of force bend away from the normal on crossing the boundary, we can reconcile the equality of the  $x$  and  $y$  components on the two sides with the decrease in their resultant. But the bending of the lines of force in the direction specified, results in a change in their density  $N_2$ , consequently this must be taken into account. Let the

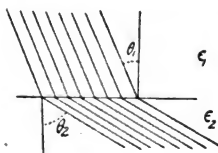


FIG. 234.

angle which the lines make with the normal in the upper medium be  $\theta_1$ , and the angle which the refracted lines make with the normal be  $\theta_2$ , as shown in Fig. 234. We will now determine the value of the  $Z$  component on the two sides of the boundary. In the upper medium the normal component of the electric force is  $Z_1 = R_1 \cos \theta_1$ ; in the lower medium,  $Z_2 = R_2 \cos \theta_2$ . The density of the lines of force parallel to the  $z$  axis (normal polarization) for the two media is given by

$$\frac{\epsilon_1 R_1 \cos \theta_1}{4\pi} \text{ and } \frac{\epsilon_2 R_2 \cos \theta_2}{4\pi}.$$

Now the normal polarization is the same in the two media, for the same number of lines pass through a plane of unit area which is perpendicular to the  $z$  axis, in whichever medium we consider the plane, consequently we can write

$$\frac{\epsilon_1 R_1 \cos \theta_1}{4\pi} = \frac{\epsilon_2 R_2 \cos \theta_2}{4\pi} \text{ or } \epsilon_1 Z_1 = \epsilon_2 Z_2,$$

which expression determines the boundary conditions of the  $z$  component.

In a similar way it may be shown that the boundary conditions for the components of the magnetic force are  $\alpha_1 = \alpha_2$ ,  $\beta_1 = \beta_2$ ,  $\mu_1 \gamma_1 = \mu_2 \gamma_2$ . Since, however,  $\mu = 1$  in practically all optical problems, we can write  $\gamma_1 = \gamma_2$ .

**Velocity of the Wave.**—To find the velocity of the wave we differentiate the first equation of (9) with respect to  $t$ ,

$$\frac{\epsilon}{c} \frac{\partial X}{\partial t} = \frac{\partial \gamma}{\partial y} - \frac{\partial \beta}{\partial z}, \quad \frac{\epsilon}{c} \frac{\partial^2 X}{\partial t^2} = \frac{\partial}{\partial y} \left( \frac{\partial \gamma}{\partial t} \right) - \frac{\partial}{\partial z} \left( \frac{\partial \beta}{\partial t} \right),$$

and substituting for  $\frac{\partial \gamma}{\partial t}$  and  $\frac{\partial \beta}{\partial t}$  the values given by the last two equations of (9),

$$\begin{aligned} \frac{\epsilon}{c^2} \frac{\partial^2 X}{\partial t^2} &= \frac{\partial}{\partial y} \left( \frac{\partial X}{\partial y} - \frac{\partial Y}{\partial x} \right) - \frac{\partial}{\partial z} \left( \frac{\partial Z}{\partial x} - \frac{\partial X}{\partial z} \right) \\ &= \frac{\partial^2 X}{\partial x^2} + \frac{\partial^2 Y}{\partial y^2} + \frac{\partial^2 X}{\partial z^2} - \frac{\partial^2 X}{\partial x^2} - \frac{\partial^2 Y}{\partial x \partial y} - \frac{\partial^2 Z}{\partial z \partial x}. \end{aligned} \quad (10)$$

We now differentiate the first three equations of (9) with respect to  $x$ ,  $y$ , and  $z$  respectively,

$$\frac{\epsilon}{c} \frac{\partial^2 X}{\partial t \partial x} = \frac{\partial^2 \gamma}{\partial y \partial x} - \frac{\partial^2 \beta}{\partial z \partial x}, \quad \frac{\epsilon}{c} \frac{\partial^2 Y}{\partial t \partial y} = \frac{\partial^2 \alpha}{\partial z \partial y} - \frac{\partial^2 \gamma}{\partial x \partial y}, \quad \frac{\epsilon}{c} \frac{\partial^2 Z}{\partial t \partial z} = \frac{\partial^2 \beta}{\partial x \partial z} - \frac{\partial^2 \alpha}{\partial y \partial z}.$$

Addition of these three equations gives

$$\epsilon \left( \frac{\partial^2 X}{\partial t \partial x} + \frac{\partial^2 Y}{\partial t \partial y} + \frac{\partial^2 Z}{\partial t \partial z} \right) = 0 \quad \text{or} \quad \frac{\partial}{\partial t} \left( \frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y} + \frac{\partial Z}{\partial z} \right) = 0.$$

Since we are dealing with periodic changes of the electric force, the differential coefficient, with respect to the time, of the quantity in the parenthesis, can be considered as proportional to the quantity, with a phase increase of  $\frac{\pi}{2}$  (since differentiating the sine gives the cos, the equivalent of a phase change of  $\frac{\pi}{2}$ ).

This gives us 
$$\left( \frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y} + \frac{\partial Z}{\partial z} \right) = 0, \quad (11)$$

and we have as our final equation

$$\frac{\epsilon}{c^2} \frac{\partial^2 X}{\partial t^2} = \frac{\partial^2 X}{\partial x^2} + \frac{\partial^2 X}{\partial y^2} + \frac{\partial^2 X}{\partial z^2} = \Delta X. \quad (12)$$

Similar equations hold for  $Y$ ,  $Z$ ,  $\alpha$ , and  $\beta$ ,

$$\begin{aligned} \frac{\epsilon}{c^2} \frac{\partial^2 X}{\partial t^2} &= \Delta X, & \frac{\epsilon}{c^2} \frac{\partial^2 Y}{\partial t^2} &= \Delta Y, & \frac{\epsilon}{c^2} \frac{\partial^2 Z}{\partial t^2} &= \Delta Z, \\ \frac{\epsilon}{c^2} \frac{\partial^2 \alpha}{\partial t^2} &= \Delta \alpha, & \frac{\epsilon}{c^2} \frac{\partial^2 \beta}{\partial t^2} &= \Delta \beta, & \frac{\epsilon}{c^2} \frac{\partial^2 \gamma}{\partial t^2} &= \Delta \gamma. \end{aligned}$$

We have seen (page 7) that differential equations of this form represent waves travelling with a velocity  $v = \frac{c}{\sqrt{\epsilon}}$ .

Now the dielectric constant of the ether equals unity, consequently our equation shows us that the velocity of the wave in space is equal to  $c$ , the ratio of the two systems of electrical units. This has been confirmed by experiment, the velocity of light determined by optical methods being  $2.9989 \cdot 10^{10}$  cms. per sec., while the velocity of  $c$  determined by electrical methods is  $3 \cdot 10^{10}$  cm./sec.



Suppose a ray of light coming from  $A$  (behind the plane of the paper) to be incident at angle  $\Phi$ , on the surface of a block of glass (Fig. 235). We will consider this ray plane-polarized, the plane of vibration making an angle of  $45^\circ$  with the plane of incidence.

Resolve the incident electric force into two components,  $E$ , perpendicular to the plane of incidence and  $E_p$ , parallel to it. We therefore write for the  $y$  component of the electric force, since

$$m = \sin \Phi, \quad n = 0, \quad \text{and} \quad p = \cos \Phi,$$

$$Y_e = E_i \cos \frac{2\pi}{T} \left( t - \frac{x \sin \Phi + z \cos \Phi}{V_1} \right), \dots \dots \dots (14)$$

in which  $V_1$  is the velocity of the light in air.

The other component  $E_p$ , which is also perpendicular to the ray, has  $x$  and  $z$  components given by  $A_x = E_p \cos \Phi$  and  $A_z = -E_p \sin \Phi$ .

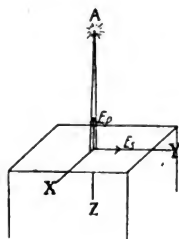


FIG. 235.

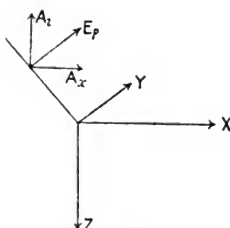


FIG. 236.

The positive direction of  $E_p$  is upwards from the boundary which makes the  $z$  component negative, as will be seen from Fig. 236, which gives us the side view of the block.

The  $x$  and  $z$  components of the electric force are therefore given by

$$\left. \begin{aligned} X_e &= E_p \cos \Phi \cdot \cos \frac{2\pi}{T} \left( t - \frac{x \sin \Phi + z \cos \Phi}{V_1} \right) \\ Z_e &= -E_p \sin \Phi \quad \text{,,} \quad \text{,,} \quad \text{,,} \end{aligned} \right\} \dots \dots \dots (14)$$

The magnetic forces associated with these are obtained at once by differentiating the above and substituting in equations (9). For example the  $x$  component  $\alpha$  is given by

$$\begin{aligned} \frac{1}{c} \frac{\partial \alpha}{\partial t} &= \frac{\partial Y}{\partial z} - \frac{\partial Z}{\partial y} = E_i \sin \frac{2\pi}{T} \left( t - \frac{x \sin \Phi + z \cos \Phi}{V_1} \right) \frac{2\pi \cos \Phi}{T V_1}, \\ \frac{\partial \alpha}{\partial t} &= c E_i \frac{2\pi \cos \Phi}{T V_1} \sin \frac{2\pi}{T} \quad \text{,,} \quad \text{,,} \quad \text{,,} \\ \alpha_e &= -c E_i \frac{\cos \Phi}{V_1} \cos \frac{2\pi}{T} \left( t - \frac{x \sin \Phi + z \cos \Phi}{V_1} \right), \end{aligned}$$

and since  $V_1 = \frac{C}{\sqrt{\epsilon_1}}$

$$\left. \begin{aligned} \alpha_r &= -E_s \cos \Phi \sqrt{\epsilon_1} \cos \frac{2\pi}{T} \left( t - \frac{x \sin \Phi + z \cos \Phi}{V_1} \right) \\ \text{Similarly, } \beta_r &= E_p \sqrt{\epsilon_1} \cos \frac{2\pi}{T} \quad \quad \quad \text{,,} \quad \quad \quad \text{,,} \\ \gamma_r &= E_s \sin \Phi \sqrt{\epsilon_1} \cos \frac{2\pi}{T} \quad \quad \quad \text{,,} \quad \quad \quad \text{,,} \end{aligned} \right\} \dots\dots\dots (15)$$

Writing for the refracted wave,

$$\left. \begin{aligned} X_2 &= D_p \cos \chi \cos \frac{2\pi}{T} \left( t - \frac{x \sin \chi + z \cos \chi}{V_2} \right) \\ Y_2 &= D_s \cos \frac{2\pi}{T} \quad \quad \quad \text{,,} \quad \quad \quad \text{,,} \\ Z_2 &= D_p \sin \chi \cos \frac{2\pi}{T} \quad \quad \quad \text{,,} \quad \quad \quad \text{,,} \end{aligned} \right\} \dots\dots\dots (16)$$

in which  $D_p$  and  $D_s$  are the components of amplitude parallel and perpendicular to the plane of incidence. If now the boundary conditions are to be complied with, there will be a reflected wave except when  $\sqrt{\epsilon_1} = \sqrt{\epsilon_2}$ . Let us take the simplest possible case of a plane-polarized vibration parallel to the  $x$  axis at normal incidence. The boundary conditions are  $X_1 = X_2$ ,  $\beta_1 = \beta_2$ .

The magnetic components of the refracted wave are obtained in the same way as those of the incident.

They are given by

$$\left. \begin{aligned} \alpha_2 &= -D_s \cos \chi \sqrt{\epsilon_2} \cos \frac{2\pi}{T} \left( t - \frac{x \sin \chi + z \cos \chi}{V_2} \right) \\ \beta_2 &= D_p \sqrt{\epsilon_2} \quad \quad \quad \text{,,} \quad \quad \quad \text{,,} \\ \gamma_2 &= D_s \sin \chi \sqrt{\epsilon_2} \quad \quad \quad \text{,,} \quad \quad \quad \text{,,} \end{aligned} \right\} \dots\dots\dots (17)$$

At the boundary we have

$$\left. \begin{aligned} X_1 &= X_2 \text{ that is, } E_p = D_p, \text{ since } \Phi = \chi = 0 \\ \beta_1 &= \beta_2 \text{ or } E_p \sqrt{\epsilon_1} = D_p \sqrt{\epsilon_2} \\ &\quad \quad \quad \sqrt{\epsilon_1} = \sqrt{\epsilon_2} \end{aligned} \right\} \dots\dots\dots (18)$$

The boundary conditions will hold for the incident and refracted wave only under the above condition, that is, when the refractive indices of the two media are the same. If  $\epsilon_1$  differs from  $\epsilon_2$ , we shall have a reflected wave, and the sum of the forces of the incident and reflected wave constitute the force at the boundary in the upper medium, which is to be equated to the force in the lower medium. The direction of the force in the reflected wave is opposite to that

in the incident, for as the reflecting power increases, the force in the lower medium must diminish.

We now write for the electric and magnetic components of the reflected wave,

$$\left. \begin{aligned} X_r &= R_p \cos \Phi' \cos \frac{2\pi}{T} \left( t - \frac{x \sin \Phi' - z \cos \Phi'}{V_1} \right) \\ Y_r &= R_s \cos \frac{2\pi}{T} \quad \quad \quad \text{,,} \quad \quad \quad \text{,,} \\ Z_r &= -R_p \sin \Phi' \cos \frac{2\pi}{T} \quad \quad \quad \text{,,} \quad \quad \quad \text{,,} \end{aligned} \right\} \dots\dots\dots (19)$$

and

$$\left. \begin{aligned} \alpha_r &= -R_s \cos \Phi' \sqrt{\epsilon_1} \cos \frac{2\pi}{T} \left( t - \frac{x \sin \Phi' - z \cos \Phi'}{V_1} \right) \\ \beta_r &= R_p \sqrt{\epsilon_1} \cos \frac{2\pi}{T} \quad \quad \quad \text{,,} \quad \quad \quad \text{,,} \\ \gamma_r &= R_s \sin \Phi' \sqrt{\epsilon_1} \cos \frac{2\pi}{T} \quad \quad \quad \text{,,} \quad \quad \quad \text{,,} \end{aligned} \right\} \dots\dots\dots (20)$$

From these equations we can deduce the laws of reflection and refraction, as well as the relation between the intensities of the reflected and refracted rays for various states of polarization.

The relations between the angles of incidence, reflection, and refraction follow at once from the boundary conditions, which are only fulfilled when for  $z=0$  we put all of the forces proportional to the same function of  $t$ ,  $x$ , and  $y$ .

This gives us 
$$\frac{\sin \Phi}{V_1} = \frac{\sin \Phi'}{V_1} = \frac{\sin \chi}{V_2},$$

or (21) 
$$\frac{\sin \Phi}{\sin \chi} = \frac{V_1}{V_2} = n, \text{ the refractive index.}$$

We will now deduce expressions for the intensities of the reflected and refracted components, the relations between the phases of the vibrations, their dependence upon the angle of incidence and the state of polarization. The force  $X$ , on the upper side of the boundary, is equal to the sum of the forces in the incident and reflected waves,  $X_i + X_r$ , which is to be equated to the force on the lower side of the boundary.

$$\left. \begin{aligned} (1) \quad X_i + X_r &= X_2 \quad \text{or} \quad E_p - R_p \cos \Phi = D_p \cos \chi \\ (2) \quad Y_i + Y_r &= Y_2 \quad \text{or} \quad E_s + R_s = D_s \\ (3) \quad \alpha_i + \alpha_r &= \alpha_2 \quad \text{or} \quad (E_s - R_s) \sqrt{\epsilon_1} \cos \Phi = D_s \sqrt{\epsilon_2} \cos \chi \\ (4) \quad \gamma_i + \gamma_r &= \gamma_2 \quad \text{or} \quad (E_p + R_p) \sqrt{\epsilon_1} = D_p \sqrt{\epsilon_2} \end{aligned} \right\} \dots\dots\dots (22)$$

The positive directions of the components  $R_p$  and  $D_p$  are shown in Fig. 237.

$$\left. \begin{aligned} \text{Add (2) and (3),} & \quad 2E_s = D_s \left( 1 + \frac{\sqrt{\epsilon_2} \cos \chi}{\sqrt{\epsilon_1} \cos \Phi} \right) \\ \text{Elim. } D_s \text{ from (2) and (3), } E_s \left( \frac{\sqrt{\epsilon_1} \cos \Phi}{\sqrt{\epsilon_2} \cos \chi} - 1 \right) &= R_s \left( \frac{\sqrt{\epsilon_1} \cos \Phi}{\sqrt{\epsilon_2} \cos \chi} + 1 \right) \\ \text{Add (1) and (4),} & \quad 2E_p = D_p \left( \frac{\cos \chi}{\cos \Phi} + \frac{\sqrt{\epsilon_2}}{\sqrt{\epsilon_1}} \right) \\ \text{Elim. } D_p \text{ from (1) and (4), } E_p \left( \frac{\cos \Phi}{\cos \chi} - \frac{\sqrt{\epsilon_1}}{\sqrt{\epsilon_2}} \right) &= R_p \left( \frac{\cos \Phi}{\cos \chi} + \frac{\sqrt{\epsilon_1}}{\sqrt{\epsilon_2}} \right) \end{aligned} \right\} \quad (23)$$

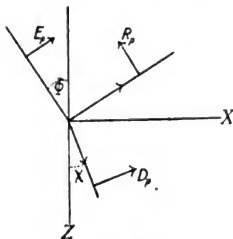


FIG. 237.

Substitute for  $\frac{\sqrt{\epsilon_2}}{\sqrt{\epsilon_1}} = n$  its equivalent  $\frac{\sin \Phi}{\sin \chi}$  and we obtain equations identical with the formulae of Fresnel, from which the phases and intensities of the reflected and refracted waves can be calculated.

Reflected amplitudes :

$$\left. \begin{aligned} R_s &= -E_s \frac{\sin(\Phi - \chi)}{\sin(\Phi + \chi)}, & R_p &= E_p \frac{\tan(\Phi - \chi)}{\tan(\Phi + \chi)} \end{aligned} \right\} \quad (24)$$

Refracted amplitudes :

$$\left. \begin{aligned} D_s &= E_s \frac{2 \sin \chi \cos \Phi}{\sin(\Phi + \chi)}, & D_p &= E_p \frac{2 \sin \chi \cos \Phi}{\sin(\Phi + \chi) \cos(\Phi - \chi)} \end{aligned} \right\}$$

It should be noticed that these formulae are unsuitable for perpendicular incidence, for when  $\Phi = 0$ ,  $\chi = 0$ , and the expression becomes indeterminate.

We will now examine these formulae in detail.

It is evident that the component perpendicular to the plane of incidence in the reflected light never vanishes, whatever be the values of  $\Phi$  and  $\chi$  in the formula for  $R_s$ .

It is different, however, in the case of the formula for  $R_p$ , the parallel component. As we increase the angle of incidence from 0, it is evident that we shall eventually reach a point at which the reflected and refracted rays are at right angles, for the angle between them is greater

than  $90^\circ$  near perpendicular incidence, and less than  $90^\circ$  at grazing incidence. At the angle in question it is evident that  $(\Phi + \chi) = 90^\circ$  and  $\tan(\Phi + \chi) = \infty$ , that is  $R_p = 0$ , or the component parallel to the plane of incidence is wholly absent. This means simply that the reflected light is plane polarized, and the angle in question is known as the angle of polarization or the Brewsterian angle.

The refractive index  $n = \frac{\sin \Phi}{\sin \chi}$ , consequently if  $\Phi'$  be the angle of polarization, we have

$$\sin \chi' = \sin\left(\frac{\pi}{2} - \Phi'\right) = \cos \Phi' \quad \text{and} \quad \frac{\sin \Phi'}{\cos \Phi} = \tan \Phi' = n,$$

a relation which has been fully discussed in the chapter on Polarization.

**Perpendicular Incidence.**—As has been stated our formulae do not hold for perpendicular incidence, for then  $\phi = \chi = 0$ .

Substituting in equations (23) for  $\frac{\sqrt{\epsilon_2}}{\sqrt{\epsilon_1}}$  the refractive index  $n$ , we have, since  $\frac{\cos \phi}{\cos \chi} = 1$ ,

$$\begin{aligned} R_s \left( \frac{1+n}{n} \right) &= E_s \left( \frac{1-n}{n} \right), \\ R_s &= E_s \left( \frac{1-n}{1+n} \right) = -E_s \left( \frac{n-1}{n+1} \right), \\ R_p &= E_p \left( \frac{n-1}{n+1} \right). \end{aligned}$$

From the first equation it is clear that if  $n > 1$ , the reflected electric amplitude is oppositely directed from the incident, since the direction of the vector depends on its sign. The second equation shows the same condition, for when  $\phi = 0$ , similar signs mean opposite directions, as will be seen by referring to Fig. 237. The presence of the reflected wave will thus reduce the amplitude of the incident wave *at the reflecting boundary*, and if the intensity of the reflected wave is equal to that of the incident, the amplitude will be reduced to zero. This was found to be the case by Wiener in his experiments upon stationary light waves, the node occurring at the reflecting surface. These experiments were fully described in the chapter on interference. The opposite condition will be found for the magnetic vectors, which are similarly directed in the incident and reflected waves. They will therefore add their effects at the boundary.

It must be remembered that the above formulae express the *amplitudes* of the reflected vibrations: the *intensity* of the reflected light, if the intensity of the incident light is 1, is given by

$$R_s^2 = I = \left( \frac{n-1}{n+1} \right)^2.$$

This formula has been verified for water by Lord Rayleigh, who found that the observed value agreed with the calculated within 1.5 %.

**Change of Phase by Passage through the Polarizing Angle.**—The formula for the reflected amplitude, parallel to the plane of incidence, shows us that the phase changes suddenly by  $180^\circ$  on passage through the angle of polarization: for  $(\Phi + \chi)$  is obtuse or acute according as the angle of incidence is less or greater than the polarizing angle. Suppose now that the incident light is plane polarized at an azimuth of  $45^\circ$ . At the polarizing angle the component of the vibration, which is parallel to the surface, will be the only one reflected, and it can be completely quenched by means of a Nicol prism held with its short diagonal vertical. On either side of the polarizing angle we shall have a reflected component perpendicular to the other, but the directions of the vectors will be opposite on opposite sides of the angle. The resultant will be in each case a plane-polarized vibration, which will, however, be turned slightly towards the plane of incidence, the direction of the rotation from the plane parallel to the surface being opposite in the two cases. This will be readily understood by drawing the horizontal component, and compounding it first with a small vertical component directed upwards, and then with one directed downwards, the change of direction corresponding to the phase change of  $180^\circ$ .

It was found by Jamin and others that in the majority of cases the light was not completely polarized by reflection at the Brewsterian angle. Moreover, if the incident light was polarized, and reflection occurred in the neighbourhood of this angle, the reflected light, instead of being plane polarized, as the formulae indicate, showed traces of elliptical polarization. This indicates that the phase change, instead of occurring abruptly at the polarizing angle, enters by degrees; Drude observed in 1889 that the elliptical polarization produced by a freshly-split surface of rock-salt was very small, but that it increased rapidly on the exposure of the surface to the air. Shortly afterward Lord Rayleigh found that the ellipticity produced by reflection from water, could be completely eliminated by removing the surface film of grease, which is always present unless special precautions are taken.

These experiments indicate that the disagreement with the formulae is caused by surface films having optical properties different from those of the body of the substance. We will now take up the investigation of the effects of these films, and the calculation of their probable thickness.

**Influence of Surface Films. Elliptical Polarization.**—The theory of reflection, applied to boundaries between media of different optical densities, has led us to the conclusion that plane-polarized light should always be reflected as plane-polarized light. As a matter of fact, we find that this is seldom the case. If the incident light is polarized at an angle of  $45^\circ$  with the plane of incidence, almost no change in intensity is seen when the reflected light is examined through a revolving Nicol, if the reflecting surface is a metal, while even in the case of transparent substances it is seldom possible to completely extinguish the reflected light with the analyzing Nicol. The cause of this we shall find is the almost universal presence of a so-called "surface-layer" within which the optical density changes gradually from that of the upper medium to that of the lower. In the previous treatment we considered that the change at the boundary was abrupt, and deduced our boundary conditions on this assumption.

The presence of the surface-layer, as we shall see, introduces a difference of phase between the two components of the reflected vibration, the resultant being an ellipse of greater or less eccentricity, as we can prove by means of a quarter-wave plate. We will now develop the boundary conditions which exist when a surface-layer is present, using the same method as before.

Consider that the dielectric constant changes gradually across the layer of thickness  $l$ , from the value  $\epsilon_1$  which it has in the upper medium to the value  $\epsilon_2$ , that of the lower medium. The two boundaries of the layer we will call 1 and 2, and the corresponding  $x$  components of the electric force  $X_1$  and  $X_2$ . In the previous case we found that these components were equal, otherwise a perpetual motion would be possible, and we could derive an unlimited amount of work by carrying an electric charge along the boundary in one medium, and back to its starting point in the other medium. We shall have to modify the treatment in the present case, because to perform the cycle it will be necessary to carry the charge a finite distance up and down parallel to the  $z$  axis through a distance equal to the thickness of the surface-layer.

If the plane of incidence is parallel to the  $x$  axis, the component  $z$  of the force is obviously a function of  $x$  (in the case of wave-motion), but not a function of  $y$ .

Let  $X_1$  and  $X_2$  be the values of the components of the force along the upper and lower surface of the layer,

and  $Z$  and  $Z + \frac{\partial z}{\partial x} dx$ , the value of the  $z$  component at the points indicated in the diagram (Fig. 238).

The work done upon, and gained from, a charge in carrying it around the cycle is

$$X_2 dx - \int_1^2 \left( X + \frac{\partial z}{\partial x} dx \right) dz - X_1 dx + \int_1^2 X dz,$$

which is to be equated to zero, since no work is gained or lost.

This, however, does not represent the whole of the work, since in the case of periodic disturbances the forces are changing with the time, which change will give rise to a magnetic flux, as shown by the arrow. This magnetic current will induce a counter electro-motive force equal to  $-\frac{1}{c} \frac{\partial N}{\partial t}$ , and since  $N = \beta$  multiplied by the area  $l dx$ , we have for our complete expression, writing  $l = \int_1^2 dz$ ,

$$X_2 dx - \int_1^2 \left( Z + \frac{\partial z}{\partial x} dx \right) dz - X_1 dx + \int_1^2 Z dz + \frac{1}{c} \frac{\partial}{\partial t} (\beta l dx) = 0.$$

Dividing through by  $dx$

$$X_2 - \int_1^2 \frac{\partial z}{\partial x} dz - X_1 + \frac{l}{c} \frac{\partial \beta}{\partial t} = 0.$$

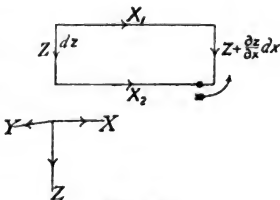


FIG. 238.

Now the induction is the same throughout the layer, or

$$\epsilon Z = \epsilon_2 Z_2, \quad Z = \frac{\epsilon_2 Z_2}{\epsilon}, \quad \frac{\partial Z}{\partial x} = \frac{\epsilon_2 Z_2}{\epsilon \partial x},$$

$$\int_1^2 \frac{\partial Z}{\partial x} dz = \int_1^2 \frac{\epsilon_2}{\epsilon} \frac{\partial Z_2}{\partial x} dz = \epsilon_2 \frac{\partial Z_2}{\partial x} \int_1^2 \frac{dz}{\epsilon}.$$

Let  $q = \int_1^2 \frac{dz}{\epsilon}$  and we have

$$X_1 = X_2 + \frac{l}{c} \frac{\partial \beta}{\partial t} - \epsilon_2 \frac{\partial Z_2}{\partial x} q,$$

the boundary conditions for the  $x$  component.

For non-periodic disturbances, where  $\beta$  does not vary with the time the equation reduces to  $X_1 = X_2$ , since  $q$  is small; that is, the surface layer plays no part when the conditions are statical.

Similarly for the  $y$  component we have, since  $z$  does not vary with  $y$  (Fig. 238a),

$$Y_2 dy - \int Z dz - Y_1 dy + \int Z dz - \frac{1}{c} \frac{\partial}{\partial t} a dy = 0,$$

and dividing by  $dy$ ,

$$Y_1 = Y_2 - \frac{l}{c} \frac{\partial a}{\partial t},$$

FIG. 238a.

the boundary conditions for the  $y$  component.

We have assumed that  $a_1 = a_2$  and  $\beta_1 = \beta_2$ , as in the previous case. This is not strictly true, but the differences between them are infinitesimals of the first order, and the effect of these differences on our values of  $X$  and  $Y$  will be second order infinitesimals, and can consequently be neglected.

In the same way, by considering  $X_1 = X_2$ , and  $Y_1 = Y_2$ , we find the boundary conditions for  $a$  and  $\beta$  to be

$$a_1 = a_2 - l \frac{\partial \gamma_2}{\partial x} - \frac{p}{c} \frac{\partial Y_2}{\partial t} \quad \text{and} \quad \beta_1 = \beta_2 + \frac{p}{c} \frac{\partial X_2}{\partial t},$$

in which

$$p = \int_1^2 \epsilon dz.$$

The physical interpretation of the new boundary conditions is that a change of phase occurs between the incident and refracted wave at the boundary. It is obvious that the maximum value which  $X$  attains (which measures the intensity) might have very nearly the same value in the two media, and yet  $X$  might have, at any given instant of time, quite different values on opposite sides of the boundary, if the maximum values were not attained at the same moment. This would of course mean a sudden phase change.

We will now investigate the change of phase between the incident and reflected waves.

If  $Y_r$  be the value of the  $y$  component of the force at time  $t$  and coordinate  $x$ , we have

$$Y_r = R_s \cos \left[ \frac{2\pi}{T} t - \frac{x \sin \phi' + z \cos \phi'}{V_1} + \delta \right],$$

in which  $\delta$  represents the increment of phase.

Writing this in exponential form, we have

$$Y_r = \text{real part of } R_s e^{i \left[ \frac{2\pi}{T} \left( t - \frac{x \sin \phi' + z \cos \phi'}{V_1} \right) + \delta \right]},$$

and writing

$$R_s e^{i\delta} = \mathbf{R}_s,$$

$$Y_r = \text{real part of } \mathbf{R}_s e^{i \frac{2\pi}{T} \left( t - \frac{x \sin \phi' + z \cos \phi'}{V_1} \right)},$$

since

$$e^{i\alpha} = \cos \alpha + i \sin \alpha.$$

By this transformation we represent the phase increment by the complex amplitude  $\mathbf{R}_s$ , the remainder of the expression having the same form as when reflection occurred without phase change.

Complex amplitudes can be used in all our former expressions for amplitude ratios. At the end of our calculations we obtain the physical significance of the equations by equating the real parts of the complex quantities.

The  $x$  component of the electric force in the upper medium at the boundary is the algebraic sum of the  $x$  components of the incident and reflected waves,

$$X_1 = X_i + X_r = X_2 + \frac{l}{c} \frac{\partial \beta_2}{\partial t} - \epsilon_2 \frac{\partial Z_2}{\partial t} q \quad \left( \begin{array}{l} \text{from the new} \\ \text{boundary conditions} \end{array} \right).$$

From the above we get (compare equations 22)

$$\begin{aligned} \mathbf{E}_p - \mathbf{R}_p \cos \phi e^{i \frac{2\pi}{T} \left( t - \frac{x \sin \phi}{V_1} \right)} \\ = \mathbf{D}_p \left[ \cos \chi e^{i \frac{2\pi}{T} \left( t - \frac{x \sin \phi}{V_1} \right)} + \frac{l}{c} \frac{\partial \beta_2}{\partial t} - \epsilon_2 \frac{\partial Z_2}{\partial t} q \right] \quad (\text{since } z = 0). \dots (25) \end{aligned}$$

Now

$$\beta_2 = \mathbf{D}_p \sqrt{\epsilon_2} e^{i \frac{2\pi}{T} \left( t - \frac{x \sin \Phi}{V_1} \right)},$$

$$\frac{\partial \beta_2}{\partial t} = \frac{2\pi i}{T} \mathbf{D}_p \sqrt{\epsilon_2} e^{i \frac{2\pi}{T} \left( t - \frac{x \sin \phi}{V_1} \right)},$$

$$Z_2 = -\mathbf{D}_p \sin \chi e^{i \frac{2\pi}{T} \left( t - \frac{x \sin \chi}{V_2} \right)},$$

$$\frac{\partial Z_2}{\partial t} = \frac{\sin \chi}{V_2} \frac{2\pi i}{T} \mathbf{D}_p \sin \chi e^{i \frac{2\pi}{T} \left( t - \frac{x \sin \chi}{V_2} \right)}.$$

Substituting these values in eq. (25) and dividing by  $e^{i\frac{2\pi}{T}\left(l - \frac{x \sin \phi}{V_1}\right)}$ ,

$$\left. \begin{aligned} (1) \quad (\mathbf{E}_p - \mathbf{R}_p) \cos \phi &= \mathbf{D}_p \left[ \cos \chi + i \frac{2\pi}{T} \left( \sqrt{\epsilon_2} \frac{l}{c} - \frac{\sin^2 \chi}{V_2} \epsilon_2 q \right) \right] \\ \text{Similarly,} \\ (2) \quad (\mathbf{E}_s + \mathbf{R}_s) &= \mathbf{D}_s \left[ 1 + i \frac{2\pi}{T} \cos \chi \sqrt{\epsilon_2} \frac{l}{c} \right] \\ (3) \quad (\mathbf{E}_s - \mathbf{R}_s) \sqrt{\epsilon_1} \cos \Phi &= \mathbf{D}_s \left[ \sqrt{\epsilon_2} \cos \chi - i \frac{2\pi}{T} \left( \frac{\sin^2 \chi}{V_2} \sqrt{\epsilon_2} l - \frac{p}{c} \right) \right] \\ (4) \quad (\mathbf{E}_p + \mathbf{R}_p) \sqrt{\epsilon_1} &= \mathbf{D}_p \left[ \sqrt{\epsilon_2} + i \frac{2\pi}{T} \cos \chi \frac{p}{c} \right] \end{aligned} \right\} \quad (26)$$

Adding (1) and (4) and subtracting (1) from (4) gives us expressions for  $2\mathbf{E}_p$  and  $2\mathbf{R}_p$ , and by division we get, writing  $Tc = \lambda$ ,  $V_2 = \frac{c}{\sqrt{\epsilon_2}}$ ,

$$\frac{\mathbf{R}_p}{\mathbf{E}_p} = \frac{\cos \phi \sqrt{\epsilon_2} - \cos \chi \sqrt{\epsilon_1} + i \frac{2\pi}{\lambda} [p \cos \phi \cos \chi - (l - q\epsilon_2 \sin^2 \chi) \sqrt{\epsilon_1 \epsilon_2}]}{\cos \phi \sqrt{\epsilon_2} + \cos \chi \sqrt{\epsilon_1} + i \frac{2\pi}{\lambda} [p \cos \phi \cos \chi + (l - q\epsilon_2 \sin^2 \chi) \sqrt{\epsilon_1 \epsilon_2}]}$$

If the surface film is absent, we have for the ratio

$$\frac{R_p}{E_p} = \frac{\frac{\cos \phi}{\cos \chi} - \frac{\sqrt{\epsilon_1}}{\sqrt{\epsilon_2}}}{\frac{\cos \phi}{\cos \chi} + \frac{\sqrt{\epsilon_1}}{\sqrt{\epsilon_2}}} = \frac{\cos \phi \sqrt{\epsilon_2} - \cos \chi \sqrt{\epsilon_1}}{\cos \phi \sqrt{\epsilon_2} + \cos \chi \sqrt{\epsilon_1}},$$

which differs from the above only by certain correction terms, which are small, since they are proportional to the thickness of the film  $l$ . Higher powers than the first of  $\frac{l}{\lambda}$  can be neglected in the development, also the squares of  $p$  and  $q$ .

The numerator of our expression for the ratio of the complex amplitudes has the form  $a' + ib'$ , the denominator  $\frac{1}{a + ib} = \frac{1}{a} - \frac{ib}{a^2}$  (neglecting terms with higher powers of  $b$ ),

$$\frac{\mathbf{R}_p}{\mathbf{E}_p} = (a' + ib') \left( \frac{1}{a} - \frac{ib}{a^2} \right) = \frac{a'}{a} \left[ 1 + \frac{i}{aa'} (ab' - ba') \right].$$

Going back to our values of  $a$ ,  $a'$ ,  $b$  and  $b'$ , we find that

$$ab' - ba' = \frac{4\pi}{\lambda} \cos \phi \sqrt{\epsilon_1} (p \cos^2 \chi - l\epsilon_2 + q\epsilon_2^2 \sin^2 \chi),$$

and

$$\left. \begin{aligned} \frac{\mathbf{R}_p}{\mathbf{E}_p} &= \frac{\cos \phi \sqrt{\epsilon_2} - \cos \chi \sqrt{\epsilon_1}}{\cos \phi \sqrt{\epsilon_2} + \cos \chi \sqrt{\epsilon_1}} \left\{ 1 + i \frac{4\pi}{\lambda} \cos \phi \sqrt{\epsilon_1} \frac{p \cos^2 \chi - l\epsilon_2 + q\epsilon_2^2 \sin^2 \chi}{\epsilon_2 \cos^2 \phi - \epsilon_1 \cos^2 \chi} \right\} \\ \text{Similarly,} \\ \frac{\mathbf{R}_s}{\mathbf{E}_s} &= \frac{\cos \phi \sqrt{\epsilon_1} - \cos \chi \sqrt{\epsilon_2}}{\cos \phi \sqrt{\epsilon_1} + \cos \chi \sqrt{\epsilon_2}} \left\{ 1 + i \frac{4\pi}{\lambda} \cos \phi \sqrt{\epsilon_1} \frac{l\epsilon_2 - p}{\epsilon_1 \cos^2 \phi - \epsilon_2 \cos^2 \chi} \right\} \end{aligned} \right\} \quad (27)$$

These two expressions suffice for the determination of the phases and amplitudes of the reflected components, with reference to the incident components, but to determine completely the nature of the reflected light we require an expression for the ratio  $\frac{R_p}{R_s}$ .

To simplify the calculation of this ratio, we will consider only the special case in which the incident light is polarized in azimuth  $45^\circ$ . Its two components  $E_p$  and  $E_s$  are therefore equal.

For brevity write

$$\frac{R_p}{E_p} = \frac{a}{b} \text{ and } \frac{R_s}{E_s} = \frac{c}{d}, \quad E_p = E_s = \frac{R_p b}{c} = \frac{R_s d}{c}, \quad \frac{R_p}{R_s} = \frac{d}{c} \frac{a}{b}.$$

Leaving out of account for the present the terms in the brackets in equations (27) (the numerators and denominators of which are represented by  $a$ ,  $b$ ,  $c$ , and  $d$ ), we have for the coefficient of  $\frac{R_p}{R_s}$ ,

$$\frac{\cos^2 \Phi \sqrt{\epsilon_1 \epsilon_2} + (\epsilon_2 - \epsilon_1) \cos \Phi \cos \chi - \cos^2 \chi \sqrt{\epsilon_1 \epsilon_2}}{\cos^2 \Phi \sqrt{\epsilon_1 \epsilon_2} - (\epsilon_2 - \epsilon_1) \cos \Phi \cos \chi - \cos^2 \chi \sqrt{\epsilon_1 \epsilon_2}}.$$

$$\text{Now } \frac{\sin \phi}{\sin \chi} = n = \frac{\sqrt{\epsilon_2}}{\sqrt{\epsilon_1}} \text{ and } \frac{\sin \phi \sin \chi}{\sin^2 \chi} = \frac{\sin \phi \sin \chi}{1 - \cos^2 \chi} = \frac{\sqrt{\epsilon_1 \epsilon_2}}{\epsilon_1};$$

$$\therefore \sqrt{\epsilon_1 \epsilon_2} \cos^2 \chi = -\epsilon_1 \sin \phi \sin \chi + \sqrt{\epsilon_1 \epsilon_2},$$

$$\sqrt{\epsilon_1 \epsilon_2} \cos^2 \phi = -\epsilon_2 \sin \phi \sin \chi + \sqrt{\epsilon_1 \epsilon_2}.$$

Substituting these values in the coefficient, we find that it reduces to

$$\begin{aligned} \frac{(\epsilon_2 - \epsilon_1) \cos \phi \cos \chi - (\epsilon_2 - \epsilon_1) \sin \Phi \sin \chi}{-(\epsilon_2 - \epsilon_1) \cos \phi \cos \chi - (\epsilon_2 - \epsilon_1) \sin \phi \sin \chi} &= \frac{(\epsilon_2 - \epsilon_1) \cos (\Phi + \chi)}{-(\epsilon_2 - \epsilon_1) \cos (\Phi - \chi)} \\ &= -\frac{\cos (\Phi + \chi)}{\cos (\Phi - \chi)}. \end{aligned}$$

This quantity is to be multiplied by the quotient obtained by dividing the bracketed term in the first of equations (27) by that of the second, an expression of the form  $\frac{1 + ib'}{1 + ib}$ .

Now  $\frac{1}{1 + ib} = 1 - ib$  (neglecting higher powers of  $b$ ), and the fraction therefore becomes  $(1 + ib')(1 - ib) = 1 + i(b' - b)$ ,

$$b' = \frac{4\pi}{\lambda} \cos \phi \sqrt{\epsilon_1} \frac{p \cos^2 \chi - l\epsilon_2 + q\epsilon_2^2 \sin^2 \chi}{\epsilon_1 - \epsilon_2 (\epsilon_1 \sin^2 \phi - \epsilon_2 \cos^2 \phi)} \text{ from eq. (27),}$$

$$b = \frac{4\pi}{\lambda} \cos \phi \sqrt{\epsilon_1} \frac{l\epsilon_2 - p}{\epsilon_1 - \epsilon_2},$$

from which we deduce the expression

$$b' - b = \frac{4\pi}{\lambda} \frac{\epsilon_2 \sqrt{\epsilon_1}}{\epsilon_1 - \epsilon_2} \frac{\cos \phi \sin^2 \phi}{\epsilon_1 \sin^2 \phi - \epsilon_2 \cos^2 \phi} (p - l(\epsilon_1 + \epsilon_2) + q\epsilon_1 \epsilon_2),$$

$$\frac{\mathbf{R}_p}{\mathbf{R}_s} = -\frac{\cos(\phi + \chi)}{\cos(\phi - \chi)} \left\{ 1 + i \frac{4\pi}{\lambda} \frac{\epsilon_2 \sqrt{\epsilon_1}}{\epsilon_1 - \epsilon_2} \frac{\cos \phi \sin^2 \phi}{\epsilon_1 \sin^2 \phi - \epsilon_2 \cos^2 \phi} \right\} \eta,$$

in which  $\eta = p - l(\epsilon_1 + \epsilon_2) + q\epsilon_1 \epsilon_2$ .

For the polarizing angle, for which  $\tan \phi = n$ , the above expression takes the form

$$\frac{\mathbf{R}_p}{\mathbf{R}_s} = i \frac{\pi}{\lambda} \frac{\sqrt{\epsilon_1 + \epsilon_2}}{\epsilon_1 - \epsilon_2} \eta.$$

Let us now see what the physical interpretation of this last equation is. Remembering that  $\mathbf{R}_p$  and  $\mathbf{R}_s$  are both complex, we have

$$\mathbf{R}_p = R_p e^{i\delta_p} \quad \text{and} \quad \mathbf{R}_s = R_s e^{i\delta_s},$$

in which  $R_p$  and  $R_s$  are the components of the electric amplitudes and  $\delta_p$  and  $\delta_s$  their respective phase accelerations, with respect to the incident wave.

$$\frac{\mathbf{R}_p}{\mathbf{R}_s} = \frac{R_p}{R_s} e^{i(\delta_p - \delta_s)} = \rho e^{i\Delta},$$

in which  $\rho$  is the amplitude ratio, and  $\Delta$  the phase difference between the two mutually perpendicular components of the reflected light.

To determine the difference of phase between the components of the reflected light, we write

$$\bar{\rho} e^{i\Delta} = i \frac{\pi}{\lambda} \frac{\sqrt{\epsilon_1 - \epsilon_2}}{\epsilon_1 - \epsilon_2} \eta,$$

or

$$\bar{\rho} \cos \Delta + i \bar{\rho} \sin \Delta = i \frac{\pi}{\lambda} \frac{\sqrt{\epsilon_1 - \epsilon_2}}{\epsilon_1 - \epsilon_2} \eta,$$

which, when we equate the real and imaginary parts, gives us

$$\bar{\rho} \cos \Delta = 0, \quad \Delta = \frac{\pi}{2},$$

and

$$i \bar{\rho} \sin \frac{\pi}{2} = i \frac{\pi}{\lambda} \frac{\sqrt{\epsilon_1 - \epsilon_2}}{\epsilon_1 - \epsilon_2}, \quad \bar{\rho} = \frac{\pi}{\lambda} \eta \frac{\sqrt{\epsilon_1 - \epsilon_2}}{\epsilon_1 - \epsilon_2}, \dots\dots\dots (29)$$

The reflected light is consequently elliptically polarized, and since  $\Delta = \frac{\pi}{2}$ , the axes of the ellipse are parallel and perpendicular to the plane of reflection. The eccentricity of the ellipse depends on the ratio  $\bar{\rho}$  of the axes, which is given by equation (29).

Going back now to our definitions of  $p$ ,  $q$  and  $l$ , we have

$$\left. \begin{aligned} \eta &= \int_1^2 \epsilon dz - (\epsilon_1 + \epsilon_2) \int_1^2 dz + \int_1^2 \frac{dz}{\epsilon} \epsilon_1 \epsilon_2 = \int_1^2 \left( \epsilon - \epsilon_1 - \epsilon_2 + \frac{\epsilon_1 \epsilon_2}{\epsilon} \right) dz, \\ \eta &= \int_1^2 \frac{(\epsilon - \epsilon_1)(\epsilon - \epsilon_2)}{\epsilon} dz, \\ \bar{\rho} &= \frac{\pi}{\lambda} \frac{\sqrt{\epsilon_1 - \epsilon_2}}{\epsilon_1 - \epsilon_2} \int \frac{(\epsilon - \epsilon_1)(\epsilon - \epsilon_2)}{\epsilon} dz. \end{aligned} \right\} \quad (30)$$

This expression shows us that  $\rho$  has a positive sign when  $\epsilon_2 > \epsilon_1$ , and the dielectric constant  $\epsilon$  of the film has an intermediate value. If the incident light is considered as coming towards us and the plane of vibration makes an angle of  $45^\circ$  with the vertical (the rotation away from the vertical being clockwise), the direction of the elliptical vibration will be clockwise for positive values of  $\bar{\rho}$  and counter-clockwise for negative values, a change of sign of the amplitude ratio, amounting to the same thing as a phase difference of  $180^\circ$  between the components. We can easily determine the value of  $\bar{\rho}$  experimentally by either of the methods given in the chapter on Elliptical Polarization. In the case of reflection at a glass surface in air it has a value not far from  $\cdot 007$ , though for flint glass with a high refractive index the value may be as high as  $\cdot 03$ . Negative ellipticity, which occurs when  $\epsilon > \epsilon_2$ , has been observed in the case of water and certain solids with very low refractive indices.

In the case of water the ellipticity is due to a very thin film of grease, as has been shown by Lord Rayleigh, which naturally has a higher refractive index than water. In the case of solids, a higher refractive index of the surface film may perhaps be explained by some action of the polishing material upon the surface.

By carefully cleaning the surface of the water, Lord Rayleigh was able to almost completely destroy all traces of elliptical polarization, the value of  $\rho$  being not more than  $\cdot 00035$ .

Drude found that freshly cleaned crystal surfaces showed no traces of elliptical polarization, but that it appeared after the surfaces had been exposed to the air for some time, owing to the formation of surface films.

**Thickness of the Surface Films.**—If we assume that the dielectric constant is uniform throughout the film, we can calculate the thickness of the film necessary to produce a given axis ratio  $\rho$ . To do this we must first find the value of  $\epsilon$ , which will give the factor  $\frac{(\epsilon - \epsilon_1)(\epsilon - \epsilon_2)}{\epsilon}$

in eq. (30) its largest value. Differentiating this expression with respect to  $\epsilon$  and equating to zero, we find that  $\epsilon = \sqrt{\epsilon_1 \epsilon_2}$  or the dielectric constant of the film must be the geometrical mean of the constants of the two media. Substituting this value in equation (30) and writing  $l$  for  $d$  we find that

$$\frac{l}{\lambda} = -\frac{\bar{\rho}}{\pi \sqrt{\epsilon_1 + \epsilon_2}} \cdot \frac{\sqrt{\epsilon_2} + \sqrt{\epsilon_1}}{\sqrt{\epsilon_2} - \sqrt{\epsilon_1}} = \frac{\bar{\rho}}{\pi \sqrt{1 + n^2}} \frac{n + 1}{n - 1},$$

$n$  being the refractive index of the second medium with respect to the first.

For glass of refractive index  $n = 1.5$  and  $\bar{\rho} = \cdot 007$  we find for  $\frac{l}{\lambda}$  the value  $\cdot 0035$ , or the thickness of the film necessary to account for the ellipticity is less than  $\frac{1}{300}$  of the wave-length of the light.

**Total Reflection.**—We have seen in the chapter on Refraction that when a ray of light is incident at the boundary separating an optically dense from a rarer medium, that the refracted ray vanishes for incidence angles greater than a certain value, the energy being totally

reflected. We will now apply our equations to this phenomenon. In this case  $\sin \chi$  turns out to be greater than unity, *i.e.*  $\chi$  is no longer real. We can study the nature of the reflected light, however, by substituting  $\frac{\sin \phi}{n}$  for  $\sin \chi$  in the equations (21) on page 293,

$$\therefore \cos \chi = \sqrt{1 - \frac{\sin^2 \phi}{n^2}},$$

which quantity is imaginary if  $\sin \phi > n$ .

We can write this in the form

$$\cos \chi = -i \sqrt{\frac{\sin^2 \phi}{n^2} - 1}. \dots\dots\dots (31)$$

Substitution of the above in equation (23), page 294, gives us reflected light with a complex amplitude, which as we have seen can be interpreted as a change of phase which results at the moment of reflection. This change of phase, if it is different for the two components of the incident vibration, will result in the transformation of a linear vibration into an elliptical one, and as total reflection is one of the methods commonly employed to produce elliptically and circularly polarized light, the subject is of some importance. To calculate this phase change we write as before  $R_p e^{i\delta_p}$  and  $R_i e^{i\delta_i}$  for the components of the reflected amplitude, and obtain, since  $\sqrt{\epsilon_2}/\sqrt{\epsilon_1} = n$ , by substitution in equations (23),

$$E_i \left( \frac{i \cos \chi}{\sqrt{\sin^2 \phi - n^2}} - 1 \right) = R_i e^{i\delta_i} \left( \frac{i \cos \chi}{\sqrt{\sin^2 \phi - n^2}} + 1 \right),$$

$$E_p \left( \frac{i \cos \chi \cdot n}{\sqrt{\sin^2 \phi - n^2}} - \frac{1}{n} \right) = R_p e^{i\delta_p} \left( \frac{i \cos \chi \cdot n}{\sqrt{\sin^2 \phi - n^2}} + \frac{1}{n} \right).$$

If we multiply these equations by their complex conjugates, obtained by writing in them  $-i$  for  $i$ , we find that  $E_i^2 = R_i^2$  and  $E_p^2 = R_p^2$ , *i.e.* the intensities of the reflected components are equal to those of the incident.

Suppose now that our incident light is plane-polarized vibrating in azimuth  $45^\circ$ . In this case  $E_i = E_p$  and  $R_i = R_p$ , and if we substitute these values in the above equations and divide, we get

$$\frac{i \cos \chi - \sqrt{\sin^2 \phi - n^2}}{i \cos \chi \cdot n - \frac{1}{n} \sqrt{\sin^2 \phi - n^2}} = e^{i(\delta_i - \delta_p)} \frac{i \cos \chi + \sqrt{\sin^2 \phi - n^2}}{i \cos \chi \cdot n + \frac{1}{n} \sqrt{\sin^2 \phi - n^2}},$$

or 
$$e^{i(\delta_p - \delta_i)} = e^{i\Delta} = \frac{\sin^2 \phi + i \cos \chi \sqrt{\sin^2 \phi - n^2}}{\sin^2 \phi - i \cos \chi \sqrt{\sin^2 \phi - n^2}},$$

and 
$$\frac{1 - e^{i\Delta}}{1 + e^{i\Delta}} = \frac{-i \cos \chi \sqrt{\sin^2 \phi - n^2}}{\sin^2 \phi}.$$

Multiplying this by its complex conjugate gives

$$\frac{1 - \cos \Delta}{1 + \cos \Delta} = \left\{ \frac{\cos \Phi - \sqrt{\sin^2 \Phi - n^2}}{\sin^2 \Phi} \right\}^2,$$

since

$$e^{i\Delta} + e^{-i\Delta} = 2 \cos \Delta;$$

therefore

$$\tan \frac{1}{2} \Delta = \frac{\cos \Phi \sqrt{\sin^2 \Phi - n^2}}{\sin^2 \Phi} \dots \dots \dots (32)$$

This expression shows us that the relative phase difference  $\Delta$  is zero for grazing incidence ( $\phi = \frac{\pi}{2}$ ), and also at the critical angle ( $\sin \Phi = n$ ), in which  $n$  is the relative refractive index. If the denser medium has a refractive index 1.51 and the reflection occurs at an air surface,  $n$  in our equations will be  $\frac{1}{1.51}$ .

To find the value of  $\Phi$  which will give  $\Delta$  its maximum value we differentiate the last equation with respect to  $\Phi$ , and obtain

$$\frac{1}{2 \cos^2 \frac{1}{2} \Delta} \frac{\partial \Delta}{\partial \Phi} = \frac{2n^2 - \sin^2 \Phi (1 + n^2)}{\sin^3 \Phi \sqrt{\sin^2 \Phi - n^2}},$$

and the maximum value of  $\Delta$  is obtained at an incidence angle  $\Phi'$  determined by

$$\sin^2 \Phi' = \frac{2n^2}{1 + n^2}.$$

The corresponding value of  $\Delta$  is given by

$$\tan \frac{1}{2} \Delta' = \frac{1 - n^2}{2n}.$$

For glass of refractive index 1.51,  $\Phi' = 51^\circ 20'$  and  $\Delta' = 45^\circ 36'$ . A value of  $45^\circ$  occurs at incidence angles  $48^\circ 37'$  and  $54^\circ 37'$ . Two reflections at this angle will give us  $\Delta = 90^\circ$ , and circularly-polarized light will result if the incident light was polarized in azimuth  $45^\circ$ . This is accomplished by the Fresnel rhomb described in the chapter on Elliptical Polarization.

**Penetration of the Disturbance into the Second Medium.**—If we apply the equations (23), page 294, which express the relation between the incident and the refracted amplitudes, to the case of total reflection, we reach the somewhat astonishing conclusion that the refracted amplitude is not zero, which appears to be inconsistent with the *total* reflection of the energy. The case is a peculiar one, for although  $D$  may have a large value close to the boundary it becomes zero at a distance of a few wave-lengths, the energy being entirely thrown back into the first medium. This decrease in the amplitude, as we advance from the boundary in the direction of the  $z$  axis, can be seen from equations (15) (16), page 292, which show that electric and magnetic forces in the second medium are proportional to the real part of the complex quantity

$$e^{\frac{2\pi}{T} \left( t - \frac{x \sin \chi - z \cos \chi}{V_2} \right)},$$

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in which if we substitute for  $\cos \chi$  the value given by equation (31), page 304, takes the form

$$e^{-\frac{2\pi}{TV_2} \sqrt{\frac{\sin^2 \Phi}{n^2} - 1} \cdot z} \cdot e^{i \frac{2\pi}{T} \left( t - \frac{x \sin \Phi}{nV_2} \right)} \dots \dots \dots (33)$$

This formula represents a wave disturbance moving parallel to the  $x$  axis, which implies that the energy stream is along the boundary, and not down into the second medium. The amplitude, which is represented by the underscored part of (33), decreases as  $z$  increases, becoming sensibly zero when  $z$  is large in comparison to the wave-length  $\lambda_2 = TV_2$ . These boundary waves possess another peculiarity, in that they are not transverse, for in a transverse disturbance moving along the  $X$  axis in the second medium,  $X_2$  must equal zero, which is not the case.

The existence of these waves can be shown experimentally by bringing a convex surface of glass of large radius of curvature, into contact with the surface at which total reflection is taking place. The light will be found to enter the lens in an annular region surrounding the point of contact. This experiment dates back to the time of Newton and Fresnel, and shows us that if the rarer medium

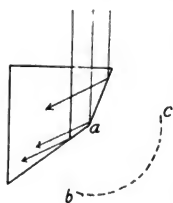


FIG. 239.

is of extreme thinness, total reflection no longer occurs. Voigt<sup>1</sup> has described an experiment designed to show the possibility of separating this surface wave from the incident and reflected waves, and allowing it to spread out into space. A prism of the form shown in Fig. 239 was constructed, and light passed into it in the direction indicated by the arrows. The angle of incidence exceeded the critical angle at both of the surfaces, and must therefore have exceeded it along the edge  $a$ , which can be regarded as a cylinder of very minute radius of curvature.

Voigt observed that the edge  $a$  was luminous, and that the intensity was greatest when viewed from a point  $b$ , decreasing steadily as the eye was moved towards  $c$ . He regarded this emission of light by the edge as due to the breaking away of the surface wave from the wave disturbances in the glass.<sup>2</sup>

Ketteler<sup>3</sup> criticised this experiment, and claimed that it was a physical impossibility to obtain a separation of the two wave disturbances in the manner imagined by Voigt. An attempt was made by the author some years ago to repeat Voigt's experiment with a prism of very nearly identical form, but no distinct evidence of the phenomenon was observed, and it seems possible that the effect observed might have been due to the fact that the edge was not in fact a perfectly polished cylinder. It is difficult, however, if not

<sup>1</sup> *Wied. Ann.*, 67, page 185, 1899.

<sup>2</sup> In Fig. 239 the directions of the reflected rays are not properly indicated.

<sup>3</sup> *Wied. Ann.*, 67, 879.

impossible, to pass judgment on an experiment of this nature without actually witnessing it.

Another method of showing the presence of a luminous disturbance in the rarer medium, which is to be recommended on account of its simplicity, is to scatter minute particles on the reflecting surface: for example, smoke the hypotenuse surface of a right angle prism *very lightly* with a flame. On sending a strong beam of light into the prism the smoked patch will be illuminated, and if viewed under a powerful microscope, each individual carbon particle will be seen to scatter light in all directions. This method was used by the author in establishing the granular nature of certain metallic films, which will be described in the chapter on Optical Resonance. The method was subsequently and independently originated by Cotton as a means of rendering visible ultra-microscopic particles.

## CHAPTER XIII.

### THE THEORY OF DISPERSION.

PREVIOUS to the discovery of anomalous dispersion, all that was required of a dispersion theory was a satisfactory explanation of a steady increase in the refractive index with decreasing wave-length, and the development of a mathematical relation between the two which should conform to the dispersion curves determined by experiment. The expression for the velocity of transverse waves  $v = C\sqrt{\frac{\epsilon}{d}}$ ,

where  $\epsilon$  is the elasticity and  $d$  the density of the medium, is developed on the assumption that the wave-length is large in comparison to the distance between the vibrating particles. If this is not the case, the velocity of propagation is a function of the wave-length, as was shown by Cauchy, who assumed that in refracting media it was not allowable to assume the wave-length large in comparison to the distance between the particles, on account of the shortening of the waves by retardation. Cauchy deduced the expression

$$n = A + \frac{B}{\lambda^2} + \frac{C}{\lambda^4},$$

which gives the refractive index in terms of the wave length and three constants, for the determination of which we require determinations of  $n$  for three different wave lengths.

This formula was found to represent the dispersion of many transparent substances with considerable accuracy, and the fact that the dispersion increases as we pass down the spectrum into the region of the short waves, makes it appear at first sight as if his assumption was justifiable. It was pointed out by Biot, however, that the shortening of the waves was not sufficient to account for the phenomenon, since if, in dispersing media, the wave length has a value comparable to the distance between the particles, the same must be true in free ether, which should, therefore, show evidences of dispersion; in other words, there is not sufficient change in the wave-length. It is evident that this must be the case, for the wave-length of red light in glass is greater than that of violet light in free ether. That the formula expresses the relation between  $n$  and  $\lambda$  in certain cases is purely accidental, and, as we shall see presently, it is a special case of a much

more elaborate formula, developed from quite different fundamental assumptions.

The discovery of anomalous dispersion, and the relation existing between absorption and dispersion, put the matter in a new light. The refractive index of a medium, which exercised strong selective absorption, was found to increase rapidly as the absorption band was approached from the region of longer wave-lengths. This made it seem extremely probable that the dispersion of so-called transparent media was due to absorption bands in the ultra-violet; in other words, that there was no essential difference between normal and anomalous dispersion, the former being only a special case of the latter, the observations being restricted to a range of wave-lengths too narrow to show any anomalies.

As has been recently pointed out by Lord Rayleigh, the foundation of the theory of modern dispersion was in reality laid by Maxwell in the form of a question propounded in an examination paper (*Camb. Calendar*, 1869, Math. Tripos Exam.). The same idea subsequently occurred to Sellmeier, who has always been regarded as the founder of the theory. Sellmeier sought for the cause of dispersion in the vibrations of the atoms of the molecule caused by the repeated impacts of the light waves. These atoms would naturally have free-periods of vibration of their own, and would be set in motion by the light waves, exactly as a tuning fork is set in vibration by waves of sound.

Sellmeier deduced a formula which is practically identical with a special case of the more recent electro-magnetic dispersion formula, and which represents the dispersion for those wave-lengths for which the medium is comparatively transparent, *i.e.* on both sides of the absorption band. Within the region of absorption it breaks down, for reasons which will appear presently. Sellmeier's formula is as follows:

$$n^2 = 1 + \frac{DT^2}{T^2 - T_m^2},$$

where  $n$  is the refractive index for light of periodic time  $T$ , and  $T_m$  is the periodic time of the vibrating atom. Substituting wave-lengths for periodic times the formula becomes

$$n^2 = 1 + \frac{D\lambda^2}{\lambda^2 - \lambda_m^2},$$

where  $\lambda_m$  is the wave-length in ether of light of the same vibration frequency as the absorbing atom. If more than one absorption band exists, the formula takes the form

$$n^2 = 1 + \sum \frac{D\lambda^2}{\lambda^2 - \lambda_m^2},$$

the summation being taken for as many terms as there are atoms of different periods. We will now compare this formula with the Cauchy formula:

$$n = A + \frac{B}{\lambda^2} + \frac{C}{\lambda^4}.$$

As the wave-length increases, the refractive index, as expressed by this formula, becomes less, approaching asymptotically the limiting value

*A*. This was formerly supposed to agree with experiment, but more recent investigations in the infra-red have shown that the dispersion curve after running in a nearly horizontal direction for a certain distance, may again descend.

In Fig. 240 *XABC* is an experimentally determined curve. The portion *AB* follows the Cauchy formula, which, if applied to values

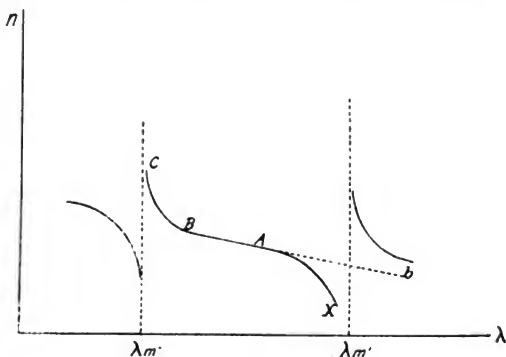


FIG. 240.

of  $\lambda$  greater than the value at *A*, would give the dotted curve *Ab* instead of the actual curve *AX*. For infinitely short waves the refractive index would be infinite.

Let us now examine the formula of Sellmeier,

$$n^2 = 1 + \frac{D\lambda^2}{\lambda^2 - \lambda_m^2}.$$

For very long waves the denominator becomes approximately equal to  $\lambda^2$ , and we have

$$n = \sqrt{1 + D},$$

which, if *D* is small, as is sometimes the case, does not differ much from unity. As  $\lambda$  decreases the value of the fraction  $\frac{\lambda^2}{\lambda^2 - \lambda_m^2}$  increases,

becoming infinite when  $\lambda = \lambda_m$ . For values of  $\lambda$  less than  $\lambda_m$  the sign of the term changes, and we have values of *n*, which are less than unity, the lowest values being for wave-lengths close to the absorption band; as  $\lambda$  decreases *n* increases, becoming unity for infinitely short waves. The form of the curve is shown in Fig. 241. As we shall see later on, the Sellmeier formula represents most perfectly the dispersion of sodium vapor, in which the refraction and dispersion are due almost entirely to the influence of a single absorption band (in reality a close double band).

If we confine our attention to the region *AB* of the curve, we see that the dispersion here is normal, the refractive index increasing with decreasing  $\lambda$ , and the curve convex towards the axis of abscissae, as is the case with all ordinary transparent media for visible radiations.

The decrease in refractive index occurs when we pass across the absorption band, in the above case the drop being very sudden. A further decrease in the value of  $\lambda$  causes an increase in  $n$ , the curve

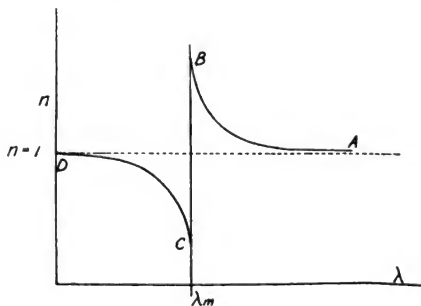


FIG. 241.

now being *concave* towards the axis of abscissae. The dispersion here is normal in that  $n$  increases as  $\lambda$  decreases, but it differs from the dispersion along the branch  $AB$  in that the rate of change of  $n$  with  $\lambda$  becomes less as  $\lambda$  becomes less, while along  $AB$  the rate of change of  $n$  with  $\lambda$  increases as  $\lambda$  decreases. This is due to the fact that in the former case we are receding from the absorption band, while in the latter we are approaching it.

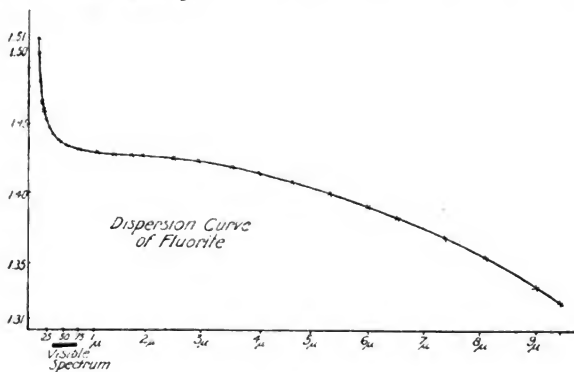


FIG. 242.

The dispersion of glass or water, for example, is represented by a curve of form similar to  $AB$ , consequently we may infer that an absorption band in the ultra-violet is responsible for the dispersion.

Curves of the form  $CD$  are never found for ordinary transparent media in the visible spectrum. If, however, the investigations are carried into the infra-red region, we frequently find that the curve, which in the visible region of the spectrum was convex towards the axis of abscissae, eventually becomes concave in this direction, indicating that an absorption band is being approached. The curve for fluorite (Fig. 242) is an example.

If we apply Sellmeier's formula to the dispersion of some substance such as glass, where the refractive index appears to approach a limiting value of say 1.5 in the extreme red, instead of unity in the first case considered, we find that the only difference lies in the larger value of the constant  $D$ , which must be equal to 1.25 if  $n = \sqrt{1 + D} = 1.5$ .

A fuller discussion of the matter can be undertaken to better advantage after we have actually developed the dispersion formula, for we are then in a position to recognize the physical significance of the constants.

Thus far all that has been attempted is to show that normal dispersion is only a special case of the so-called anomalous dispersion, and that the general form of the curve can be represented by a formula in which the difference between the squares of the observed wave-length, and the wave-length at the center of the absorption band occurs in the denominator.

The formula of Sellmeier was incomplete in that it was inapplicable to values of  $\lambda$  very near the value of  $\lambda_m$ . The curve which the formula represents, runs to infinity on one side of the center of the absorption band, and to zero on the other. On the fundamental assumptions made by Sellmeier there would in fact be no absorption, for he introduced no term which provided for a transformation of the radiant energy into energy of some other form, which must occur if light is actually absorbed by the medium.

The conception of something akin to friction accompanying the vibration of the atom was introduced by Helmholtz, who formed separate differential equations for the vibration of the ether and that of the ponderable atom, introducing a term representing a frictional loss into the latter. The absorption of the light is here accounted for as a frictional transformation of the radiant energy into heat, and the final formula which expresses the variation of  $n$  with  $\lambda$  shows that the dispersion curve is continuous through the absorption band. The factor expressing friction enters into the formula in such a way that its tendency is to *decrease* the value of  $n$  as the absorption band is approached from the region of longer waves. This factor becomes larger as we near the center of the band, consequently the curve, instead of running off to infinity, turns as we enter the band, and running down through it meets the other branch, the whole curve being continuous.

We will now proceed with the development of the complete dispersion formula, first on the mechanical theory and then on the electromagnetic theory. The former treatment will be made independent of complex quantities, and though longer, is perhaps easier to follow; the latter will involve the use of imaginaries, and though the final equation is not very different from the other, it will help us to understand the

physical significance of the constants, and the relations existing between the optical and electrical properties of various media

**The Helmholtz Dispersion Formula.**—In the following treatment we shall regard the ether in the nature of an elastic solid, *i.e.* made up of small particles, which when displaced are urged back into their original position by forces of restitution. The refracting medium we assume to be made up of molecules, between which the ether penetrates freely. The atoms of these molecules are capable of vibrating in periods of their own like pendulums, and any displacement of an ether particle is assumed to cause a displacement of one or more atoms; in other words, forces exist between the ether particles and atoms, similar to those existing between the ether particles themselves. When a wave enters the medium, we consider it propagated through the agency of the ether alone—that is, there is no direct propagation of a disturbance from molecule to molecule or from atom to atom. Helmholtz considered that the molecule remained at rest, but that the atom could be displaced from its position of equilibrium by the vibration of the ether, and when so displaced was drawn back by a force of restitution proportional to the displacement. Though the atoms are independent of each other, and each is free to vibrate by itself, they will, when disturbed in succession by a passing wave, have displacements which collectively form a wave curve, just as chips floating on water-waves, though not transmitting the waves, will be arranged in the form of the wave-curve, the difference in this case being an absence of any force of restitution tending to draw back the displaced chips.

Thus far the conception is not very different from that of Sellmeier. Helmholtz next assumes the vibration of the atom to be accompanied by friction, for if this were not the case the energy taken from the ether would be given back to the ether by the vibrating atom and no absorption would take place. The velocity with which waves are propagated through a medium can be determined in terms of the elasticity and density of the medium. The measure of the elasticity is the force of restitution exerted upon a displaced particle by the neighboring particles. In the case which we are about to consider, the displaced ether particle is urged back into its position of equilibrium not only by the forces exerted by neighboring ether particles, but also by the forces existing between the ether and the atoms. The atoms are, however, not stationary, but are set in motion to a greater or less degree by the waves. If the waves have the same period as the atom, the latter will be violently agitated, in a manner analogous to that of a tuning fork when subjected to sound waves of the same period as its own. If the period of the light-waves be different from the free period of the atom, the latter will be forced to vibrate with a period different from its natural period, and the amplitude of the vibration will be less than before. We have to determine the velocity of waves of different length, as influenced by the reaction upon the ether of the vibrating atoms, taking into account the damping which they experience as the result of friction.

Let  $\mu$  = the mass of the ether particle and  $\eta$  its displacement at time  $t$

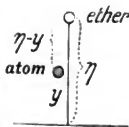


FIG. 243.

(Fig. 243). Let  $m$  = the mass of the atom and  $y$  its displacement at time  $t$ . If  $\epsilon$  is the elasticity of the ether, the equation of motion of the ether alone, the atom considered absent, would be

$$\frac{d^2\eta}{dt^2} = a \frac{\epsilon}{\mu} \frac{d^2\eta}{dx^2}.$$

To this acceleration must be added that due to the force exerted on the ether particle by the atom. If  $B$  is the force for unit displacement, the force in this case is  $-B(\eta - y)$ , the negative sign being given since the force acts in a direction opposite to the displacement. The expression for the acceleration of the ether particle now becomes

$$(1) \quad \frac{d^2\eta}{dt^2} = a \frac{\epsilon}{m} \frac{d^2\eta}{dx^2} - \beta(\eta - y).$$

This expression is not sufficient to determine the motion of the ether particle; we require in addition an expression for the motion of the atom.

The ether particle exerts on the atom, in the direction of the displacement, the force  $B(\eta - y)$ . The atom is drawn in the other direction by a force  $-\gamma y$ , where  $\gamma$  is the force exerted when the displacement is unity.

From this is to be subtracted a term representing the effect of friction, which is proportional to the velocity. The equation for the motion of the atom of mass  $m$  is

$$(2) \quad \frac{d^2y}{dt^2} = \frac{\beta}{m}(\eta - y) - \frac{\gamma y}{m} - \frac{\delta}{m} \frac{dy}{dt},$$

where  $\delta$  is the frictional resistance for unit velocity.

These two equations are sufficient to determine the motion of the ether and the atom.

We shall now get values for  $\frac{d^2\eta}{dt^2}$  and  $\frac{d^2y}{dt^2}$  by twice differentiating the usual expression for the displacement in the case of wave-motion, and substitute the values thus found in the above equations.

Let  $t$  be the time which has elapsed since the disturbance reached the boundary of the medium,  $T$  the period of the vibration of the wave,  $x$ , the distance of the particle in question from the boundary,  $\lambda$  the wave-length in the medium and  $a$  the amplitude of the particle.

$$\text{Then} \quad \eta = a \sin 2\pi \left( \frac{t}{T} - \frac{x}{\lambda} \right),$$

in which  $\eta$  is the displacement of a particle at distance  $x$  from the boundary at time  $t$

As we are dealing with a medium in which absorption may take place, we must modify this expression somewhat. The amplitude will decrease as the disturbance penetrates into the medium, owing to the friction of the vibrating atoms, and it will decrease by the same fractional amount for equal increments of  $x$ .

Let  $A$  be the amplitude at the boundary. At distance  $x$  from the boundary the amplitude  $a$  will be,  $a = Ae^{-kx}$ , in which  $e$  is the base of the natural logarithm system and  $k$  is a constant.

Substituting this in our last equation we have

$$(3) \quad \eta = Ae^{-kx} \sin 2\pi \left( \frac{t}{T} - \frac{x}{l} \right).$$

In the same way, if  $B$  is the amplitude of the atom at the boundary, we have

$$(4) \quad y = Be^{-kx} \sin 2\pi \left( \frac{t}{T} - \frac{x}{l} - \Delta \right),$$

where  $\Delta$  is a possible difference of phase between the ether particle and the atom.

We now twice differentiate  $\eta$  and  $y$  with respect to  $t$ , regarding  $x$  as constant, and twice with respect to  $x$ , regarding  $t$  as constant, and substitute these values in equations (1) and (2).

$$\frac{d^2\eta}{dt^2} = -\frac{4\pi^2}{T^2} Ae^{-kx} \sin 2\pi \left( \frac{t}{T} - \frac{x}{l} \right),$$

$$\frac{d^2y}{dt^2} = -\frac{4\pi^2}{T^2} Be^{-kx} \sin 2\pi \left( \frac{t}{T} - \frac{x}{l} - \Delta \right),$$

$$\frac{d^2\eta}{dx^2} = k^2 Ae^{-kx} \sin \Phi + \frac{2\pi}{l} k (Ae^{-kx} \cos \Phi) + k \frac{2\pi}{l} Ae^{-kx} \cos \Phi - \frac{4\pi^2}{l^2} Ae^{-kx} \sin \Phi,$$

in which 
$$\Phi = 2\pi \left( \frac{t}{T} - \frac{x}{l} \right).$$

The expression for  $\frac{d^2\eta}{dt^2}$  is substituted as it stands, while in the case of the expression for  $\frac{d^2\eta}{dx^2}$  we group the  $\sin \phi$  terms and  $\cos \phi$  terms thus :

$$\left( k^2 Ae^{-kx} - \frac{4\pi^2}{l^2} Ae^{-kx} \right) \sin \Phi = \left( k^2 - \frac{4\pi^2}{l^2} \right) Ae^{-kx} \sin \Phi,$$

which, when multiplied by  $a \frac{\epsilon}{\mu}$ , gives

$$-a \frac{\epsilon}{\mu} \left( \frac{4\pi^2}{l^2} - k^2 \right) Ae^{-kx} \sin \Phi.$$

Grouping the  $\cos \phi$  terms in the same way gives

$$2a \frac{\epsilon}{\mu} \left( \frac{4\pi^2}{l^2} - k^2 \right) Ae^{-kx} \sin \Phi.$$

We require also an expression for  $(\eta - y)$ ,

$$\begin{aligned} \frac{\beta}{\mu} (\eta - y) &= \frac{\beta}{\mu} (Ae^{-kx} \sin \Phi - Be^{-kx} \sin (\Phi - \Psi)) \\ &= \frac{\beta}{\mu} Ae^{-kx} \sin \Phi - \frac{B}{A} \sin (\Phi - \Psi) \end{aligned}$$

in which 
$$\Psi = 2\pi\Delta.$$

Substituting in equation (1) for  $\frac{d^2\eta}{dt^2}$ ,  $\frac{d^2\eta}{dx^2}$  and  $(\eta - y)$  the values obtained from the equations (3) and (4), we have

$$-\frac{4\pi^2}{T^2}Ae^{-kx}\sin\Phi = -a\frac{\epsilon}{\mu}\left(\frac{4\pi}{l^2} - k^2\right)Ae^{-kx}\sin\Phi + 2a\frac{\epsilon}{\mu}\frac{2\pi}{l}kAe^{-kx}\cos\Phi - \frac{\beta}{\mu}Ae^{-kx}\left[\sin\Phi - \frac{B}{A}\sin(\Phi - \Psi)\right].$$

If  $n$  is the refractive index of the medium for a disturbance of period  $T$ , the wave-length of which in free ether is  $\lambda$ , then  $l = \frac{\lambda}{n}$ ; we next introduce a new constant  $\kappa$  such that  $k = \kappa\frac{2\pi}{\lambda}$ , and put  $\zeta^2 = a\frac{\epsilon}{\mu}$ .

We have seen that  $a\frac{\epsilon}{\mu}$  is the square of the velocity of propagation of the disturbance;  $\therefore \zeta = \frac{\lambda}{T}$

Dividing each term in the expression by  $Ae^{-kx}\zeta^2$  or its equivalent, we get

$$\begin{aligned} -\frac{4\pi^2}{\lambda^2}\sin\Phi &= -\left(\frac{4\pi^2n^2}{\lambda^2} - \kappa^2\frac{4\pi^2}{\lambda^2}\right)\sin\phi + \frac{4\pi n\kappa}{\lambda}\frac{2\pi}{\lambda}\cos\phi \\ &\quad - \frac{4\pi^2}{\lambda^2}\sin\Phi - \frac{\beta}{\mu\zeta^2}\left[\sin\Phi - \frac{B}{A}\sin(\phi - \Psi)\right] - \frac{4\pi^2}{\lambda^2}(n^2 - \kappa^2)\sin\phi \\ &\quad + \frac{4\pi^2}{\lambda^2}2n\kappa\cos\Phi - \frac{\beta}{\mu\zeta^2}\left(\sin\Phi - \frac{B}{A}\sin(\Phi - \Psi)\right). \end{aligned}$$

Writing now for  $\sin(\phi - \Psi)$  the equivalent  $\sin\phi\cos\Psi - \cos\phi\sin\Psi$ , we get

$$\begin{aligned} \left\{(n^2 - \kappa^2 - 1)\frac{4\pi^2}{\lambda^2} + \frac{\beta}{\mu\zeta^2}\left(1 - \frac{B}{A}\cos\Psi\right)\right\}\sin\Phi \\ - \left(2n\kappa\frac{4\pi^2}{\lambda^2} - \frac{\beta}{\mu\zeta^2}\frac{B}{A}\sin\Psi\right)\cos\Phi = 0. \end{aligned}$$

This equation must hold for every value of  $\phi$ , which is only possible when the coefficients of  $\sin\phi$  and  $\cos\phi$  both equal zero; otherwise, with increasing  $\Phi$  in the first quadrant, the first member would increase and the second diminish and the equation no longer hold.

The equation therefore falls into two, and dividing through by  $\frac{4\pi}{\lambda^2}$  we get (5)

$$n^2 - \kappa^2 - 1 = -\frac{\beta\lambda^2}{4\pi^2\mu\zeta^2} + \frac{\beta\lambda^2}{4\pi^2\mu\zeta^2}\frac{B}{A}\cos\Psi,$$

and (6)

$$2n\kappa = \frac{\beta\lambda^2}{4\pi^2\mu\zeta^2}\frac{B}{A}\sin\Psi.$$

These equations give us the refractive index and extinction coefficient  $\kappa$  in terms of the ratio of the vibration amplitudes of the atom and ether, and the phase between them. We have next to determine

these two quantities. Substituting in equation (2) the values found for the first and second derivatives, and for  $(\eta - y)$  and  $y$ , gives

$$-\frac{4\pi^2}{T^2} B e^{-kx} \sin(\Phi - \Psi) = \frac{\beta}{m} B e^{-kx} \left( \frac{A}{B} \sin \Phi - \sin(\Phi - \Psi) \right) \\ - \frac{\gamma}{m} B e^{-kx} \sin(\Phi - \Psi) - \frac{\delta}{m} B e^{-kx} \frac{2\pi}{T} (\cos(\Phi - \Psi)).$$

Divide through by  $B e^{-kx}$ , and writing as before for  $\sin(\Phi - \Psi)$  its equivalent, we find that this equation also breaks up into two:

$$-\frac{4\pi^2}{T^2} \cos \Psi = \frac{\beta}{m} \frac{A}{B} - \frac{\beta + \gamma}{m} \cos \Psi - \frac{\delta}{m} \frac{2\pi}{T} \sin \Psi \\ \text{and } \frac{4\pi^2}{T^2} \sin \Psi = \frac{\beta + \gamma}{m} \sin \Psi - \frac{\delta}{m} \frac{2\pi}{T} \cos \Psi,$$

which gives us the amplitude ratio and the phase difference in terms of the constants of equation (2).

Multiply the first equation by  $\cos \Psi$ , and the second by  $\sin \Psi$ , and subtract the first from the second, and we have

$$\frac{4\pi^2}{T^2} = -\frac{\beta}{m} \frac{A}{B} \cos \psi + \frac{\beta + \gamma}{m}.$$

Multiply the first by  $\sin \Psi$  and the second by  $\cos \Psi$  and add, and we have

$$0 = \frac{\beta}{m} \frac{A}{B} \sin \Psi - \frac{\delta}{m} \frac{2\pi}{T}.$$

The first equation gives  $\frac{B}{A} = \frac{\frac{\beta}{m} \cos \Psi}{\frac{\beta + \gamma}{m} - \frac{4\pi^2}{T^2}}.$

The second  $\frac{B}{A} = \frac{\frac{\beta}{m} \sin \Psi}{\frac{\delta}{m} \frac{2\pi}{T}}.$

The quantity  $\beta + \gamma$  in the first of these two equations is the sum of the forces acting on the atom when displaced unit distance, the ether being regarded as at rest (or  $\eta = 0$ ). The equation of motion of the atom under this condition is, disregarding friction,  $\frac{d^2 y}{dt^2} = -\frac{\beta + \gamma}{m} y$ , and its period of oscillation we will designate by  $T_m$ .

$\frac{\beta + \gamma}{m} = \frac{4\pi^2}{T_m^2} = \frac{4\pi^2 \zeta^2}{\lambda_m^2}$ , where  $\lambda_m$  is the wave length in ether of a disturbance of periodic time  $T_m$ ,  $\zeta$  being the velocity of propagation. (See last equations in section on periodic motion, chap. i.)

Substituting this value in the equations for the amplitude ratio gives us

$$(9) \quad \frac{B}{A} = \frac{\beta}{4\pi^2 \zeta^2 m} \frac{\cos \Psi}{\frac{1}{\lambda_m^2} - \frac{1}{\lambda^2}} = \frac{\beta}{4\pi^2 \zeta^2 m} \frac{\lambda_m^2 \lambda^2}{\lambda^2 - \lambda_m^2} \cos \Psi.$$

$$(10) \quad \frac{B}{A} = -\frac{\beta}{4\pi^2\xi^2m} \frac{\sin \Psi}{\frac{\delta}{2\pi m\xi} \cdot \frac{1}{\lambda}},$$

in which we recognize the term  $\frac{\lambda^2}{\lambda^2 - \lambda_m^2}$ , which we have seen accounts for anomalous dispersion.

We will now write  $\alpha$  for  $\frac{\delta}{2\pi m\xi} \lambda_m^2$  (all the quantities being constants), and (10) becomes  $\frac{B}{A} = \frac{\beta}{4\pi^2\xi^2m} \frac{\lambda_m^2\lambda}{\alpha} \sin \Psi$ , and dividing this by (9) we have  $\frac{1}{\tan \Psi} = \frac{\lambda_m\lambda}{\alpha} \cdot \frac{\lambda^2 - \lambda_m^2}{\lambda_m^2\lambda^2}$ .

$$(11) \quad \tan \Psi = \frac{\alpha\lambda}{\lambda^2 - \lambda_m^2},$$

an expression for the phase-difference between the ether and the atom in terms of the constants and the wave-length of the light.

We shall presently require an expression for  $\sin^2\Psi$  and  $\cos^2\Psi$ , which we can get from

$$\tan^2\Psi = \frac{\sin^2\Psi}{\cos^2\Psi} \quad \text{and} \quad \sin^2\Psi = 1 - \cos^2\Psi;$$

$$\therefore \frac{1 - \cos^2\Psi}{\cos^2\Psi} = \frac{\alpha^2\lambda^2}{(\lambda^2 - \lambda_m^2)^2},$$

$$\alpha^2\lambda^2 \cos^2\Psi = (\lambda^2 - \lambda_m^2)^2 - (\lambda^2 - \lambda_m^2)^2 \cos^2\Psi.$$

$$(12) \quad \cos^2\Psi = \frac{(\lambda^2 - \lambda_m^2)^2}{(\lambda^2 - \lambda_m^2)^2 + \alpha^2\lambda^2},$$

$$(13) \quad \sin^2\Psi = \frac{\alpha^2\lambda^2}{(\lambda^2 - \lambda_m^2)^2 + \alpha^2\lambda^2}, \quad \text{since} \quad \sin^2\Psi + \cos^2\Psi = 1,$$

expressions which we shall presently substitute in (9). We now multiply equation (9) by  $\cos \Psi$  and (10) by  $\sin \Psi$ , and substitute for the  $\cos^2\Psi$  and  $\sin^2\Psi$  the expressions given by (12) and (13).

$$\frac{B}{A} \cos \Psi = \frac{\beta}{4\pi^2\xi^2m} \cdot \frac{\lambda_m^2\lambda^2(\lambda^2 - \lambda_m^2)}{(\lambda^2 - \lambda_m^2)^2 + \alpha^2\lambda^2},$$

$$\frac{B}{A} \sin \Psi = \frac{\beta}{4\pi^2\xi^2m} \cdot \frac{\alpha\lambda_m^2\lambda^3}{(\lambda^2 - \lambda_m^2)^2 + \alpha^2\lambda^2}.$$

Substitute these values in equations (5) and (6),

$$n^2 - \kappa^2 - 1 = -\frac{\beta}{4\pi^2\xi^2\mu} \cdot \lambda^2 + \frac{\beta^2}{(4\pi^2\xi^2)^2\mu m} \cdot \frac{\lambda_m^2\lambda^4(\lambda^2 - \lambda_m^2)}{\lambda^2 - \lambda_m^2 + \alpha^2\lambda^2},$$

$$2n\kappa = \frac{\beta}{(4\pi^2\xi^2)^2\mu m} \cdot \frac{\alpha\lambda_m^2\lambda^5}{(\lambda^2 - \lambda_m^2)^2 + \alpha^2\lambda^2}.$$

$$\text{Let} \quad P = \frac{\beta}{4\pi^2\xi^2\mu} \quad \text{and} \quad Q = \frac{\beta^2\lambda_m^2}{(4\pi^2\xi^2)^2\mu m},$$

which we may do since only constants are involved, and we have the final equations,

$$n^2 - \kappa^2 = 1 - P\lambda^2 + Q \frac{\lambda^4(\lambda^2 - \lambda_m^2)}{(\lambda^2 - \lambda_m^2)^2 + a^2\lambda^2}, \quad 2n\kappa = Q \frac{a\lambda^5}{(\lambda^2 - \lambda_m^2)^2 + a^2\lambda^2}.$$

In these equations  $n$  is the refractive index of the medium for wave-length  $\lambda$ ,  $2\pi\kappa$  is the fraction of the light lost by absorption in distance  $\lambda$  within the medium (since  $k = \kappa \frac{2\pi}{\lambda}$ ). (See eq. (2a).)

The dispersion formula gives us not only the relation between the refractive index and the wave-length, but also the relation between the absorption and the wave-length. If we have atoms with different free periods, we must place a summation sign before each of the terms in the right-hand member of the equation. They are then applicable to media with more than one absorption band.

$$n^2 - \kappa^2 - 1 = - \sum P\lambda^2 + \sum Q \frac{\lambda^4(\lambda^2 - \lambda_m^2)}{(\lambda^2 - \lambda_m^2)^2 + a^2\lambda^2},$$

$$2n\kappa = \sum Q \frac{a\lambda^5}{(\lambda^2 - \lambda_m^2)^2 + a^2\lambda^2}.$$

We will now proceed to examine the equations in some detail, applying them to transparent and then to absorbing media.

**Application to Transparent Media.**—As we have seen, we must define transparent media as media with absorption bands outside of the visible spectrum. For such media, if we confine our attention to values of  $\lambda$  in the freely transparent region we can put  $\kappa = 0$  in the equation, since there is no absorption. If  $\kappa = 0$ , then  $a$  must also equal zero, since if there is no absorption there is no friction.

The first equation now reduces to

$$n^2 = 1 - P\lambda^2 + Q \frac{\lambda^4}{\lambda^2 - \lambda_m^2}.$$

This expression differs somewhat from the Sellmeier formula,

$$n^2 = 1 + D \frac{\lambda^2}{\lambda^2 - \lambda_m^2},$$

and from the electromagnetic dispersion formula, which we shall presently develop, and which, for transparent media, reduces to a form identical with the Sellmeier equation. The term  $-P$  reminds us of a similar term in the complete dispersion formula of Ketteler which we shall take up presently, which represents the effect of an absorption band in the infra-red. There is, however, no connection between the two, as is at once evident when we remember that the expression was developed for a *single* absorption band. The  $\lambda^4$  in the numerator makes it seem at first sight as if the two formulae must be quite different. This apparent difference disappears as soon as we calculate the values of the constants  $P$  and  $Q$ .

These constants cannot be determined from theory, but they can be calculated if we determine three values of the refractive index  $n_1, n_2, n_3$  for wave-lengths  $\lambda_1, \lambda_2, \lambda_3$ .

Writing the equation  $n^2 - 1 = -P\lambda^2 + Q\frac{\lambda^4}{\lambda^2 - \lambda_m^2}$  in the form

$$\left(\frac{n^2 - 1}{\lambda^2} + P\right)\left(1 - \frac{\lambda_m^2}{\lambda^2}\right) - Q = 0,$$

and letting  $\frac{n_1^2 - 1}{\lambda_1^2} = r$ ,  $\frac{n_2^2 - 1}{\lambda_2^2} = s$ , and  $\frac{n_3^2 - 1}{\lambda_3^2} = t$ ,

we have three equations

$$(r + P)\left(1 - \frac{\lambda_m^2}{\lambda_1^2}\right) - Q = 0, \quad (s + P)\left(1 - \frac{\lambda_m^2}{\lambda_2^2}\right) - Q = 0,$$

$$(t + P)\left(1 - \frac{\lambda_m^2}{\lambda_3^2}\right) - Q = 0,$$

from which the three constants  $P$ ,  $Q$ , and  $\lambda_m$  are determined.

The constants of the Cauchy formula can be determined from two observed values of  $n$  and  $\lambda$ ; thus

$$n_1 = A + \frac{B}{\lambda_1^2}, \quad n_2 = A + \frac{B}{\lambda_2^2},$$

$$B = \frac{n_2 - n_1}{\frac{1}{\lambda_2^2} - \frac{1}{\lambda_1^2}}, \quad A = \frac{n_1 \frac{1}{\lambda_2^2} - n_2 \frac{1}{\lambda_1^2}}{\frac{1}{\lambda_2^2} - \frac{1}{\lambda_1^2}}.$$

For water at 19°·5 C. the constants are as follows :

Helmholtz formula.

$$\lambda_m^2 = \cdot 87979$$

$$P = \cdot 865895$$

$$Q = \cdot 865767$$

Cauchy formula.

$$A = 1\cdot 324137$$

$$B = \cdot 30531$$

These constants once determined, we can test the formula by calculating the values of  $n$  for other wave-lengths, and compare them with observed values. In the following table are given the values for water obtained by Wüllner, the values of  $\lambda$  being designated by the Fraunhofer lines. The differences between the observed and calculated values are given in the last two columns :

$\lambda$ .	$n$ observed.	Calculated values differ by	
		(Helmholtz)	(Cauchy)
<i>B</i>	1·33048	0	+·00012
<i>C</i>	1·33122	-·00005	-·00001
<i>D</i>	1·33307	0	-·00012
<i>E</i>	1·33527	+·00005	-·00015
<i>F</i>	1·33720	-·00000	-·00003
<i>G</i>	1·34063	+·00001	-·00002
<i>H</i>	1·34350	+·00004	+·00014

This shows the accuracy with which the two formulae represent the dispersion.

The constants  $P$  and  $Q$  of the Helmholtz formula are seen to be very nearly equal, a fact which is true for other substances than water.

If we put  $P=Q$ , the formula  $n^2 = 1 - P\lambda^2 + Q \frac{\lambda^4}{\lambda^2 - \lambda_m^2}$  reduces at once to  $n^2 = 1 + Q \frac{\lambda^2 \lambda_m^2}{\lambda^2 - \lambda_m^2}$ , which is identical with the formula of Sellmeier if we write  $D$  for  $Q$ . We can, moreover, by a different transformation cause the Helmholtz formula to take the form of the Cauchy series.

If  $\lambda_m > \lambda$ , which must be the case, otherwise we should have values of  $n$  less than 1, we can write  $Q \frac{\lambda^4}{\lambda^2 - \lambda_m^2} = \frac{Q\lambda^2}{\left(1 - \frac{\lambda_m^2}{\lambda^2}\right)}$ .

By the binomial theorem,  $\frac{1}{1 - \frac{\lambda_m^2}{\lambda^2}} = 1 + \frac{\lambda_m^2}{\lambda^2} + \frac{\lambda_m^4}{\lambda^4} + \dots$ .

Substituting this series in our dispersion formula gives us

$$n^2 = 1 - P\lambda^2 + Q\lambda^2 + Q\lambda_m^2 + Q\lambda_m^2 \frac{\lambda_m^2}{\lambda^2} + Q\lambda_m^2 \frac{\lambda_m^4}{\lambda^4}$$

or 
$$n^2 = 1 + Q\lambda_m^2 - (P - Q)\lambda^2 + Q\lambda_m^2 \frac{\lambda_m^2}{\lambda^2} + Q\lambda_m^2 \frac{\lambda_m^4}{\lambda^4}.$$

Since  $P=Q$ , the term  $-(P-Q)\lambda^2$  falls out, and we have the Cauchy formula, writing  $A = 1 + Q\lambda_m^2$ ,  $B = Q\lambda_m^4$ , etc. This explains why the Cauchy formula is capable of representing a dispersion curve as well as it does, its agreement with the Helmholtz formula being accidental of course.

**Calculation of the Position of the Absorption Bands of Transparent Media.**—The Helmholtz equation was modified by Ketteler, who obtained a formula containing a term, the square root of which represented the refractive index of the medium for infinitely long waves. This formula, being essentially identical with the electro-magnetic dispersion formula which we shall develop later on, will for the present be assumed. It enables us to push our investigations over a wider range of wave-lengths than was possible with the other equation, and is commonly spoken of as the Ketteler-Helmholtz dispersion formula.

The two equations are

$$n^2 - \kappa^2 - n_\infty^2 = \sum \frac{M(\lambda^2 - \lambda_m^2)}{(\lambda^2 - \lambda_m^2)^2 + a^2\lambda^2},$$

$$2n\kappa = \sum \frac{Ma\lambda}{(\lambda^2 - \lambda_m^2) + a^2\lambda^2}.$$

This formula has been verified over a wide range by the investigations of Paschen, Rubens and others, who have measured the dispersion of various substances in the infra-red, visible and ultra-violet regions, and compared the observed values with those calculated from the formula, finding most perfect agreement.

By measuring the dispersion in the visible spectrum and determining the constants, it is possible to calculate the positions of the absorption

bands in the infra-red and ultra-violet, even if we cannot observe them.

In this way bands of absorption have been definitely located in the infra-red region of various media, and subsequently found by experiment.

We will now examine in some detail the application of the Ketteler-Helmholtz formula to the study of the optical properties of quartz.

For transparent media the equation becomes

$$n^2 = n_\infty^2 + \sum \frac{M_m}{\lambda^2 \lambda_m^2}.$$

This formula must hold over the entire spectrum with the exception of the small gaps where the absorption bands lie.

It was found sufficient to take but two terms of the member  $\sum \frac{M_m}{\lambda^2 - \lambda_m^2}$  in all cases where in one term  $\lambda_m^2$  was considerably smaller than  $\lambda^2$ , and in the other considerably larger; in other words, to consider but two absorption bands, one in the ultra-violet, the other in the infra-red.

The expression  $n^2 = n_\infty^2 + \overset{\text{Ultra-violet.}}{\frac{M_1}{\lambda^2 - \lambda_1^2}} + \overset{\text{Infra-red.}}{\frac{M_2}{\lambda^2 - \lambda_2^2}}$  can be written in the form

$$n^2 = n_\infty^2 + \frac{M_1}{\lambda^2 - \lambda_1^2} - \left( \frac{M_2}{\lambda_2^2} + \frac{M_2 \lambda^2}{\lambda_2^4} + \frac{M_2 \lambda^4}{\lambda_2^6} \right),$$

the expansion being made by change of sign and division. The members in the parenthesis form a rapidly convergent series, and for very diathermous substances, in which the absorption band is far out in the infra-red (*i.e.*  $\lambda_2$  very large in comparison to  $\lambda$ ), a sufficiently good approximation is obtained when we write

$$n^2 = a^2 - \frac{M_1}{\lambda^2 - \lambda_1^2} - k\lambda^2, \text{ in which } a = n_\infty^2 + \frac{M_2}{\lambda_2^2}, k = \frac{M_2}{\lambda_2^4}.$$

If the diathermism is less, or if we are working nearer to the absorption band, we add a second member of the series and get

$$n^2 = a^2 + \frac{M_1}{\lambda^2 - \lambda_1^2} - k\lambda^2 - k_1\lambda^4.$$

Now  $k = \frac{M_2}{\lambda_2^4}$  and  $k_1 = \frac{M_2}{\lambda_2^6}$ ;  $\therefore \frac{k}{k_1} = \lambda_2^2$ , and by determining the constants  $k$  and  $k_1$ , we have at once determined the position of the infra red absorption band.

Quartz is a substance exceedingly transparent to the ultra-violet, visible, and infra-red, and is consequently well adapted for a verification of the dispersion formula over a wide range of wave-lengths.

The dispersion is measured in the visible spectrum with the spectrometer, in the ultra-violet by means of photography and in the infra-red either with the thermopile, bolometer or radiometer. In these instruments the heating effect of the rays is the means by

which they are detected, the thermopile furnishing a current, and deflecting a galvanometer needle when the rays strike it, while the bolometer gives evidence of the rays by the change in the resistance of a fine strip of blackened platinum due to the heating. The radiometer, which was brought to a high degree of perfection by E. F. Nichols, has the advantage of being uninfluenced by magnetic disturbances, but is not quite as convenient to work with. It consists of a double vane of mica suspended in a vacuum by a quartz fibre. The rays, falling upon one of the vanes, cause a deflection, which is measured by a mirror and scale. A fuller description of this instrument will be found in the chapter on Radiation.

Rubens employed the bolometer in his earlier investigations. The arrangement of his apparatus is shown in Fig. 244. The radiation

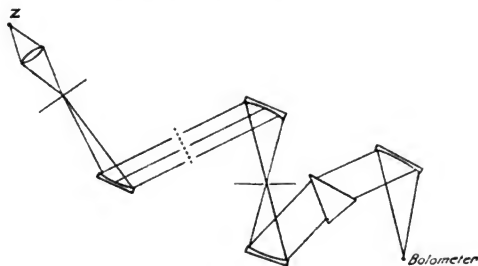


FIG. 244.

from a zirconia burner  $Z$  was concentrated by a rock-salt lens, which is very transparent to the long waves, on the slit of a reflecting spectrometer, dispersed by a wire diffraction grating, and focused on the slit of a second reflecting spectrometer, on the table of which the quartz prism was mounted. By means of the grating spectrometer, heat radiation of approximately a single wavelength was thrown on the slit of the spectrometer, and the deviation of these rays by the quartz prism was determined by means of the bolometer, which took the place of the cross-hairs in the eye-piece of the ordinary spectrometer. With this apparatus Rubens investigated the dispersion of fluorite to  $8.9\mu$ , and of quartz to  $4.26\mu$ . Taking Sarasin's measurements in the ultra-violet and visible spectrum, and his own in the infra-red, Rubens compiled the following table. The differences between the observed and calculated values are given in the  $\delta$  columns. The values in column 1 were calculated from the formula, making use of one member of the convergent series. The calculated values agree with the observed, from the extreme ultra-violet up to nearly  $2\mu$ , beyond which a rapidly increasing difference is found, due to the fact that we are getting so near the absorption band that it is no longer sufficient to take two members of the series. Three members of the series were used in compiling the values of column 2, and it is seen that the agreement is perfect out to  $4.26\mu$ ,

which was as far as it was possible to obtain measurements at the time. The constants of the equation were determined as follows :

$$a^2 = 2.35681, \quad k = .01113, \quad k_1 = .0001023, \\ M_1 = .010654, \quad \lambda_1 = .010627.$$

		I.		II.			
		$\lambda$ .	m. obs.	m. cal.	$\delta$ I.	m. cal.	$\delta$ II.
Visible Spectrum.	{	.198	1.65070	1.65077	+ .7	1.65077	+ .7
		.274	1.5850	1.58757	+ .7	1.58757	+ .7
		.358	1.56400	1.56395	- .5	1.56395	- .5
		.434	1.553869	1.5539	+ .3	1.5539	+ .3
		.534	1.54663	1.5466	+ .3	1.5466	+ .3
		.656	1.541807	1.5419	+ .9	1.5419	+ .9
		1.160	1.5329	1.5330	+ .1	1.5329	0
		1.617	1.5271	1.5271	0	1.5269	- 2
		1.969	1.5216	1.5221	+ 5	1.5216	0
		2.32	1.5156	1.5162	+ 6	1.5152	- 4
		2.60	1.5099	1.5111	+ 12	1.5096	- 3
		2.86	1.5039	1.5057	+ 18	1.5034	- 5
		3.06	1.4985	1.5013	+ 28	1.4983	- 2
		3.21	1.4942	1.4980	38	1.4944	+ 2
		3.42	1.4877	1.4926	49	1.4879	+ 2
		3.67	1.4790	1.4861	63	1.4799	+ 1
		3.84	1.4739	1.4812	73	1.4738	- 1
		4.01	1.4678	1.4761	83	1.4683	+ 5
4.15	1.4619	1.4717	98	1.4616	- 3		
4.26	1.4567	1.4682	115	1.4570	+ 3		

Calculation of the position of the infra-red absorption band from  $k$  and  $k_1$  showed it to be at  $10.4\mu$ .

The approximate position of the ultra-violet band is at  $.103\mu$ , the two being  $6\frac{1}{2}$  octaves apart. Of this region  $4\frac{1}{2}$  octaves could be measured, and the agreement between the observed and calculated values within this range was found to be perfect.

**Radiometric Observations on Quartz.**—The behavior of quartz with respect to very long heat waves was investigated by Nichols with a radiometer.

He found a strong absorption band between 8 and  $9\mu$  which agreed fairly well with the position predicted by Rubens from the constants of the formula. If, however, we use this experimentally determined value of  $\lambda_m$  for calculating the refractive indices, we no longer find agreement between the two sets.

The reason of this appeared upon the completion of a subsequent piece of experimental work by Rubens and Nichols.

Nichols had found that for the wave-length corresponding to the center of the absorption band, ( $8.5\mu$ ), quartz reflected almost as powerfully as a metal. This, as we shall see in the chapter on Absorption, is universally true of substances showing powerful selective absorption. The phenomenon is commonly spoken of as "Surface-color," the aniline dyes exhibiting it to a very marked degree. Quartz was found to reflect about 80 % of the incident radiation of wave-length  $8.5\mu$ , and only about 2 % of the radiation at  $4\mu$ . The transmission and reflection curves are shown in Fig. 245.

On this property of quartz Rubens and Nichols based the very beautiful method of isolating heat waves of great wave-length. In brief,

the method consisted in reflecting the radiant energy, coming from a zirconia button heated in the oxy-hydrogen flame, in succession from several polished surfaces of quartz. It is apparent that, if the quartz reflects like a metal for wave-length  $8.5\mu$  and like a transparent medium for all other wave-lengths, the radiant energy after a sufficient number of reflections will contain practically nothing but the metallicly reflected waves. The energy after five reflections was examined with a wire diffraction grating, and found to consist principally of waves of length  $8.5\mu$ . In addition to these waves, the grating showed that wave-length  $20\mu$  was also present in the reflected energy, conclusive

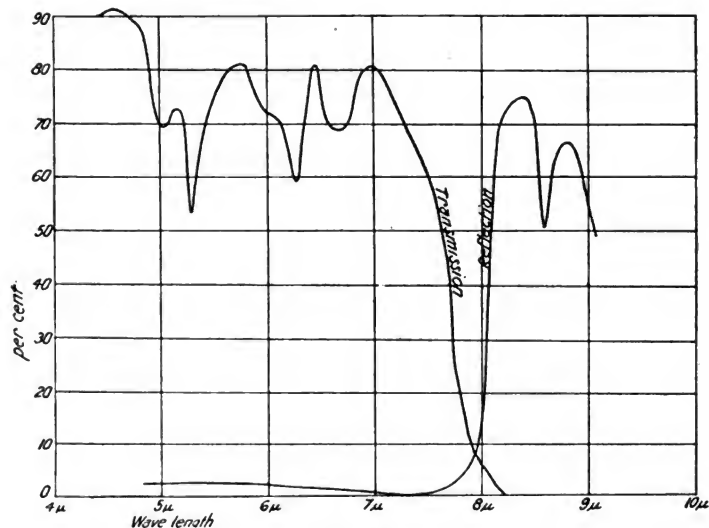


FIG. 245.

proof that there was a second absorption band at  $20\mu$ . Taking  $8.5\mu$  and  $20\mu$  as values for  $\lambda_2$  and  $\lambda_3$  in the formula, adding a term  $\frac{M_3}{\lambda^2 - \lambda_3^2}$  for the new absorption band, it was found that perfect agreement was again obtained between the observed and calculated values.

The error in the calculated position of the band ( $10.4$  against  $8.5\mu$ ) was due to the fact that the second band had been neglected. This shows us how the presence of an absorption band beyond the reach of observation may be detected.

A further remarkable verification of the formula has been found by Rubens and Aschkinass (*Ann. der Physik*, 67, 459, 1899), who have measured the refractive index of quartz for heat-waves of length  $56\mu$ . These waves are isolated by means of repeated reflections from surfaces

of fluorite. Inasmuch as they lie on the side of the infra-red bands towards the region of longer wave-length, we should expect a higher value of the refractive index than in the visible spectrum. Calculating the refractive index from the dispersion formula, we find the extraordinary value 2.20, higher even than for the ultra-violet. The value found experimentally by Rubens was 2.18, a remarkably close agreement.

**Application to Absorbing Media and Anomalous Dispersion.**—The first investigations in anomalous dispersion having been made with solutions of the aniline dyes, we will begin with the application of the formula to media of this nature.

We have defined  $\lambda_m$  as the wave-length in ether of a disturbance of a period equal to the period of the atom. We will now see whether the Helmholtz formula indicates that waves of this length are the ones most strongly absorbed; in other words, if  $\lambda_m$  is at the center of the absorption band, as we have assumed it to be. Since  $\kappa$  is the measure of absorption, we must determine that value of  $\lambda$  which gives  $\kappa$  its largest value in the Helmholtz formula. Since the values of  $n$  do not vary greatly, we shall obtain a sufficiently close approximation if we seek for the value of  $\lambda$  which gives to  $2n\kappa$  its maximum value. Substituting  $\frac{k\lambda}{2\pi}$  for  $\kappa$ , the second formula becomes

$$2n\kappa = 2\pi Q \frac{\alpha\lambda^4}{(\lambda^2 - \lambda_m^2)^2 + \alpha^2\lambda^2}.$$

Differentiating with respect to  $\lambda$ , and equating to zero, gives

$$\frac{d(2n\kappa)}{d\lambda} = 2\pi Q \frac{\{(\lambda^2 - \lambda_m^2)^2 + \alpha^2\lambda^2\} 4\alpha\lambda^3 - \alpha\lambda^4 \{4\lambda(\lambda^2 - \lambda_m^2) + 2\alpha^2\lambda\}}{\{(\lambda^2 - \lambda_m^2)^2 + \alpha^2\lambda^2\}^2} = 0.$$

If we equate this to zero the numerator must equal zero also, which will be true for  $\lambda=0$ , in which case  $\kappa=0$  (the minimum value of the function), and also if

$$4\{(\lambda^2 - \lambda_m^2)^2 + \alpha^2\lambda^2\} = \lambda\{4\lambda(\lambda^2 - \lambda_m^2) + 2\alpha^2\lambda\},$$

which, if we perform the multiplications and cancel, gives

$$\lambda^2 = \frac{\lambda_m^4}{\lambda_m^2 - \frac{\alpha^2}{2}},$$

or since  $\alpha$  has a very small value  $\lambda^2 = \lambda_m^2$  and  $\lambda_m$  is the wave-length most strongly absorbed.

The Helmholtz formulae for a solution of a dye with one absorption band in a transparent solvent are :

$$n^2 - \kappa^2 - 1 = -P_1\lambda^2 + Q_1 \frac{\lambda_m^4}{\lambda^2 - \lambda_{m_1}^2} - P_2\lambda^2 + Q_2 \frac{\lambda^4(\lambda^2 - \lambda_{m_2}^2)}{(\lambda^2 - \lambda_{m_2}^2)^2 + \alpha^2\lambda^2},$$

$$2n\kappa = Q_2 \frac{\alpha\lambda^5}{(\lambda^2 - \lambda_{m_2}^2)^2 + \alpha^2\lambda^2}.$$

It will be seen that we have added two terms  $P_2\lambda^2$  and  $Q_2$ , etc., to the terms representing the dispersion of the transparent solvent, which

have been simplified as before by letting  $\kappa = 0$  and  $\alpha = 0$ . We cannot put  $\kappa = 0$  for the terms belonging to the dye, for we are to carry our investigation through its absorption band.

The second equation applies only to the region where  $\kappa$  has some value greater than zero, and hence is free from the constant of the solvent, which is equivalent to saying that the second equation does not apply to transparent media.

Pflüger (*Ann. der Phys.*, 65, p. 113, 1898) was the first to undertake a careful series of observations of  $n$  and  $\kappa$  in the case of strongly absorbing substances, with a view of testing the dispersion formula, near to and within the absorption band. He employed solid prisms of small angle obtained by evaporating an alcoholic solution of an aniline dye between a glass plate and a curved segment of a glass tube. The refractive indices could be obtained by means of these prisms even at the center of the absorption band, though in this case the image of the

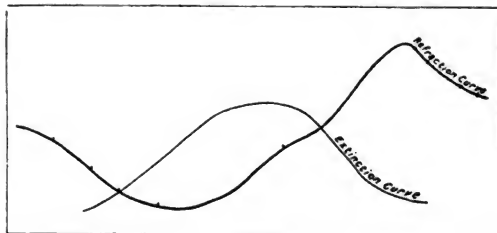


FIG. 246.

slit of the spectrometer was greatly broadened by diffraction, since only the extreme edge of the prism transmitted the light. The values of the constant  $\kappa$  for the different values of  $\lambda$  were determined by means of the spectrophotometer, thin films of different thicknesses being used. The general form of the curves found in the case of cyanine is shown in Fig. 246.

The dispersion and absorption were found to be well represented by the Ketteler-Helmholtz formulae in the forms

$$n^2 - \kappa^2 - 1 = \sum \frac{D\lambda^2(\lambda^2 - \lambda_m^2)}{(\lambda^2 - \lambda_m^2)^2 + g^2\lambda^2}, \quad 2n\kappa = \sum \frac{Dg\lambda^3}{(\lambda^2 - \lambda_m^2)^2 + g^2\lambda^2}.$$

The dispersion of cyanine was also investigated by Wood and Magnusson (*Phil. Mag.*, Jan. 1901) by means of cyanine prism of larger angles than the ones employed by Pflüger, as well as by means of observations of the shift of the fringes formed by the Michelson interferometer, caused by the introduction of a thin film of solid cyanine into one of the optical paths. The continuity of the dispersion curve through the absorption band was well brought out in the photographs obtained with the interferometer.

The most complete investigation of the dispersion and absorption of solutions of aniline dyes is that made by Stöckl. The refractive indices were determined by measuring with the spectrometer the

deviations produced by fluid prisms of small angle. In work of this kind, when observations are required near and within the absorption band, prisms of extremely small angle are necessary, and even then light is only transmitted by a narrow strip along the refracting edge. This limitation of the transmitted beam interferes seriously with the resolving power of the instrument, the image of the slit being broadened by diffraction. Stöckl used a prism made of a pair of glass plates, the angle between which could be varied to suit the conditions. When observing at a distance from the absorption band, larger angles can be used which naturally give more accurate results.

**Determination of the Extinction Coefficient.**—The determination of  $\kappa$  is not as easy as the determination of  $n$ . By our definition of the constant, the amplitude of the light-wave of length  $\lambda$  decreases in the ratio  $1 : e^{2\pi\kappa}$  in traversing a layer of thickness  $\lambda$ . If the thickness of the layer is  $d$ , the ratio expressing the decrease of amplitude is  $1 : e^{-2\pi\kappa \frac{d}{\lambda}}$ .

Now the intensity of the light is measured by the square of the amplitude, and the intensity therefore decreases in the ratio  $1 : e^{-4\pi\kappa \frac{d}{\lambda}}$ . To avoid the error due to reflection from the surfaces of the layer, or the glass plates between which it is held, it is customary to employ layers of different thicknesses. Let these thicknesses be  $d_1$  and  $d_2$ , and the intensity of the incident light be  $J_0$ . The intensities after traversing the two layers will be  $J_1 = J_0 e^{-4\pi\kappa \frac{d_1}{\lambda}}$  and  $J_2 = J_0 e^{-4\pi\kappa \frac{d_2}{\lambda}}$ .

$$\log J_1 = \log J_0 - 4\pi\kappa \frac{d_1}{\lambda} \log e,$$

$$\log J_2 = \log J_0 - 4\pi\kappa \frac{d_2}{\lambda} \log e,$$

$$\log J_1 - \log J_2 = \frac{4\pi\kappa}{\lambda} \log e (d_2 - d_1),$$

$$\log \frac{J_1}{J_2} = \frac{4\pi\kappa}{\lambda} (d_2 - d_1) \log e.$$

From this equation we can calculate the extinction coefficient  $\kappa$ , by measuring the intensities of the transmitted beams with a spectrophotometer. The layers of different thickness are best obtained by pouring the liquid into a glass cell containing a glass plate which reduces the thickness of the layer along the bottom of the cell. The spectrophotometer best adapted to the purpose is the instrument designed by Vierordt. It is provided with a double slit, before which the cell is placed in such a position that the dividing line between the two layers coincides with the junction of the two slits. The intensities of the two spectra, which lie one above the other, can be made equal for any value of  $\lambda$  by altering the widths of the slits. Equal illumination is obtained when the slit widths  $b_1$  and  $b_2$  are inversely proportional to the intensities of the illuminating beams, that is when

$$\frac{J_1}{J_2} = \frac{b_2}{b_1}.$$

Our equation now takes the form

$$\log \frac{b_2}{b_1} = \frac{4\pi\kappa}{\lambda} (d_2 - d_1) \log e,$$

$$\kappa = \frac{1}{4\pi(d_2 - d_1) \log e} \lambda \log \frac{b_2}{b_1}.$$

The absorption coefficient which we have called  $k$  is given by

$$k = \frac{1}{(d_2 - d_1) \log e} \log \frac{b_2}{b_1}, \text{ since } \kappa = \frac{k\lambda}{4\pi}.$$

Further particulars regarding the measurements will be found in Stöckl's paper. His measurements were made with solutions of varying concentrations, and the results plotted as curves. The curve for cyanine in alcohol is shown in Fig. 247. The dispersion curve for pure alcohol is represented by an unknown line, the solution dispersion curves by dotted lines. It will be seen that on the red side of the absorption band, the cyanine increases the refractivity of the alcohol, while on the blue side it decreases it up to a certain point, and then increases it again. The family of curves pass through two common points, which are the intersections of the curve for alcohol with that of solid cyanine, the refractive indices of which have been measured by Pflüger and Wood.

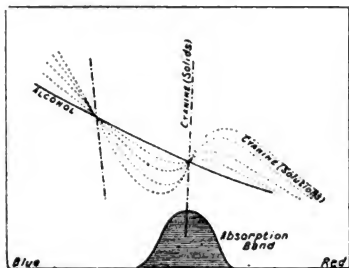


FIG. 247.

The curves for  $\kappa$  are given below the dispersion curves. The wavelength for which the maximum absorption occurs lies further towards the red end of the spectrum than in the case of the solid dye. Stöckl calculated the position of the maximum of the curve for solid cyanine from these observations of  $n$  and  $\kappa$  by employing the formula for  $2n\kappa$ , and obtained a value which agreed closely with the value observed by Pflüger.

**Dispersion of Nitroso-dimethyl-aniline.**—The remarkable optical properties of this substance were investigated by Wood (*Phil. Mag.*, 1903). It is of especial interest in that, while fairly transparent for wave-lengths comprised between the red and blue, it has its band of metallic absorption in the violet. This circumstance gives it an enormous dispersive power in the yellow and green, a prism of the substance yielding a spectrum about fifteen times as long as the spectrum given by a glass prism of the same angle.

The substance melts at  $85^\circ \text{C.}$  and can be formed into prisms between small strips of thin plate glass. The strips should be about two centimeters long, and are best fastened together with one of the small clamps used with rubber tubing. It is best to melt the material on the end of one of the strips, the other being warmed over the same flame, and then

clamp the two together with a piece of a match between the ends. to give the required prismatic form. A candle flame viewed through the prism is spread out into a most remarkable spectrum. It is instructive to have a prism of the same angle made of Canada balsam or some such substance pressed out between two similar glass strips.

The refractive indices were measured with a spectrometer, the slit of which was illuminated with approximately monochromatic light obtained from a spectroscopie furnished with a narrow slit in place of the eye-piece. In the more transparent region a prism of  $8^{\circ} 7'$  was used, while in the vicinity of the absorption band it was necessary to employ one of less than one degree, on account of the opacity of the substance. The values found are given in the following table:

Prism Angle $8^{\circ} 7'$ .				Prism Angle $53'$ .	
$\lambda$ .	$n$ .	$\lambda$ .	$n$ .	$\lambda$ .	$n$ .
508	2.025	636	1.647	497	2.140
516	1.985	647	1.758	500	2.114
525	1.945	659	1.750	506	2.074
536	1.909	669	1.743	513	2.020
546	1.879			577	1.826
557	1.857			647	1.754
569	1.834			669	1.743
584	1.815			696	1.723
602	1.796			713	1.718
611	1.783			730	1.713
620	1.778			749	1.709
627	1.769			763	1.697

The curve for carbon bi-sulphide, which has the strongest dispersion of any transparent substance in common use, is given for the sake of comparison (Fig. 249). The extraordinary dispersion of the nitroso in the visible region is at once apparent. Carbon bi-sulphide absorbs strongly below wave-length 36, and its dispersion can only be measured in the ultra-violet by employing very acute prisms.

Green. Violet. Ultra-violet.



FIG. 248.

The substance was found to become transparent again on the ultra-violet side of the absorption band, and measurements were made in this region by means of photography. A small quartz spectrograph was used, the nitroso prism being mounted with its refracting edge horizontal, immediately behind the quartz prism of the instrument. This device will be at once recognized as the method of crossed prisms.

The undeviated spectrum was photographed by the rays which passed below the edge of the small prism, and by measuring the distances between it and the deviated portion, it was possible to

calculate the refractive index. One of these photographs is reproduced in Fig. 248. It will be seen that the deviation is a maximum in the

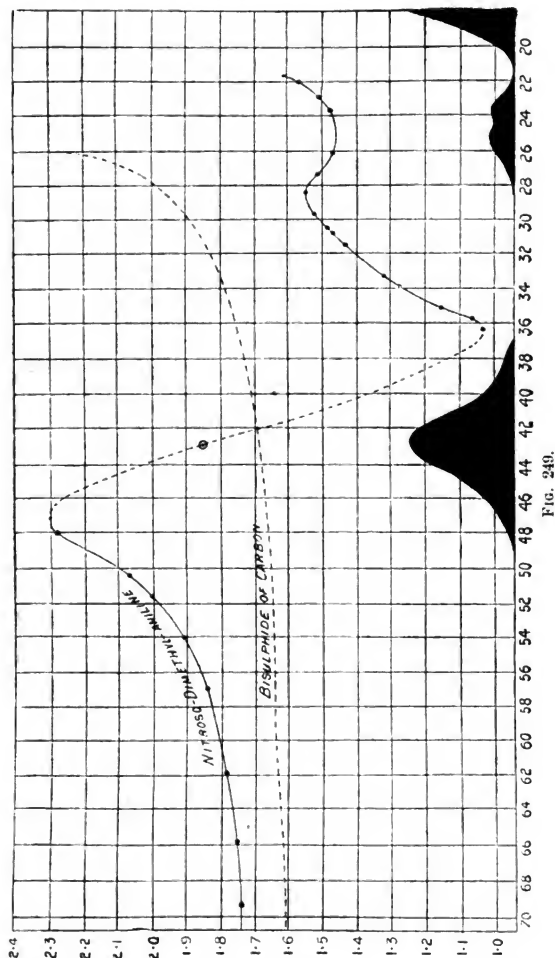


FIG. 249.

green at the edge of the absorption band, while on the other side of the band the deviation is zero, *i.e.* the refractive index equals unity for this wave-length. The continuity of the dispersion curve can be traced

through the absorption band, though the deviated spectrum is so broadened by diffraction that accurate measurements cannot be made in this region. The vertical lines are the bright spectrum lines of the cadmium spark which served as a source of light. The complete dispersion curve is shown graphically in Fig. 249, the position of the absorption bands being recorded as well.

**Electro-Magnetic Theory of Dispersion.**—In the previous treatment we have considered the atoms set in motion by waves in a medium having the properties of an elastic solid. We will now take up the development of the electro-magnetic dispersion formula, which is not unlike the one which we have already examined. Instead of assuming the atoms to be the oscillating particles, we will consider that the medium contains charged electrons. These may or may not be identical with the electrons which we consider as the carriers of electricity in metallic conduction. They are at all events not the same as the ions of electrolysis. It is necessary to consider them charged either positively or negatively, if they are to be set in motion by the rapidly reversing electric force which constitutes light. We must also think of them as having fixed positions of equilibrium with reference to the atoms to which they belong. The application of a steady electric force will displace them, but this displacement once produced, there will be no further movement until the force ceases, when they will resume their former positions. It is clear that these electrons are incapable of conduction, *i.e.* there will be no current produced by the application of a constant difference of potential, for they are bound to the atoms, and their displacement by an electric force is accompanied with something that corresponds to a force of restitution, which we shall consider as proportional to the displacement, as in the Helmholtz treatment.

As an introduction to the present treatment, the reader is advised to glance over the derivation of the expression for the dielectric constant in absorbing media, on pages 363-365 of the chapter on the Optical Properties of Metals. The electrons behave differently of course in the case of metals, but we shall require one or two of the expressions representing their effects in the present chapter.

If the electron is displaced by a force which instantly ceases, it will be drawn back by the elastic force of restitution, and vibrate with a definite period, depending on its mass, charge, and the force of restitution. Forces akin to friction may damp this vibration and eventually bring the electron to rest. A damping due to radiation may also occur; though this is comparatively small, and will not be considered in the present chapter. The electron is analogous in every respect to the Sellmeier vibrator, and, as we shall see, will cause the medium to absorb radiant energy of a period similar to its own. This period depends on the chemical constitution of the medium; in other words, upon the arrangement of the atoms, which makes it appear doubtful if we can consider the ion, at least in some cases, as a minute part of the atom. For example, the aniline dyes, complicated organic compounds, with powerful selective absorption, are made up of atoms which, when existing as elements, or when entering into the composition of other compounds, do not show this absorption at all.

A certain knowledge of the nature of the electron is not necessary for the construction of a satisfactory theory of dispersion. We have merely to assume that an electrical vibration of some sort can be set up within the molecule, it being quite immaterial whether this consists of the to and fro excursion of a group of atoms, or of an electron within an atom.

Calling  $m$  the mass of the electron,  $e$  its charge, and  $\xi$  its displacement along the  $x$  axis, we have its motion represented by an equation similar to the fundamental equation of the Helmholtz treatment,

$$m \frac{\partial^2 \xi}{\partial t^2} = eX - \frac{4\pi e^2}{\theta} \xi - r e^2 \frac{\partial \xi}{\partial t}. \quad \dots\dots\dots (1)$$

In this equation  $eX$  is the force applied by the wave,  $\theta$  may be defined as the reciprocal of the elastic force which urges the electron back, when displaced unit distance. If the electron was in equilibrium under the action of steady force  $X$ , we should have  $r\xi = \frac{\theta}{4\pi} X$ . In the case of metals, where the electrons are free to move continuously under the action of a steady force, we have  $\theta = \infty$ .

The last term of the equation represents the action of some damping factor analogous to friction, which is proportional to the velocity  $\frac{\partial \xi}{\partial t}$  and a constant  $r$ . In the two last terms  $e^2$  is written to show that the direction of the force is independent of the sign of the charge.

The current along the  $x$  axis will consist of two parts, a displacement current in the ether

$$(j_x)_0 = \frac{1}{4\pi} \frac{\partial X}{\partial t},$$

and a convection current due to the motion of the electrons, proportional to the number in unit volume and their velocity,

$$(j_x)_1 = eN \frac{\partial \xi}{\partial t},$$

in which  $N$  is the number of electrons in unit volume and  $\frac{\partial \xi}{\partial t}$  their velocity. The total current will be

$$j_x = (j_x)_0 + (j_x)_1 = \frac{1}{4\pi} \frac{\partial X}{\partial t} + \frac{\partial}{\partial t}(eN\xi). \quad \dots\dots\dots (2)$$

For periodic disturbances we have  $\xi = Ae^{\frac{t}{\tau}}$ , in which  $\tau = \frac{T}{2\pi}$ ,  $T$  being the periodic time of the disturbance which enters the medium, and  $\xi$  the real part of the complex quantity to which it is equated. The calculations can be much simplified by the introduction of complex quantities, and we can return at the end to the physical conception, *i.e.* the real part of the complex. Differentiating we have

$$\frac{\partial \xi}{\partial t} = \frac{i}{\tau} \xi, \quad \frac{\partial^2 \xi}{\partial t^2} = -\frac{1}{\tau^2} \xi.$$

Multiplying (1) through by  $\frac{\theta}{e4\pi}$ , and substituting the above derivatives, gives

$$e\xi + \frac{rei\xi\theta}{\tau 4\pi} - \frac{m\xi\theta}{\epsilon 4\pi\tau^2} = \frac{\theta}{4\pi} X,$$

$$e\xi \left( 1 + \frac{i}{\tau} \frac{r\theta}{4\pi} - \frac{1}{\tau^2} \frac{m\theta}{4\pi e^2} \right) = \frac{\theta}{4\pi} X.$$

Writing

$$a = \frac{r\theta}{4\pi}, \quad b = \frac{m\theta}{4\pi e^2},$$

gives us

$$e\xi = \frac{1}{4\pi} X \frac{\theta}{1 + \frac{i}{\tau} a - \frac{b}{\tau^2}} \dots\dots\dots(3)$$

If we have a number of ions with different values of  $r$  and  $\theta$  the expression for the current (eq. 2) takes the form, by substituting for  $\frac{Xi}{\tau}$  its equivalent  $\frac{\partial X}{\partial t}$  (see page 364, eq. (2)),

$$i_x = \frac{1}{4\pi} \frac{\partial X}{\partial t} \left( 1 + \sum \frac{\theta N}{1 + \frac{i}{\tau} a - \frac{b}{\tau^2}} \right), \dots\dots\dots(4)$$

an equation similar to the one which we derived for an insulator (page 286), except that the dielectric constant  $\epsilon$  is, in this case, represented by the complex quantity in the parenthesis.

It will be seen that the complex dielectric constant is dependent on the period  $T = 2\pi\tau$  of the light. The relation which this value of the dielectric constant bears to the value  $\epsilon$  determined electrically, can be found as follows.

In the electrical determinations we make use either of very long periods (method of electrical waves) or static charges, for either of which we can write  $\tau = \infty$ . Substituting this value of  $\tau$  in the expression for the complex dielectric constant, which we may call the optical dielectric constant  $\epsilon'$ , gives us

$$\epsilon = \epsilon' = 1 + \Sigma \theta N.$$

We can define  $\theta N$  as the dielectric constant of one of the groups of electrons.

The constant  $b$  is identified with the natural free-period of the electron, the friction coefficient  $a$  being neglected. For this case  $X = 0$  and  $a = r = 0$ , under which conditions equation (1) becomes

$$m \frac{\partial^2 \xi}{\partial t^2} = - \frac{4\pi e^2}{\theta} \xi$$

$$- \frac{m\theta}{4\pi e^2} \frac{1}{\tau_m^2} \xi = - \xi, \text{ since } \frac{\partial^2 \xi}{\partial t^2} = - \frac{1}{\tau_m^2} \xi,$$

$$b = \tau_m^2 \text{ and } \tau_m = \frac{T_m}{2\pi},$$

$T_m$  being the period of the electron.

In the chapter on the Optical Properties of Metals we shall see that a complex dielectric constant means absorption, and if we substitute the value which we have obtained for  $\epsilon'$  namely  $1 + \sum \frac{\theta N}{1 + \frac{ia}{\tau} + \frac{b}{\tau^2}}$  (see

equation (4)) in equation (6) (Optics of Metals), page 365, we obtain an expression connecting the refractive index, and the index of absorption, with the period of the light vibration and the natural free period.

The equation of wave-motion can be applied here if we substitute for  $\epsilon$  the complex value  $\epsilon'$ ,

$$\frac{\epsilon'}{c^2} \frac{\partial^2 X}{\partial t^2} = \frac{\partial^2 X}{\partial z^2} \dots \dots \dots (5)$$

To integrate this, we write

$$X = Ae^{i \frac{2\pi}{T} (t - mz)} \dots \dots \dots (6)$$

$X$  has here the significance of the real part of the imaginary. The calculations are much simplified by the introduction of imaginaries, and we can arrive at the physical significance of the equations at the end of the calculations by separating the real quantities from the imaginary. The equation gives us the value of the electric force parallel to the  $x$  axis of a plane-polarized disturbance travelling along the  $z$  axis, at any time  $t$ , and at any point on  $z$ ,  $m$  being of course the reciprocal of the velocity with which the disturbance travels in the medium.

$$\text{Differentiating (6), } \frac{\partial^2 X}{\partial t^2} = -\frac{4\pi^2}{T^2} X, \quad \frac{\partial^2 X}{\partial z^2} = -\frac{4\pi^2 m^2}{T^2} X,$$

and substituting in (5),

$$\frac{\epsilon'}{c^2} \frac{4\pi^2}{T^2} X = \frac{4\pi^2 m^2}{T^2} X, \quad \frac{\epsilon'}{c^2} = m^2.$$

Since  $\epsilon'$  is complex,  $m$  must also be complex, and we can write  $m = \frac{1 - i\kappa}{V}$ , in which  $V$  is the velocity of propagation and  $\kappa$  a constant.

Substituting this value in (6),

$$X = Ae^{\frac{2\pi i}{T} \left( t - \frac{(1 - i\kappa)}{V} z \right)} = Ae^{\frac{2\pi i t}{T} - \frac{2\pi i z}{TV} - \frac{2\pi \kappa z}{TV}},$$

in which  $TV = \lambda$ ,

$$X = Ae^{-2\pi \kappa \frac{z}{\lambda}} e^{\frac{2\pi i}{T} \left( t - \frac{z}{V} \right)} \dots \dots \dots (7)$$

In this expression  $Ae^{-2\pi \kappa \frac{z}{\lambda}}$ , which represents the maximum value which  $X$  has during a complete reversal, is seen to diminish as  $z$  increases. This means that the intensity falls off as we proceed along the  $z$  axis, or that absorption takes place.

The light, after traversing a thickness equal to the wave-length  $\lambda$ , is decreased in amplitude by an amount  $e^{-2\pi \kappa}$ . The constant  $\kappa$  is the measure of absorption, and is called the absorption index.

We are now in a position to get the equations which connect the refractive index  $n$  and the absorption index  $\kappa$ , with the constants of the medium.

$$\frac{\epsilon'}{c^2} = \left( \frac{1 - i\kappa}{V} \right)^2, \text{ and, since } n = \frac{c}{V}, \quad \epsilon' = \frac{c^2}{V^2} (1 - \kappa^2 - 2i\kappa);$$

$$\therefore \epsilon' = n^2 (1 - \kappa^2 - 2i\kappa). \dots\dots\dots (8)$$

By equations (4) and (8) we have

$$n^2 (1 - i\kappa)^2 = 1 + \sum \frac{\theta N}{1 + \frac{i}{\tau} a - \frac{\tau_m^2}{\tau^2}} \dots\dots\dots (9)$$

By separating the real and imaginary parts of this equation we can derive two relations, from which  $n$  and  $\kappa$  can be determined.

**Normal Dispersion.**—In the case of normal dispersion we are dealing with a range of frequencies which does not include the free-period of the electron. The term  $\frac{i}{\tau} a$  can in this case be neglected, since it represents friction, and friction is not brought into play, since the electron is not thrown into vibration. This makes the right-hand member of the equation real, and  $\kappa = 0$ . The expression for the refractive index reduces to

$$n^2 = 1 + \sum \frac{\theta'_h}{1 - \left( \frac{\tau_m}{\tau} \right)^2}, \dots\dots\dots (10)$$

in which  $\theta'_h = \theta N$ .

For a medium with two absorption bands, one in the infra-red, the other in the ultra-violet, the formula may be written

$$n^2 = 1 + \frac{\theta'_v}{1 - \left( \frac{\tau_v}{\tau} \right)^2} + \frac{\theta'_r}{1 - \left( \frac{\tau_r}{\tau} \right)^2},$$

in which  $\theta'_v$  is the value of  $\theta N$  for the electrons vibrating with ultra-violet periods, and  $\theta'_r$  the value of  $\theta N$  for the infra-red electrons.  $\tau_v = \frac{T'_v}{2\pi}$ ,  $\tau_r = \frac{T'_r}{2\pi}$ ,  $T'_v$  and  $T'_r$  being the free-periods of the two sets of electrons.

This formula can be shown to be the equivalent of a Cauchy series, with four constants,

$$n^2 = -A'T^2 + A + \frac{B}{T^2} + \frac{C}{T^4},$$

in which  $T$  is the period of the light.

If  $T$  differs considerably from  $T'_v$  and  $T'_r$ , as is the case when the radiations belong to the visible spectrum, and the medium is transparent,  $\frac{\tau_v}{\tau}$  will be a small fraction, and we have, by division,

$$\frac{1}{1 - \left( \frac{\tau_v}{\tau} \right)^2} = 1 + \left( \frac{\tau_v}{\tau} \right)^2 + \left( \frac{\tau_v}{\tau} \right)^4 + \dots$$

For the infra-red electrons  $\frac{\tau}{\tau_r}$  is a small fraction, and

$$\frac{1}{1 - \left(\frac{\tau_r}{\tau}\right)^2} = \frac{\tau^2}{\tau^2 - \tau_r^2} = \frac{\tau^2}{\tau_r^2} \frac{1}{\frac{\tau^2}{\tau_r^2} - 1} = -\frac{\tau^2}{\tau_r^2} \frac{1}{1 - \left(\frac{\tau}{\tau_r}\right)^2}.$$

Developing the fraction into a series as before, the expression becomes

$$-\frac{\tau^2}{\tau_r^2} \left\{ 1 + \left(\frac{\tau}{\tau_r}\right)^2 + \left(\frac{\tau}{\tau_r}\right)^4 \right\} + \dots$$

The dispersion formula now takes the form (writing in place of  $\tau$  the actual period  $T$ )

$$n^2 = 1 + \theta'_r + \frac{\theta'_r T_r^2}{T^2} + \frac{\theta'_r T_r^4}{T^4} - \frac{\theta'_r T^2}{T_r^2} - \frac{\theta'_r T^4}{T_r^4},$$

which is identical with the four-constant Cauchy formula just given. The term  $A$  of this formula, which is independent of  $T$ , has the physical significance

$$A = 1 + \theta'_r.$$

The dielectric constant  $\epsilon = 1 + \Sigma \theta'$ ;  $\therefore \epsilon - A = \theta_r$ ,

or the difference between the dielectric constant and the term of the dispersion formula which is free from  $T$ , represents the dielectric constant of the group of electrons with periods corresponding to those of infra-red radiations.

The coefficient  $A'$  in the formula represents the effect of the electrons with infra-red periods, and in the case of substances with dispersions represented by the three-constant formula  $n^2 = A + \frac{B}{\tau^2} + \frac{C}{\tau^4}$ , though we can not be sure that there are no absorption bands in the infra-red region, we know that the dielectric constant of the electrons is small. For such substances  $A$  should represent the dielectric constant.

The coefficient  $A'$  has a larger value for water than for any other transparent substance, which is what we should expect, from the circumstance that water is the least diathermous of all the transparent media. If we assume that but one absorption band exists in the infra-red, we can calculate its position from the experimentally determined value of the constants  $A'$  and  $\epsilon - A$ . Referring to the formula we find

that  $A'$  corresponds to  $\frac{\theta'_r}{T_r^2}$  and  $\epsilon - A = \theta_r$ , from which we have for the period of the infra-red electron

$$T_r = \sqrt{\frac{\epsilon - A}{A'}}.$$

$A'$  and  $A$  are calculated from observations of the dispersion, while  $\epsilon$  is the electrically determined dielectric constant.

The absorption band determined in this way is situated at a point in the spectrum corresponding to wave-length .078 mm. Rubens and Aschkinass found (*Wied. Ann.*, 65, 252, 1898) that the long heat waves obtained by multiple reflections from sylvite ( $\lambda = 61\mu$ ) were more strongly absorbed by water-vapor than the rays from rock-salt

( $\lambda = 51\mu$ ). This makes it seem probable that the infra-red band is somewhere beyond  $61\mu$ , which is in agreement with the value  $78\mu$  calculated by Drude.

Equation (10) can be written

$$n^2 = 1 + \sum \frac{\theta'_a \tau_a^2}{\tau_a^2 - \tau_a^2} = 1 + \sum \theta'_a + \sum \frac{\theta'_a \tau_a^2}{\tau_a^2 - \tau_a^2}$$

or 
$$n^2 = b^2 + \sum \frac{M_a}{\lambda_a^2 - \lambda_a^2} \dots\dots\dots (11)$$

in which  $b^2$  represents the dielectric constant. In this expression we have substituted wave-lengths for periods, as they are more convenient to work with,  $\lambda_a$  being the wave-length in ether of a disturbance of the same period as that of the electron and  $M_a = \theta'_a \lambda_a^2$ . The dispersion of the ordinary ray in quartz is well represented if we take the summation for three terms, *i.e.* for three absorption bands.

The constants have been determined as follows:

$$\begin{array}{lll} M_1 = 0.106, & \lambda_1^2 = 0.106, & \\ M_2 = 44.224, & \lambda_2^2 = 78.22, & b^2 = 4.58. \\ M_3 = 713.55, & \lambda_3^2 = 430.56. & \end{array}$$

From equation (11) we see that these constants must fulfil the relation

$$b^2 - 1 = \sum \theta'_a = \frac{M_1}{\lambda_1^2} + \frac{M_2}{\lambda_2^2} + \frac{M_3}{\lambda_3^2}.$$

The sum of the fractions is 3.2, while  $b^2 - 1 = 3.58$ .

The discrepancy can be accounted for by assuming the existence of one or more other absorption bands in the remote ultra-violet. The wave-lengths corresponding to the positions of these bands are so small that we can consider them equal to zero.

If the sum of the dielectric constants of these groups of electrons is  $\theta'_0$ , we have  $b^2 = 1 + \theta'_0 + \sum \theta'_a$ ,  $M_a = \theta'_a \lambda_a^2$ ,

$$\theta'_0 = b^2 - 1 - \sum \frac{M_a}{\lambda_a^2}.$$

In the present case  $\theta'_0 = 0.38$ .

The dielectric constant for quartz has been found to be in the neighborhood of 4.6, which agrees well with the value of  $b^2$  determined optically.

**Anomalous Dispersion of Sodium Vapor.**—We will begin by considering the remarkably interesting case presented by the vapor of metallic sodium. This vapor has two very strong absorption bands (the *D* lines) which profoundly affect the velocity with which light waves traverse the vapor. The anomalous dispersion of the vapor was first observed by Kundt, who noticed, when projecting a continuous spectrum upon a screen, a sodium flame having been placed in front of the lens to exhibit the reversal of the lines, that the edges of the spectrum immediately adjacent to the dark absorption lines were slightly curved in opposite directions, indicating abnormally high and low

refracting power of the prismatic absorbing flame. The phenomenon has been subsequently studied by Becquerel, Julius, Ebert and Wood.

An application of the Sellmeier formula, which is identical with the electro-magnetic formula which we have just developed, was made by Lord Kelvin (*Phil. Mag.*, 47, 1899), which will be given presently.

If we write the formula in the form  $n^2 = 1 + \frac{m\lambda^2}{\lambda^2 - \lambda_m^2}$ , in which  $\lambda_m = 5893$ , the mean wave-length of the *D* lines, it will represent fairly well the dispersion of the vapor, the two absorption lines being so close together that they can be considered as one, at least when the vapor is very dense. If we give to  $\lambda$  values slightly larger and slightly smaller than  $\lambda_m$ , we shall find that the denominator grows less as we approach the absorption band from the red side, resulting in a rapid increase of  $n$ . When  $\lambda = \lambda_m$  the denominator becomes 0 and  $n = \infty$ . If, however, we approach the absorption band from the side of the shorter wave-lengths the denominator of the fraction becomes negative, which gives us values of  $n$  which are less than unity. The value of the fraction will obviously become greater than one eventually, which will give us for  $n$  the square root of a negative quantity. Such a value of the refractive index has no meaning however, though we may interpret it as Lord Kelvin does in his paper, by saying that it indicates that no light of such wave-lengths as give an imaginary value to the refractive index can enter the medium: they are selectively reflected. It is perhaps questionable whether this interpretation is allowable, since the formula, as we are using it, has been simplified on the assumption that it is only to be applied to regions of the spectrum for which the substance is transparent, a region which obviously does not contain the wave-lengths in question, owing to the finite width of the absorption bands.

The conception is useful, however, in that it shows us that the failure of light to penetrate a medium may result from other causes than the stirring up of vibrations within the medium which are accompanied by friction. In the latter case the energy is used up in heating the medium, in the former it is thrown back or reflected. It is clear that as the density of the sodium vapor is increased, the numerator of the fraction increases. For a very dense vapor the range of wave-lengths which give to  $n$  an imaginary value stretch farther away from the center of the absorption line towards the more refrangible end of the spectrum. We should therefore expect that the dark band seen in the spectrum of the transmitted light would widen out on this side as the density of the vapor increased. As a matter of fact it widens out on both sides. Moreover, it widens in a similar way if we increase the length of the absorbing column, holding the density constant, which makes it seem probable that the broad band is caused by true absorption, and not by selective reflection.

A study of the dispersion of the vapor of metallic sodium has been made by Wood, with a view of testing the simplest form of the dispersion formula which we have, namely the one given above.<sup>1</sup>

<sup>1</sup> Wood, "A Quantitative Determination of the Anomalous Dispersion of Sodium Vapor," *Phil. Mag.*, Sept. 1904.

As the phenomena exhibited by this vapor are among the most beautiful in physical optics, it may be well to consider them in some detail.

If we place a number of pieces of clean sodium in a tube of hard Jena glass, the ends of which are closed with small pieces of thin plate glass, and exhaust the tube on a mercurial pump, on heating the under side strongly with small Bunsen flames, the sodium vapor shows very little inclination to distil to the cold parts of the tube. It condenses, to be sure, on the upper side of the tube, but is given off so much more rapidly from the surface of the molten metal than it can diffuse to the upper portion that the density gradient is very steep. Observations on the deviation produced by the non-homogeneous cylinder show that the equivalent prism has a form similar to that shown in Fig. 250, the

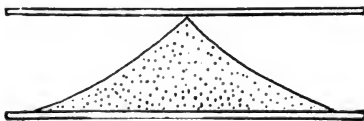


FIG. 250.

density gradient being steeper near the bottom of the tube. To secure good definition it is therefore necessary to place in front of the tube an opaque screen perforated with a wide horizontal slit. The tube thus prepared, used in the manner to be presently described, shows the strong anomalous dispersion in the vicinity of the *D* lines with great distinctness.

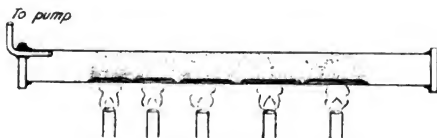


FIG. 251.

These tubes can be very easily prepared, and give no trouble at all. They are, however, apt to crack if reheated, after having been allowed to cool. It is not necessary to have the sodium in separate small lumps, as the action is not the formation of a number of prisms of sodium vapor, but a cylinder of vapor, very dense along the heated floor of the tube, and decreasing to very nearly zero along the top. If it is desired to prepare a tube for purposes of demonstration, a tube of thin steel such as can be procured at any bicycle repair shop is preferable, as it can be used over and over again. In this case it is necessary to lay a thick pad of wet asbestos paper along the top of the tube, since the more uniform heating which results from the better conductivity of the steel is detrimental to the formation of the non-homogeneous cylinder of vapor, which only takes place when the temperature gradient between the top and bottom of the tube is very steep. The

plate glass ends are fastened on with sealing-wax, a small glass tube being sealed in through which the air is exhausted. See Fig. 251.

The tube is heated on the under side by a row of small Bunsen burners, turned down low, the tips of the flames just touching the metal. The amount of dispersion can be regulated by turning the flames up or down. The apparatus for exhibiting the dispersion consists of a horizontal slit on which the crater of an arc lamp is focused. If a Nernst lamp is available it may advantageously be substituted for the illuminated slit, the glower being mounted in a horizontal position, and the collimator dispensed with. The light from the slit is made parallel by a lens, and after traversing the sodium tube, is focused on the slit of a small spectroscope by means of a second lens. It is almost as satisfactory to use a single lens passing the convergent beam through the dispersion tube.

Blue.

Green.

Yellow.

Red.

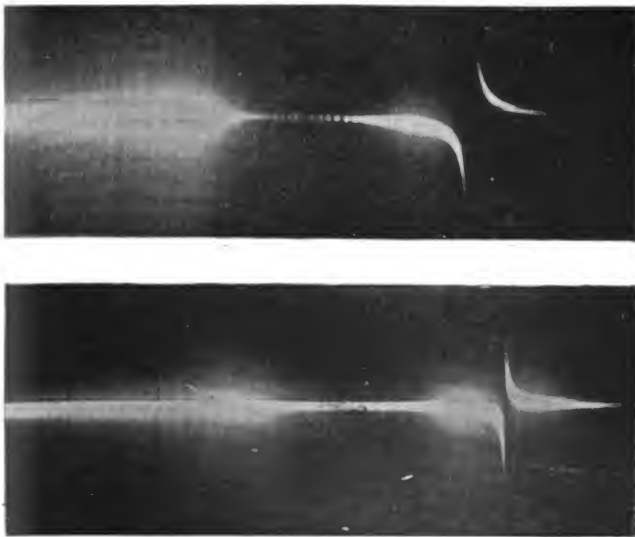


FIG. 252.

If it is desired to show the anomalous *spectrum*, a low power eyepiece is substituted for the spectroscope. The appearance of this spectrum is shown in the colored frontispiece (Fig. 1).

To obtain sufficient dispersion the heated portion of the tube should be above 40 cms. long, when the spectrum presents a superb sight, the colors being of great purity and intensity. If the electric arc is employed as the source of light, the extreme violet will be found to occupy the position of the undeviated image of the slit. Then comes the blue, sometimes in contact with the violet and sometimes slightly separated

by a fine dark line, owing to the fact that the violet light comes from the fluted carbon band of the arc, which is separated from the blue by a comparatively dark region. Then comes a wide gap corresponding to light absorbed by the sodium vapor in the blue-green region (the channelled spectrum), and above this a beautiful flare of color ranging from blue-green through grass-green to yellow. The red and orange portion of the spectrum is on the other side of or below the undeviated image, forming another brilliant flare of color. It is separated from the violet by a wide dark band, due to the absorption in the vicinity of

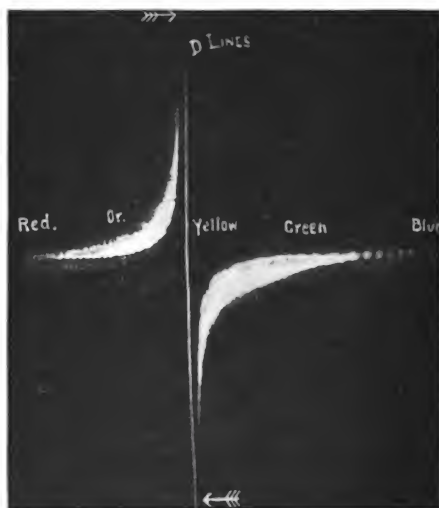


FIG. 253.

the *D* lines. If the density of the vapor is increased by heating the tube to a higher temperature the red flare extends lower down, grows fainter, and finally fades away, owing to the presence of the fluted absorption bands in the red. The green and blue persist, however, becoming more widely separated, but finally the green disappears almost entirely. It is best to arrange the gas-cock so that the height of the flames can be controlled without leaving the eye-piece, for it is surprising how slight a change is necessary to completely alter the general appearance of the spectrum.

The glass tube should not be allowed to cool until the experiment is at an end, otherwise it will immediately fly to pieces as soon as the flame is again applied to it.

If the spectroscope is placed with its slit in the position of the eye-piece, *i.e.* with the image of the horizontal slit or Nernst glower sharply focused on it, the spectrum appears as in Figs. 252, 253, which

are from photographs. The anomalously dispersed spectra are also shown in the colored frontispiece (Figs. 2).<sup>1</sup>

If a quartz spectrograph is used, and the tube closed with quartz plates, the light being collimated and focused by means of concave silvered reflectors, this spectrum can be photographed well down into the ultra-violet, when anomalous dispersion will be found to occur at the ultra-violet absorption lines, though it is much feebler here, as shown by Fig. 254.

Though we can measure the relative indices of the vapor by this method, we have no means of determining the absolute values, for we have no means of knowing the angle of the prism of vapor, which is the equivalent of the non-homogeneous cylinder.

Absolute values have, however, been obtained by means of the interferometer, by comparing and measuring the shifts of the fringes obtained by introducing a given amount of sodium vapor into the path of one of the interfering beams. A full description of the methods and apparatus employed will be found in the paper referred to. In brief, it consisted in placing a sodium tube, electrically heated, between two of the mirrors of a Michelson interferometer. Uniform heating was



FIG. 254.

necessary in this case, as no prismatic action was desired. The instrument was illuminated with two sources of light, one a helium spectrum tube which gives us a bright yellow light,  $D_3$ , very near the sodium absorption band, the other a spectroscope arranged to furnish a beam of approximately monochromatic light in any desired part of the spectrum. Two sets of fringes were thus formed, and the drifts of the systems were recorded by two observers as sodium vapor was formed in the tube. To obtain the dispersion very near the absorption band, the helium tube was placed in a powerful magnetic field, which causes the line to become double (with suitable arrangement of the apparatus). The two components were of very nearly the same wave-length, the distance between them being about  $\frac{1}{10}$  of the distance between the  $D$  lines, yet the dispersion of the vapor was so powerful that the two sets of fringes were displaced at rates so different that the fringes disappeared entirely at regular intervals, owing to their "out-of-step" superposition.

By employing the method of "crossed prisms," relative determinations were made still closer to the  $D$  lines than the helium line. To see the effect close to and between the  $D$  lines, the tube should only be slightly heated, and a grating spectroscope employed. As the

<sup>1</sup> The lowest branch of the dispensed spectra is a little too far to the left in the frontispiece.

vapor prism forms, we see presently the portions of the spectrum adjacent to the absorption lines curve away in opposite directions, as shown in Fig. 255. As the vapor becomes denser, the light disappears between the  $D$  lines, and we have the stage previously described.

Absolute values of the refractive index were obtained by heating the tube to a known temperature, measuring the length of the vapor column, and counting the fringe shift produced when monochromatic light of known wave-length was used to illuminate the instrument.

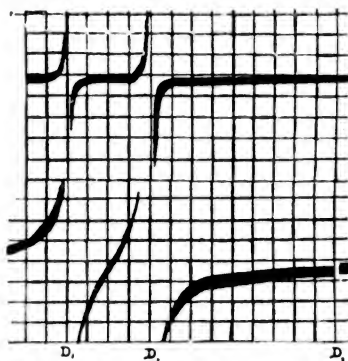


FIG. 255.

**The Refraction and Dispersion of Sodium Vapor of Great Density.**—A knowledge of the absolute value of the refractive index of the vapor, and its dispersion, enables us to compile a table of the refractive indices for all wave-lengths, for vapors of varying density. This has been done for the very dense vapor obtained by heating a vacuum tube containing the metal to the temperature of  $644^{\circ}\text{C}$ . A column of the vapor at this temperature 8 cms. in length examined by transmitted light has a distinct blue color, as a result of the channelled absorption spectrum. The values are given in the following table, and will be spoken of in future as "observed values" to distinguish them from values calculated from the dispersion formula. It must be remembered that sodium vapor as dense as that with which we are dealing in the present case has an absorption band at the  $D$  lines broad enough to completely cut out everything down to and even below the helium line, at least for all thicknesses with which it is possible to work. On this account we are obliged to calculate the refractive indices within this region from observations made with a less dense vapor, a method which in the present case is probably allowable within certain limits. A thin enough sheet of the vapor would probably transmit light within this region with a velocity indicated by the calculated indices. The question of selective

reflection at the surface, and refusal to transmit the radiation will be considered presently.

In the following table the wave-lengths are given in the first column, the fringe displacements in comparison with helium light in the second column. These values, with the exception of those in the extreme red, blue-violet, and ultra-violet, were obtained with the interferometer.

In the third column are given the actual fringe shifts which would be found for a layer of vapor 8 cms. thick (effective thickness 16 cms.), and in the fourth column the refractive indices. The indices calculated from the dispersion formula are given in the fifth column, for the sake of comparison with the observed values. More will be said of them in the next section.

$\lambda$ .	Relative Shift.	Total Shift.	Ref. Index Obs.	Ref. Index Cal.
7500	2	25	1.000117	
6310	4	50	1.000197	
6200	6	75	1.000291	1.000285
6137	7	88	1.000335	
6055	11	138	1.000523	1.00052
6013	14	175	1.000658	
5977	20	250	1.000934	
5960	25	313	1.001164	
5942	33	413	1.001532	
5916	60	750	1.002972	
5875	100	1250	0.995410	0.9958
5867	67	834	0.996929	0.99692
5858	50	625	0.997711	
5850	40	500	0.998172	0.99815
5843	33	413	0.998492	
5827	25	313	0.998862	
5807	20	250	0.999093	
5750	11	138	0.999505	
5700	9	113	0.999599	
5650	7.4	92	0.999650	
5460	4	50	0.9998294	0.999829
5400	3.6	45	0.9998481	
5300	2.9	36	0.9998807	0.999885
4500	1.4	17.5	0.9999508	0.999965
3610	1.09	13.6	0.9999698	
3270	0.9	11.4	0.9999768	0.999987
2260	0.7	8.7	0.9999877	0.999995

#### REFRACTIVE INDICES IN THE VICINITY OF THE *D* LINES.

$\lambda$ .	$n$ Cal.	$n$ Obs.
5875	0.9958	0.9954
5882	0.9890	0.9908
5885	0.9830	0.9870
58866	0.9750	0.9740
58884	0.9450	0.9443
58896	0.697	0.614
5991	1.0046	
5904	1.0092	
5901	1.0138	
58994	1.0184	
58976	1.0557	
5897	1.094	
58964	1.386	

For  $\lambda = \infty$  the value of  $n$  is calculated to be 1.0000275, while for  $\lambda = 4167$  it is 0.999975, the deviations being the same in both cases, but in opposite directions.

For wave-lengths immediately adjoining the  $D$  lines the refractive index has been found to have a value as high as 1.38, as great as that of some liquids, while the dispersion is so great even at the position of the  $D_3$  line of helium, that, could we form a prism of the vapor giving the same deviation as a  $60^\circ$  glass prism, we could by its aid separate a double line in the spectrum, with components twenty times as close as the sodium lines by an amount as great as the distance between the red and blue of the spectrum formed by the glass prism.

The vapor is remarkable in that all waves on the blue side of the  $D$  lines travel in it with a higher velocity than in a vacuum. In other words they are accelerated. All light of wave-length greater than the  $D$  lines is retarded as in ordinary media. This is seen at once from our first experiment, the sodium prism deviating one half of the spectrum in one direction, the other in the opposite.

**Application of the Results to the Dispersion Formula.**—The simplest form of the dispersion formula, developed from electro-magnetic considerations for a medium with a single absorption band, is

$$n = 1 + \frac{m\lambda^2}{\lambda^2 - \lambda_m^2},$$

in which  $m$  is a constant,  $\lambda$  the wave-length of the light employed, and  $\lambda_m$  the wave-length at the center of the absorption band.

The vapor of sodium has, of course, a pair of very close absorption bands (the  $D$  lines), which are chiefly effective in modifying the refractivity of the medium. The ultra-violet bands affect the dispersion in their immediate vicinity, but their effect can be neglected in comparison with the stronger band, except for wave-lengths very close to them.

The first question to consider is whether we are justified in considering the  $D$  lines as a single absorption band in the case of the very dense vapor. If we consider the medium as having a single band, and assign to  $\lambda_m$  the value 5893 (a point midway between the  $D$  lines), we shall find that the observed and calculated values of the refractive index agree very closely, up to within a distance of the band about equal to the distance of the  $D_3$  line of helium. If we attempt to go closer than this we immediately find discrepancies, which become larger as we approach the  $D$  lines.

The constant  $m$  in the above formula can be determined from a single observation of the refractivity. It was determined from two values, namely, the refractivity for the green line of mercury ( $\lambda = 546$ ) and that for  $\lambda = 585$ , the latter wave-length being quite close to the  $D$  lines. Since the refractivity has widely different values for these two wave-lengths, we should expect the difference between the two calculated values to be a maximum in this case, in the event of the dispersion being incorrectly represented by the formula. The values for  $m$  found in the two cases were 0.000056 and 0.000054, a surprisingly close agreement. The mean value  $m = 0.000055$  was taken,

and the indices for a number of wave-lengths calculated. Some of these values are given in the table of refractive indices, and they will be found to agree very closely with the observed values. The values calculated between the helium line and the  $D$  lines are given in the second part of the table.

An inspection of the formula shows us that, according as we are on the red or blue side of the absorption band, the refractive index is given by adding to (or subtracting from) unity, the value of the constant  $m = 0.000055$  multiplied by the fraction  $\frac{\lambda^2}{\lambda^2 - \lambda_m^2}$ . In the case of all other substances showing anomalous dispersion, aniline dyes for example, to the dispersion of which a formula has been applied, the value of this fraction never exceeds 3 or 4, owing to the impossibility of applying it to wave-lengths very close to the center of the band. For example, in the case of the dispersion of nitroso dimethylaniline, with its strong absorption band at  $\lambda = 43$ , we cannot obtain accurate data nearer than  $\lambda = 50$ . In this case  $\frac{\lambda^2}{\lambda^2 - \lambda_m^2} = 3.9$ .

In the case of sodium vapor the value of the fraction may be several hundred or even thousand. In the case of  $\lambda = 5882$  the fraction is 367, and yet the observed and calculated values agree closely. For  $\lambda = 58884$  the fraction is 1940; and for  $\lambda = 5889.6$  we have a value as high as 3944. The product of these very large numbers and the small fraction 0.000055 give, however, values of the index which are in close agreement with the observed values.

Discrepancies occur in the immediate vicinity of the  $D$  lines which can be explained in the following way. To get values in any way consistent with the observed values it was necessary to assign to  $\lambda_m$  the value of the  $D_2$  line, the mean value 5893 being too far removed from the wave-lengths in question to give the requisite steepness to the curve. The calculated values, therefore, apply to a medium with a single band at  $D_2$  and with a constant  $m = 0.000055$ . This gives us a pretty good approximation to the observed curve, but the latter is due to the combined effects of the bands  $D_1$  and  $D_2$ , the presence of the  $D_1$  band tending to make the observed curve flatter than the calculated. A more correct approximation could be obtained by assigning to  $\lambda_m$  a value intermediate between  $D_2$  and 5893. The proper method of procedure would, of course, be to make use of two members in the dispersion formula, one for  $D_1$  and the other for  $D_2$ , thus:

$$n^2 = 1 + \frac{m\lambda^2}{\lambda^2 - \lambda_m^2} + \frac{m'\lambda^2}{\lambda^2 - \lambda_{m'}^2}$$

If  $m$  and  $m'$  were each assigned the same value obtained by dividing our original value by 2, in all probability a very close approximation would be obtained in the region in question. This has not been done for two reasons. In the first place it does not appear as if much would be learned by the procedure, and in the second place  $m$  and  $m'$  are not equal, as is shown by the stronger dispersion near  $D_2$ , and until the relative values have been determined we are not in a position to write the two-member formula accurately. It is doubtful whether

anything new would come out of such a determination, and it was on that account not attempted.

Another matter of considerable interest is the question of the indices represented by the square root of a negative quantity in the immediate vicinity of the absorption band on the blue side. Lord Kelvin interprets this as indicating that no light of such wave-lengths enters the medium; in other words, it is metallically reflected.

It is in this way that he has explained the apparent greater broadening of the  $D$  lines on the more refrangible side in some of Becquerel's photographs. In the case with which we are dealing the second term of our original formula does not become less than unity

until we reach wave-length 58898, which we get by equating  $\frac{m\lambda^2}{\lambda^2 - \lambda_m^2}$

to unity and solving for  $\lambda$ .

This shows us that, even with a vapor so dense that both  $D$  lines run together and broaden out into a wide band, we do not get values of the index which are imaginary until we are within 0.2 of an Angstrom unit of the  $D$  line, or, in other words, until we are within a distance of  $D$  equal to  $\frac{1}{30}$  of the distance between  $D_1$  and  $D_2$ .

In the case of the comparatively rare vapor employed by Becquerel we should have to approach much closer than this to get the imaginary values. This makes it appear certain that the greater broadening on the more refrangible side, if it exists, must be assigned to some other cause than imaginary values of the refractive index.

The medium is exceptionally interesting in that its dispersion can be represented throughout the entire range of wave-lengths without taking the extinction coefficient into account, as is always necessary in the case of solids and liquids when in the vicinity of the absorption band.

## CHAPTER XIV.

### ABSORPTION OF LIGHT.

THE transmission of light through a material medium is always accompanied by a certain amount of absorption, regardless of the color or wave-length of the light. Media which we commonly speak of as transparent, if not employed in too great thickness, transmit without appreciable absorption the range of wave-lengths comprised within the region of the visible spectrum.

In general, however, they exercise powerful absorption in the infra-red and ultra-violet regions, and if a sufficiently great thickness is employed, absorption will be found present even in the range of visible radiations. Pure water, which is one of the most transparent substances which we have, in long columns appears distinctly blue, showing that it absorbs more or less completely the red end of the spectrum. The same is true of most varieties of glass. The definition "transparent" is thus seen to be purely arbitrary, there being no such thing in nature as a perfectly transparent substance.

The character of the absorption exerted by any substance can be best observed by receiving the transmitted light on the slit of a spectroscope, when dark regions will be observed in the spectrum, corresponding in position to the wave-lengths absorbed. If the absorbing medium is moulded into the form of a wedge, which is placed in contact with the slit of the instrument, we can observe at once the effect of increased thickness, the form of the absorption curve being pictured in the spectrum. In general, it will be found that as the thickness increases the absorption band widens out. One edge of the spectrum shows us the absorption of a thin layer, the other edge that of a thick layer, intervening portions corresponding to intermediate thicknesses. The resultant curve is sometimes symmetrical, but more often not so, and we shall see, when we come to consider the theory of absorption, that the form of this curve depends upon a number of different factors. The absorption spectra of about 150 aniline dyes have been photographed and published in the form of an atlas by Wood and Uhler. A wedge-shaped layer of the liquid was used contained in a quartz cell which was placed in contact with the slit of a large grating spectroscope. Photographs were in this way obtained showing the position and forms of all absorption bands, both in the visible and ultra-violet regions. These photographs are extremely useful in the preparation of screens for absorbing particular regions of the spectrum. Three of these pictures are reproduced in Fig. 256,

and show the absorption of nitroso dimethyl-aniline, auramine, and potassium permanganate. We shall first examine the phenomenon of absorption in a general way, and then in its relation to other closely related phenomena, such as dispersion, emission, and the transformation of the absorbed radiations into other types of energy.

At the beginning of the subject we shall find it convenient to distinguish between two types of absorption: general, in which the absorbing power is very nearly the same for all wave-lengths, at least over a fairly wide range, and selective, when the absorbed region is more or less limited in extent. The absorption of metal films and lamp-black represents the first type fairly well. The light transmitted

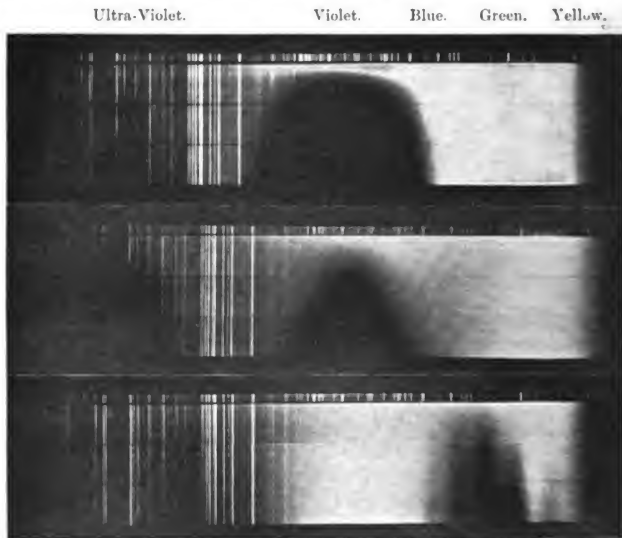


FIG. 256.

through thin layers differs but slightly in its composition from the original light, and exhibits therefore but little color. Of course there are exceptions, for, as is well known, thin films of gold transmit an excess of green light, while silver is fairly transparent to the ultra-violet. Aniline dyes, and in fact all colored media, represent the second type, certain colors being freely transmitted, while others are strongly absorbed. When we come to consider the theory of the phenomena, we shall see that the causes of the absorption are radically different in the two cases, though in many cases both conditions may occur simultaneously in one and the same medium.

**Coefficient of Transmission: Dichromatism.**—If the absorbing medium is homogeneous, the quantity of light of a given wave-length

which is absorbed will be proportional to the thickness of the medium traversed. If we represent the intensity of the light that enters the front surface of the medium by  $I$ , the intensity after transmission through unit thickness can be represented by  $Ia$ , in which  $a$  is a fraction depending on the nature of the medium, and the wave-length of the light. If the same fraction is absorbed by each successive layer, it is clear that the intensity, after traversing a thickness  $\epsilon$  of the medium, will be  $Ia^\epsilon$ , the quantity  $a$  being called the coefficient of transmission.

The coefficient of transmission varies with the color, and the emergent light is therefore colored. In the case of most absorbing media the color of the transmitted light does not depend to any great degree on the thickness, the depth or saturation merely increasing. In some cases, however, the color depends on the thickness, thin layers, for example, appearing green and thick layers red. Such substances are said to exhibit dichromatism. Some of the aniline dyes, or mixtures of them, show the phenomenon. Thin layers of a solution of cyanine appear blue: thick layers red. The addition of a little nitroso-dimethyl-aniline to the solution gives us a green-red dichromatic liquid, as has been shown by Pflüger. The explanation of the change of tint is very simple. Suppose we have a substance which absorbs the yellow and blue. The transmitted light then consists of a mixture of red and green. Let us assume, as is usually the case, that the intensity of the green is greater than that of the red. Writing for these intensities  $I_g > I_r$ , and assuming that the coefficient of transmission of the green is greater than that of the red,  $a_g > a_r$ , it follows that for small thicknesses  $I_g a_g^\epsilon$  will be greater than  $I_r a_r^\epsilon$ , while the reverse will be true for thick layers. This is at once apparent if we call the original intensities of the green and red 100 and 50, and the coefficients of transmission .5 and .8, and calculate the intensities of the transmitted colors for several different thicknesses. They will be equal for a thickness  $\epsilon$  given by the equation

$$I_g a_g^\epsilon = I_r a_r^\epsilon,$$

or taking the logarithms of both sides,

$$\epsilon = \frac{\log I_r - \log I_g}{\log a_g - \log a_r}.$$

For this thickness the intensities of the red and green will be equal, and the color of the transmitted light will appear to be yellow, for a mixture of red and green light produces the sensation of yellow when mixed in proper proportions. An excellent mixture for illustrating this can be formed by dissolving "brilliant green" and "naphthalene yellow" in hot Canada balsam and pressing the mixture between two glass plates in the form of an acute prism. The balsam should be previously boiled down until a drop solidifies on cooling, and the dyes should not be added until the fluid has cooled somewhat, otherwise they are apt to decompose. The thin edge of the wedge will appear green, the thick edge red, and the intermediate portions, where we have equality of transmission, yellow.

If some of the same mixture is moulded into a prism of twenty or thirty degrees angle, the mechanism of dichromatism can be beautifully shown by observing a lamp flame through it. The prism will show the red image well separated from the green, and the latter will be found to be extinguished more rapidly than the former as the prism is moved laterally before the eye.

Our equations for color show us as well that the color of the transmitted light, for a given thickness, will vary with the composition of the original light. If the plate of stained balsam is examined by gas-light and then by day-light, it will be found that parts of it will appear red in the former and green in the latter case. A solution of cyanine and nitroso-dimethyl-aniline in alcohol appears red by lamp-light and bottle-green by day-light. The same phenomenon is exhibited by the gem Alexandrite, found in the Urals.

**Body Color and Surface Color.** — The colors of most natural objects result from absorption. The light penetrates their surfaces, and then suffers internal reflections or refractions and emerges robbed of the rays which are most strongly absorbed. If this is to happen it is clear that the substance must not be homogeneous, otherwise the reflections and refractions, which return the unabsorbed light, will not occur. It is thus incorrect to say that colored pigments reflect certain colors more strongly than others. If the pigment particles formed a continuous and homogeneous medium, no color whatever would appear in the reflected light, which would be white. If any color appeared, as it might if the pigment were a very powerful absorbent, it would be the tint complementary to the one exhibited by the powder. Cases of this nature we shall consider presently.

Since pigments produce color by absorption it is at once apparent why a mixture of two pigments does not exhibit the color which we should obtain if we actually mixed the colored lights which they appear to reflect. The light reflected from the mixture is the residual color which remains after the dual absorption has taken place. For example, if we mix yellow light and blue light we get white, while a mixture of a blue and yellow pigment appears green. The reason of this is, that the yellow pigment absorbs the blue and violet, the blue pigment the red and yellow, the mixture absorbing everything except the green.

The nature of pigments can be well studied by preparing a number of beads of fused borax, colored with varying amounts of cobalt. If we powder a bead which appeared bright blue by transmitted light, we shall find that the powder is white, the reason being that the light in this case does not penetrate a sufficient thickness of the absorbing medium. A bead colored so dense as to appear black will, however, furnish us with a blue pigment when it is reduced to powder. Pigments then are very powerful absorbing media, and if they could be obtained in homogeneous masses would be intensely opaque, even in fairly thin sheets.

If we go on increasing the absorbing power we shall finally observe a phenomenon of a different nature. The color, instead of being absorbed, is selectively reflected. Substances which possess this property are said to exhibit surface color. The aniline dyes are excellent

examples. A dye which in solution absorbs green light, appearing purple by transmitted light, in the solid state reflects green light selectively. Absorption is, however, not the only factor which determines this selective reflection, and we often find misleading statements in text-books on optics, it being frequently stated that the wave-lengths most copiously reflected are the ones most strongly absorbed. This is by no means the case. Cyanine, for example, has a strong absorption band in the yellow, while the color of the selectively reflected light is purple, not so very different in hue from that of the transmitted light. If we examine the spectrum of the reflected light, we find a very dark band in the green, the center being not far from wave-length  $\cdot 0005$ . The distribution of intensity in the rest of the spectrum is not very different from what it would be in the case of reflection from glass, which shows that the peculiar color of the dye is not so much due to a very powerful reflection of certain waves as it is to its almost complete refusal to reflect a certain region of the spectrum.

In the next chapter we shall see that, in the case of absorbing media, the reflecting power depends both upon the refractive index and the coefficient of absorption. Now, absorbing media have a high refractive index on the red side of the absorption band and a low index on the blue side; consequently the spectrum of the reflected light will be brightest on the red side of the absorption band, since for these wave-lengths we have a large coefficient of absorption and a high refractive index. On the blue side, however, the low value of the index diminishes the reflecting power more than the augmentation due to the powerful absorption. The hue of the surface color thus depends on the refractive index of the medium in which the substance is immersed, for it is the relative and not the absolute refractive index with which we are concerned. Cyanine in contact with glass exhibits a yellowish-green surface color, much more nearly resembling the hue of the absorbed light. If the dye could be brought in contact with a transparent substance having the same dispersion, the wave-lengths selectively reflected would be identical with those absorbed, since in this case the relative refractive index would be unity for all wave-lengths.

An excellent way of showing the variable reflecting power of a film of cyanine is to compare it with glass in different parts of the spectrum. A little of the melted dye is pressed between two plates of hot glass, which are separated when cold. A spot is selected where the film has a good optical surface, and this spot is left on the glass, the rest being cleaned off. By holding the plate in the spectrum formed by a prism or grating, the reflecting power of the two surfaces can be studied. In some parts of the spectrum the cyanine reflects more strongly than glass, in other regions the reverse is true, while at wave length  $\cdot 0005$  the cyanine refuses to reflect to such a degree that the film appears as a black spot on the blue field reflected by the glass.

It is interesting to note the difference in the surface color of the dye when the reflection takes place at the surface in contact with the glass. A very convenient way of showing the yellowish green color in this case is to press out a film of the molten dye on one surface of a prism

of 8 or 10 degrees angle.<sup>1</sup> In this way the light reflected from the dye can be obtained uncontaminated with the light reflected from the first glass surface. The method is analogous to that employed by Lippmann in mounting his color photographs. The calculation of the curve of reflected intensities under these conditions makes a good exercise for the student.

**Influence of Solvent on Position of Absorption Bands.**—Kundt (*Pogg. Ann.*, Jubelband, page 615, 1874) made an extensive study of the influence of the nature of the solvent upon the position of the absorption bands of the dissolved substance, and established the following law, which has been known as Kundt's Law.

If one transparent solvent has a higher refractive and greater dispersive index than another, the absorption bands of a dissolved substance lie nearer the red end of the spectrum when it is dissolved in the former than when it is dissolved in the latter.

Kundt endeavoured to determine whether the shift of the absorption band towards the red was due to the increased refractive index of the solvent or its greater dispersive power, but as dispersion and refractivity go hand-in-hand, so to speak, he was unable to arrive at any definite conclusion.

There appears to be evidence also that the position of an absorption band depends upon the concentration of the solution. The salts of didymium exhibit very sharp and intense bands, and Becquerel (*Compt. Rend.*, 102, page 106, 1886) found that one of the bands occurred at  $\lambda = 5790$  when the refractive index of the solution was 1.439 and at 5745 when the index was 1.345.

A similar result was found by Stöckl in the case of potassium permanganate, except that the bands were shifted towards the blue in the dilute solution.

Very little is known about the subject either theoretically or experimentally. An excellent resumé of the whole subject will be found in the 3rd volume of Kayser's *Spectroscopy*.

**Absorption of Light by Gases and Vapors.**—The absorption bands of solids and liquids, with the one or two exceptions mentioned, are broad and more or less ill defined. Gases and vapors, on the contrary, usually exhibit absorption lines of extremely small width, the spectrum of the transmitted light being crossed with fine black lines. This type of absorption was first observed by Brewster (*Pogg. Annalen*, xxviii.) in the case of nitric oxide ( $\text{NO}_2$ ), in the absorption spectrum of which he found over 2000 dark lines, resembling the Fraunhofer lines in the solar spectrum. Similar lines are shown by many other vapors, bromine and iodine for example. The vapor of sodium shows a pair of lines in the yellow, corresponding in position to the *D* lines of the solar spectrum: if the vapor is denser, as when evolved by heating the metal in an iron or glass tube, a host of other lines appear in the red and green portions of the spectrum, while at a full red heat practically

<sup>1</sup> A suitable prism can be made in half an hour by grinding down a piece of thick plate window glass. A strip of thick glass cemented along one edge will be all that is necessary to make the glass take the required form. Grind on a piece of glass with very coarse emery at first, then use finer grades, polishing with rouge at the end. Small scratches do no harm, and a high polish is not necessary.

all of the red, yellow and green is absorbed, the color of the transmitted light being deep violet. Vapors also exhibit broad bands resembling those shown by liquids and solids. The yellow vapor of nitrosodimethylaniline has a broad absorption band in the violet, and shows no trace of any fine lines. Other gases show both types of bands simultaneously, chlorine for example, which has a broad band in the violet and a large number of fine lines in the blue, green and yellow regions.

Many vapors, which under ordinary circumstances show no trace of absorption, and appear colorless, exhibit the lines when great thicknesses are used. Jannsen observed them in the spectrum of light transmitted through a tube 37 meters long filled with dry steam.

**Absorption by Porous Surfaces.**—The absorption of light at surfaces formed of lamp black or finely divided metals such as platinum black is accompanied by very little reflection. The question naturally arises as to why a metal with a high reflecting power can, under certain conditions, appear near dead black. The roughness of the surface will not account for the fact, for matt surfaces of electrolytically deposited silver appear as white as plaster. Chemically precipitated silver, on the other hand, appears black.

The phenomenon is probably to be referred to the condition of the surface. Consider a bunch of polished steel needles, turned with their points towards the light. Rays falling upon the surface formed by the points will be reflected down into the interstices between the needles, and practically none of it will escape or be reflected back. A portion is absorbed at each reflection, and the large number of reflections reduce the intensity rapidly to zero. Surfaces of lamp black (soot) and platinum black can be considered as porous, the pores acting as light "traps." The energy penetrates into the spongy mass by multiple reflection, and is speedily transformed into heat by absorption. If the pores are closed up by compressing the mass, its reflecting power is increased or wholly restored. If, too, the angle of incidence is too large to admit of downward reflection into the mass, the light is more or less completely reflected. A surface of smoked glass reflects very perfectly at large angles of incidence, and at the same time yields a sharply defined image of the source of light, as we have seen in the chapter on Huygens's Principle.

**Absorption by Metals.**—While metals possess in general a high reflecting power, a considerable portion of the incident energy penetrates the surface and is absorbed. In the case of steel, for example, nearly one half of the light is lost by absorption, while even silver absorbs 5%. In the majority of cases the absorption is general, that is it is not confined to a narrow range of wave-lengths, as is the case with the substances which we have examined thus far. It is to a certain extent selective, however, as is best illustrated by gold-leaf, which appears green by transmitted light, and thin films of chemically deposited silver, which appear blue. These silver films are fairly transparent to a limited range of ultra-violet radiations, which lies just beyond the limit of the visible spectrum. It is probable that the absorption of metals is due to the presence of two types of electrons:

conducting ones, which are free to move indefinitely under the influence of a steady electric force, and non-conducting ones, which are similar to those which we have considered in the chapter on Dispersion. We shall examine the effects of these two types more in detail in the chapter on the Optical Properties of Metals.

**Absorption Spectra of the Rare Earths.**—While the absorption spectra of solids and liquids show in general only broad diffuse bands, some marked exceptions occur in the case of salts of the rare earths,

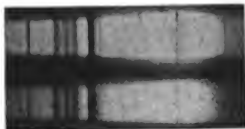


Fig. 256a.

erbium, didymium, europium, holmium, etc. These show both in solution and in the solid state very narrow bands or even sharp lines. Most remarkable is the absorption spectrum of a solution of europium, which is shown in Fig. 256a. The chloride of this element has finally been obtained in the pure state by Urbain, after 6000 fractional crystallizations, a labor of 10 years. Two of the

absorption lines in the green are as sharp and narrow as the Fraunhofer lines in the solar spectrum. Neodymium has an equally fine line in the ultra violet. The bands of erbium are broader. They appear not only in the absorption-spectra solutions, but even in the spectrum of the oxide when illuminated by sunlight. Dip a platinum wire into a concentrated solution of erbium chloride, and heat it in a Bunsen burner. Hold the oxide bead in the sunlight and examine it with a spectroscope. If the bead is heated white hot in the flame, *bright* bands appear in place of the dark ones.

**Theories of Absorption.**—In the chapter on Dispersion we have seen that the presence of electrons of definite periods of vibration, gives to a medium the property of absorbing radiations, the frequencies of which agree with those of the electrons.

The introduction of a term which expressed the vibration of the electron as accompanied by friction was sufficient to explain the absorption of energy. Such a treatment is rather loose, however. If the energy is transformed into heat by this assumed "friction," we must explain how the average molecular velocity (which is our definition of temperature) is raised. Clearly friction, as we ordinarily understand it, occurring within the molecule cannot affect the velocity of the molecule. Moreover, a vibration of the electron excited by the light waves, can be conceived as affecting the molecular velocity only, in virtue of some action occurring at the moment when two molecules are in collision. It is conceivable that, at the moment of impact, the energy stored in the molecule in the form of vibrational energy of the electron may be consumed in increasing the velocity of rebound of the molecules. Practically nothing is really known about the exact nature of the transformation of absorbed radiations. It can be shown, however, that a damping of the vibration of the electron may result from molecule impacts, the result being similar to that which would follow if its vibrations were accompanied by friction. The introduction of the friction term into the equations is thus not wholly unwarranted. The absorbed energy may be spent in effecting chemical changes

within the substance, as we shall see in the chapter on the Transformation of Absorbed Radiations. There is, however, another action which may well be expected to happen, namely a remission of energy by the vibrating electron in the form of ether waves of the same period as those absorbed.

Unfortunately experimental confirmation of such re-emission is very meagre. Fluorescence is a totally different phenomenon, for in this case the lengths of the emitted waves are different from those of the exciting ones. The vapor of sodium, however, appears to exhibit the phenomena, for when illuminated by a powerful beam of sodium light it scatters in all directions a feeble light of apparently the same wavelength. This case will be further considered under "Fluorescence."

If the electrons become centers of radiation, giving back their energy to the ether, it would appear at first sight as if no absorption would result, for by Huygens's principle the secondary waves originating from their vibration would re-constitute a wave of a type similar to the exciting wave. We are obliged to assume, however, that the electrons would send out energy in *all* directions; consequently this case would differ from that in which we determined the resultant of the secondary disturbances on a wave-front, in that we should have a back wave travelling in the reversed direction, as well as a forward wave. A theoretical treatment of absorption and dispersion has been given by Planck,<sup>1</sup> based solely on this assumed radiation of the electrons. The reduction in the intensity of the advancing wave can be explained perfectly by it, but there is no true absorption as ordinarily understood, the energy being sent back in the opposite direction. This lateral radiation is assumed by Planck to be the only cause of the damping of the vibration of the electrons, an assumption which is hardly justifiable if we require a complete explanation of all of the phenomena of absorption, but which is perfectly allowable if we wish merely to find out how far such a radiation can account for the observed effects.

One great advantage of this conception is, that it neither involves the introduction of any new constant into the equations, nor ascribes the damping to some action of which the physical significance is obscure.

There are, however, strong objections which can be brought up against the theory. In the first place, as has already been said, this lateral emission is only found in one or two rare instances. If the electrons lie close enough together, we might explain this by applying the principle of Huygens to their radiations, a lateral emission failing to take place for the same reason that a beam of light radiates no lateral disturbance. In this case there must be a return of a wave of the same type as the exciting wave, *i.e.* selective reflection. This occurs, as we have seen, in the case of solid films of strongly absorbing media, giving rise to surface color. If the molecules are too far apart for the application of the above principle, then the light should be scattered in all directions. Yet solutions of strongly absorbing media show absolutely no trace of such a lateral emission.

Planck's theory is instructive, however, in that it gives us a clear

<sup>1</sup> *Berlin Acad.*, Ber. 1903-1904.

idea of the effect of such secondary radiations arising from the electrons, upon the propagation of the exciting wave.

It might at first sight appear as if such a treatment would lead us to the conclusion that the resultant of all of the wavelets coming from the electrons would be identical with the original wave, which is the same thing as saying that there is no absorption. Planck has, however, taken into account the fact that the phase of a resonator lags a quarter of a period behind that of the exciting waves, and that there is in addition a quarter period difference between the phase of the resonator and that of the wave which it emits. The resultant wave emitted by the collection of resonators will thus be half a wave-length behind the exciting wave, which will gradually be reduced in intensity by interference with the wave originating in the resonator system. The resonators, however, emit spherical disturbances, consequently there will be an envelope propagated in the backward direction, and since there is no primary wave travelling in this direction there will be nothing to interfere with its propagation. The medium thus sends back towards the source of light a frequency corresponding to the frequency of the vibrating electrons. This is nothing more than selective reflection.

While Planck's treatment of the subject cannot very well be given in full, we can study to advantage some of the results to which it leads. His final equations, expressing  $n$  the refractive index, and  $\kappa$  the extinction coefficient, are

$$n^2 = \frac{\sqrt{(a^2 + \beta^2 - \alpha)^2 + \beta^2} + (a^2 + \beta^2 - \alpha)}{2(a^2 + \beta^2)},$$

$$\kappa^2 = \frac{\sqrt{(a^2 + \beta^2 - \alpha)^2 + \beta^2} - (a^2 + \beta^2 - \alpha)}{2(a^2 + \beta^2)},$$

in which

$$\alpha = \frac{\lambda_0^2 - (1 - g)\lambda^2}{3g\lambda^2} \quad \text{and} \quad \beta = \frac{\sigma\lambda_0}{3\pi g\lambda}.$$

$\lambda$  is the wave-length of the incident light,  $\lambda_0$  the wave-length in vacuum which the resonators would emit if thrown into vibration,  $\sigma$  the logarithmic decrement of the resonators, and  $g = \frac{\sigma N \lambda_0^3}{4\pi^3}$ ,  $N$  being the number of resonators in unit volume.

Since the last quantity depends on the distribution of the resonators in space, we are able to trace the relation existing between the refraction and absorption and the density of the medium. Planck's equation has shown that the form of the extinction curve depends on  $N$ , i.e. on the proximity of the resonators, and that its maximum lies on the longer wave-length side of  $\lambda_0$ . The equation calls for three different types of curves, according to the values assigned to  $\frac{g}{\sigma}$ . Consider first the case where  $\frac{g}{\sigma}$  has a large value, which will occur when  $N$  is large. The curve found in this case shows that the extinction

coefficient rises gradually with increasing  $\lambda$ , attains the value 1 for  $\lambda^2 = \frac{\lambda_0^2}{1 + \frac{g}{2}}$ , the value  $\sqrt{2}$  for  $\lambda^2 = \lambda_0^2$ , reaching its maximum just before  $\lambda^2 = \frac{\lambda_0^2}{1 - g}$ , beyond this point it descends much more rapidly than it rose, the curve being unsymmetrical (Fig. 257).

Planck defines the region of metallic absorption as the region within which  $\kappa > 1$ . It extends from  $\lambda^2 = \frac{\lambda_0^2}{1 + \frac{g}{2}}$

to  $\lambda^2 = \frac{\lambda_0^2}{1 - g}$ . Its width is seen to

depend only upon  $g$ , while the value of the maximum absorption depends upon  $\sigma$  as well. Considering  $\lambda_0$  and  $\sigma$  as constant, and increasing  $g$  by

bringing the resonators closer together, we obviously increase the width of the band of metallic absorption, the band widening unsymmetrically however. It spreads towards the region of shorter wave-lengths, but cannot pass the point determined by  $\lambda^2 = \frac{2}{2} \lambda_0^2$ , while

in the other direction there is no limit. At the same time the point of maximum extinction is shifted towards the region of longer wave-lengths, and the maximum value of  $\kappa$  is increased.

The interesting question now arises as to whether the shift of the maximum point may bring it about that the curve for a large value of  $N$ , instead of completely enclosing the curve for a small value of  $N$ , may cut the latter at two points, as shown in Fig. 257. If this were the case, we should have the value of the extinction coefficient decreased for a certain value of  $\lambda$  to the left of  $\lambda_0$ , by increasing the density of the medium, a circumstance which would be in violation of Beer's law of the proportionality between extinction and density. Planck finds this to be the case, for  $\kappa = \sqrt{2}$  for  $\lambda = \lambda_0$  regardless of the value of  $g$ . The family of curves obtained by assigning to  $N$  different values thus pass through a common point situated at  $\lambda_0$ . For any two curves there is a second point of intersection, and between these two points an increase of density is accompanied by a decrease in the value of the extinction coefficient.

If  $\frac{g}{\sigma}$  has a small value, as will be the case when  $N$  is small, the curve is found to be symmetrical, with its maximum at  $\lambda_0$ . Metallic reflection does not occur,  $\kappa$  being less than unity for all values of  $\lambda$ , and if  $g$  is gradually increased,  $\kappa$  increases proportionally for all wave-lengths, and Beer's law of absorption is followed. The absorption band is narrow and no shift is produced by increasing the density. A third type of curve, intermediate between the other two, is found for intermediate values of  $\frac{g}{\sigma}$ .

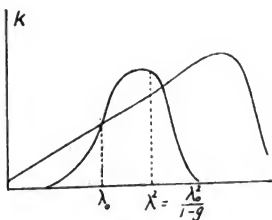


FIG. 257.

Planck's treatment differs from that of Drude and Lorentz in that the damping is referred solely to radiation. Drude's formula calls for a maximum value of  $\kappa$  for wave-length  $\lambda_0$ , and an increase in the width of the band of metallic reflection towards the region of shorter wave-lengths only.

Sufficient experimental observations are not yet available to make a choice between the different treatments possible. An experimental proof of Planck's theory would be difficult, since he assumes at the outset that the resonators are at rest and separated by distances which are large in comparison to their dimensions.

A mathematical discussion of absorption by Lamb (*Camb. Phil. Soc. Trans.*, vol. xviii., Stokes Commemoration, 1900) is extremely interesting as showing the enormous checking power which a single resonator (gas molecule) can exert upon advancing radiation of a frequency very nearly but not quite that of its own free period. Lamb draws attention also to the fact that it has not been possible to represent the dissipation of radiant energy by an absorbing medium except vaguely by means of a frictional coefficient. He considers, as does Planck, that the energy is scattered, *i.e.* remitted by the resonator, which he assumes to be a spherical molecule of enormous specific inductive capacity, with one or more free periods of vibration.

The main result of his investigation is stated as follows: "For every free period of vibration (with a wave-length sufficiently large in comparison with the diameter of a molecule) there is a corresponding period (almost exactly but not quite coincident with it) of maximum dissipation for incident waves. When the incident waves have precisely this latter period, the rate at which energy is carried outwards by the scattered waves is, in terms of the energy-flux in the primary waves,

$$\frac{2n+1}{2\pi} \lambda^2, \dots\dots\dots (1)$$

where  $\lambda$  is the wave-length and  $n$  is the order of the spherical harmonic component of the incident waves which is effective. In the particular case of  $n=1$ , this is equal to  $\cdot 477\lambda^2$ . Hence in the case of exact synchronism, each molecule of a gas would, if it acted independently, divert per unit time nearly half as much energy as in the primary waves crosses a square whose side is equal to the wave-length. Since under ordinary atmospheric conditions a cube whose side is equal to the wave-length of sodium light would contain something like  $5 \times 10^6$  molecules, it is evident that a gaseous medium of the constitution here postulated would be practically impenetrable to radiations of the particular wave length."

"It is found, moreover, on examination that the region of abnormal absorption in the spectrum is very narrowly defined, and that an exceedingly minute change in the wave length enormously reduces the scattering."

"It may be remarked that the law expressed by the formula (1) is of very general character, and is independent of the special nature of the conditions to be satisfied at the surface of the sphere. It presents itself in the elastic-solid theory; and again in the much simpler acoustical problem, where there is synchronism between plane-waves of sound and a vibrating sphere on which they impinge."

## CHAPTER XV.

### OPTICAL PROPERTIES OF METALS.

THE laws which govern the optical behavior of metals are very different from those which hold in the case of transparent substances and substances showing strong selective absorption. Metals exhibit in general a very high reflecting power, and in some cases also a powerful absorbing action. We must distinguish carefully between absorption and reflection. If we examine a thin film of silver deposited on glass, we find that it transmits little or no light, and we might therefore come to the conclusion that the metal absorbs strongly. A little further investigation will show, however, that over 90 % of the light has been reflected, the remaining 10 % having been absorbed. Platinum, however, has a much lower reflecting power, combined with equally great opacity; consequently we may regard platinum as possessing a stronger absorbing power than silver. Gold is, however, as compared to silver and platinum, extremely transparent, ordinary gold leaf transmitting no inconsiderable amount of green light.

While metallic absorption is not in general characterized by such marked selective action as is the case with the colored media which we have studied, the phenomenon of selective absorption is by no means absent, as the strong coloration of the light transmitted by gold leaf proves: silver too, while it appears to favor equally all wave-lengths in the visible spectrum, is fairly transparent to ultra-violet radiation comprised within the range 305-320: its reflecting power is correspondingly low for these same wave-lengths.

The colors which metals exhibit are due to a selective reflecting power, which is especially marked in the case of gold and copper. If two gilded glass plates are mounted parallel at a distance of a centimeter or so, with the reflecting surfaces opposed and a beam of light caused to travel back and forth between them, suffering a number of reflections, the surface color is still more marked, the filament of an incandescent appearing as red as if seen through ruby glass. This method of bringing out the surface color is analogous to the method of Rubens and Nichols for isolating long heat waves by multiple reflections from surfaces of quartz and rock salt. The same phenomenon can be seen in a less marked degree by looking into the interior of a gilded goblet, the bottom of which appears of a fairly deep-red color.

Glass plates can be easily gilded by exposing them to the discharge

from a gold cathode in a vacuum tube: plates gilded by the application of gold leaf, in the manner employed for lettering on windows, would doubtless answer as well, and can be readily obtained from a sign painter.

The phenomenon of elliptical polarization is exhibited in a high degree when light polarized in an azimuth of  $45^\circ$  is reflected obliquely from a metal surface.

In the case of transparent substances, we have seen that this result can occur if the surface is contaminated with a film, but in the case of metallic reflection the presence of a surface film is not necessary. The first attempt to establish equations for the reflection of light by metallic surfaces was made by MacCullagh, who assumed that the reflection was of a nature similar to that of total reflection in the case of transparent substances, and assigned therefore to the metals an imaginary refractive index. Cauchy also developed an equation practically identical with MacCullagh's, and more elaborate treatments were subsequently given by Beer, Eisenlohr, and Lundquist (*Pogg. Ann.*, xcii., p. 402; civ., p. 368; clii., p. 398).

Against the methods employed by these investigators, Ketteler raised the objection that the development of the equation for the reflected wave necessitated the existence of a longitudinal disturbance in the ether, against the existence of which there exists the strongest experimental evidence. Ketteler (*Wied. Ann.*, B. i. and iii.) developed an equation along different lines, and by adopting different boundary conditions avoided the necessity of a longitudinal wave. These earlier treatments, based upon the elastic solid theory, have been completely supplanted by the methods of the electro-magnetic theory, which are much more intelligible for the reason that the physical actions which are going on are definitely specified. According to our present views we regard metals as substances in which electrons exist, which are capable of continuous movement under the action of a steady electromotive force. Heretofore we have regarded the electrons as bound to positions of equilibrium by forces of restitution, experiencing only a slight change of position under the action of a steady electric force. Upon the removal of the force the electron returns to its original position.

We will now investigate the behavior of free electrons under the action of the rapidly alternating electrical forces of light waves.

**Electro-Magnetic Theory of Absorption.**—We have already discussed the propagation of waves in a medium which is a perfect insulator, in which the current is proportional to  $\frac{\partial X}{\partial t}$ . In such a medium

the current may consist of two parts, a displacement current in the ether represented by  $\frac{1}{4\pi} \frac{\partial X}{\partial t}$ , and a convection current due to the motions

of the electrons inside the atoms. It is clear that the current will cease as soon as the electric force ceases to vary; for example, if the electric force rises from 0 to  $X$  the electron will be displaced a certain amount, the motion constituting a convection current, but if the force then remain steady there will be no further motion of the electron, and the current will cease.

We will now investigate the propagation of waves in a medium which is not a perfect insulator. In such a medium a current will be set up under the influence of a steady electric force, which will be proportional to the force  $X$  instead of to the rate of change of  $X$ . We may think of this current as due to the motion of free electrons, which will drift along under the influence of the force, giving rise to a conduction current represented by  $\sigma X$ , in which  $\sigma$  represents the absolute conductivity measured electro-statically. The current in an imperfect insulator will then be made up of two parts, one proportional to  $\frac{\partial X}{\partial t}$  and the other proportional to  $X$ , the former vanishing under the influence of a steady field.

If we have a periodic electric force, as in light waves, both currents will be present, and we may have absorption or a transformation of energy from two distinct causes. If the vibration of the electrons which are not free is accompanied with something akin to friction there will be a heating similar to the heating of the dielectric of a condenser when it is rapidly charged and discharged. This type of absorption has been discussed in the chapter on Dispersion. There may in addition be an ohmic heating, similar to the heating of wires by steady currents. This we may think of as due perhaps to the impacts of the free electrons or changes in the potential energy as electrons are torn off of atoms under the influence of the electric force.

If we limit ourselves to plane-polarized plane-waves we may write for the current parallel to the  $x$  axis

$$(1) \quad j_x = \frac{\epsilon}{4\pi} \frac{\partial X}{\partial t} + \sigma X.$$

The current is thus seen to be made up of two parts: a displacement current in the ether, resulting only from an electrical force which changes with the time, and a conduction current proportional to  $X$  which does not depend for its existence upon fluctuations of  $X$ . It will be remembered that the modern theory of metallic conduction regards the electrical current as a streaming motion of negatively charged electrons, which are free to move in the metallic conductor under the action of a steady electro-motive force.

No further modifications than the one introduced into equation (1) are needed, and the fundamental Maxwell equations  $\frac{4\pi j_x}{c} = \frac{\partial \gamma}{\partial z} - \frac{\partial \beta}{\partial y}$ , etc., and  $\frac{1}{c} \frac{\partial \alpha}{\partial t} = \frac{\partial Y}{\partial z} - \frac{\partial Z}{\partial y}$ , etc., still hold, if we write the permeability  $\mu = 1$ .

This we are justified in doing, even in the case of the strongly magnetic metals such as iron, nickel and cobalt, for experiments indicate that the magnetic molecules are unable to follow the very rapid changes involved in the case of light waves.

The boundary conditions may be written as before,  $X_1 = X_2$ ,  $Y_1 = Y_2$ ,  $a_1 = a_2$ ,  $\beta_1 = \beta_2$ .

For the present we shall concern ourselves only with the absorption due to ohmic heating, *i.e.* resulting from the term  $\sigma X$ , which represents the conduction current.

If  $j_z = \frac{\epsilon}{4\pi} \frac{\partial X}{\partial t}$ , we have  $\frac{\partial^2 X}{\partial t^2} = \frac{c^2}{\epsilon} \frac{\partial^2 X}{\partial z^2}$  (page 289, eq. (12)).

If  $j_z = \sigma X$ ,  $\frac{\partial X}{\partial t} = \frac{1}{\sigma} \frac{\partial j_z}{\partial t}$ , and since  $\frac{\partial j_z}{\partial t} = \frac{c^2}{4\pi} \frac{\partial^2 X}{\partial z^2}$  (page 285, eq. (7))  
(and page 289, eq. (12), and  $\frac{4\pi j_z}{c} = \frac{\epsilon}{c} \frac{\partial X}{\partial t} = \frac{\partial \gamma}{\partial y} - \frac{\partial \beta}{\partial z}$ ),  
$$\frac{\partial X}{\partial t} = \frac{c^2}{4\pi\sigma} \frac{\partial^2 X}{\partial z^2}.$$

From (1) we then have

$$\frac{\epsilon}{c^2} \frac{\partial^2 X}{\partial t^2} + \frac{4\pi\sigma}{c^2} \frac{\partial X}{\partial t} = \frac{\partial^2 X}{\partial z^2}.$$

When applied to harmonic motion, this equation has for its solution

$$(2) \quad X = A e^{i \frac{2\pi}{T} (t - mz)}, \text{ in which } m \text{ is complex.}$$

Differentiating (2),  $\frac{\partial X}{\partial t} = i \frac{2\pi}{T} X$ , and substituting in (1),

$$j_z = \frac{\epsilon}{4\pi} \frac{\partial X}{\partial t} + \sigma \frac{\partial X}{\partial t} \frac{T}{i 2\pi} = \left( \frac{\epsilon}{4\pi} + \frac{\sigma T}{i 2\pi} \right) \frac{\partial X}{\partial t}.$$

Multiplying the numerator and denominator of the second term in the parenthesis by  $i$  gives us  $j_z = \left( \frac{\epsilon}{4\pi} - \frac{2i\sigma T}{4\pi} \right) \frac{\partial X}{\partial t}$ ,

$$j_z = \frac{\epsilon - 2i\sigma T}{4\pi} \frac{\partial X}{\partial t}.$$

For perfect insulators  $j_z = \frac{\epsilon}{4\pi} \frac{\partial X}{\partial t}$ , the only difference being that in the case of absorbing media the real constant  $\epsilon$  of the equation for insulators is replaced by the complex constant  $\epsilon' = \epsilon - 2i\sigma T$ .

Substituting this complex dielectric constant in equation 12, page 289, gives us

$$(3) \quad \frac{\epsilon'}{c^2} \frac{\partial^2 X}{\partial t^2} = \frac{\partial^2 X}{\partial z^2}.$$

$$\text{Differentiating (2), } \frac{\partial^2 X}{\partial t^2} = -\frac{4\pi^2}{T^2} X, \quad \frac{\partial^2 X}{\partial z^2} = -\frac{4\pi^2 m^2}{T^2} X,$$

$$\text{and substituting in (3), } \frac{\epsilon'}{c^2} \frac{4\pi^2}{T^2} X = -\frac{4\pi^2 m^2}{T^2} X.$$

$$(4) \quad \frac{\epsilon'}{c^2} = m^2,$$

and since  $\epsilon'$  is complex,  $m$  is also complex.

Now  $m$  has the dimension of a reciprocal velocity, and we may write  $m = \frac{1 - i\kappa}{V}$ , in which  $V$  is the velocity of propagation of the wave in the absorbing medium.

Substituting this value in (2),

$$X = Ae^{\frac{2\pi i}{T} \left( t - \frac{(1-i\kappa)z}{V} \right)} = Ae^{\frac{2\pi i}{T} \left( t - \frac{z-i\kappa z}{V} \right)} = Ae^{\frac{2\pi i t}{T} - \frac{2\pi i z}{TV} - \frac{2\pi \kappa z}{TV}},$$

in which  $TV = \lambda$ .

$$(5) \quad X = Ae^{-\frac{2\pi \kappa z}{\lambda}} e^{\frac{2\pi i}{T} \left( t - \frac{z}{V} \right)}.$$

In this expression  $Ae^{-\frac{2\pi \kappa z}{\lambda}}$  represents the amplitude, which clearly decreases as  $z$  increases.

After traversing a thickness equal to the wave-length  $\lambda$  the amplitude has decreased by the amount  $e^{-2\pi \kappa}$ . The constant  $\kappa$  is the measure of the absorption, and is called the absorption index.

If we call  $\frac{c}{V} = n$  the refractive index of the medium, we have from equation (4)

$$\frac{\epsilon'}{c^2} = \left( \frac{1-i\kappa}{V} \right)^2,$$

$$\epsilon' = \frac{c^2}{V^2} (1 - \kappa^2 - 2i\kappa);$$

$$(5) \quad \therefore \epsilon' = n^2 (1 - \kappa^2 - 2i\kappa);$$

and since

$$\epsilon' = \epsilon - i2\sigma T = n^2 - n^2\kappa^2 - 2n^2i\kappa$$

we get by equating the real and imaginary parts,

$$(5a) \quad n^2(1 - \kappa^2) = \epsilon, \quad n^2\kappa = \sigma T.$$

This last relation is not in agreement with facts, however, as we shall see presently. The reason of this is that, in the present treatment, we have not taken into account the influence which the vibrating electrons have upon the propagation of the disturbance. We have shown that in the case of imperfect insulators we have a complex dielectric constant, due to the conduction term  $\sigma X$ . In the treatment of dispersion we have seen that a complex dielectric constant results from the presence of vibrating electrons, even when the term  $\sigma X$  is absent, *i.e.* when no conducting electrons are present, and absorption will occur if the vibration of the electron is accompanied with friction. There are thus two distinct types of absorption, one caused by conducting electrons, the other by vibrating electrons, the motion of which is accompanied by friction.

**Metallic Reflection.**—Consider now the case in which plane-polarized light is incident at an angle of  $45^\circ$  on a polished metal surface, the plane of polarization making an angle of  $45^\circ$  with the plane of incidence. Referring to the treatment of reflection given on page 293, and making use of the same symbols, we have in the present case  $E_p = E_v$ , and

$$(6) \quad \sin \chi = \frac{\sin \Phi}{\sqrt{\epsilon'}}.$$

This is simply the previous equation with the complex dielectric constant  $\epsilon'$  substituted for  $\epsilon$ .

By equation (24), page 294,  $\frac{R_p}{R_i} = -\frac{E_p}{E_i} \frac{\cos(\phi + \chi)}{\cos(\Phi - \chi)}$ ,

we have, since  $E_p = E_i$ ,  $\frac{R_p}{R_i} = -\frac{\cos(\Phi + \chi)}{\cos(\Phi - \chi)}$ ,

in which  $R_p$  and  $R_i$  are complex quantities.

Let  $R_p = R_p e^{i\delta_p}$  and  $R_i = R_i e^{i\delta_i}$ , then

$$\frac{R_p}{R_i} = \rho e^{i\Delta}, \text{ in which } \delta_p - \delta_i = \Delta \text{ and } \rho = \frac{R_p}{R_i};$$

$$(7) \quad \therefore \frac{R_p}{R_i} = \rho e^{i\Delta} = -\frac{\cos(\Phi + \chi)}{\cos(\Phi - \chi)},$$

in which  $\delta_p - \delta_i = \Delta$  and  $\rho = \frac{R_p}{R_i}$ .

Since the right-hand member of the above equation is complex,  $\Delta$  must differ from zero, and there is a phase-difference between the two components of the reflected light, which produces elliptical polarization.

We will now determine how this phase-difference and the accompanying elliptical polarization vary with the angle of incidence.

$$\rho e^{i\Delta} = -\frac{\cos \Phi \cos \chi - \sin \phi \sin \chi}{\cos \Phi \cos \chi + \sin \phi \sin \chi};$$

multiplying this equation by the denominator and transposing the terms gives us

$$\frac{1 + \rho e^{i\Delta}}{1 - \rho e^{i\Delta}} = \frac{\sin \Phi \sin \chi}{\cos \Phi \cos \chi} = \tan \Phi \frac{\sin \chi}{\cos \chi}$$

$$= \tan \Phi \frac{\sin \Phi}{\sqrt{\epsilon' \cos \chi}} = \frac{\tan \Phi \sin \Phi}{\sqrt{\epsilon' \cos^2 \chi}} = \frac{\tan \phi \sin \Phi}{\sqrt{\epsilon' \left(1 - \frac{\sin^2 \Phi}{\epsilon'}\right)}};$$

$$(8) \quad \therefore \frac{1 + \rho e^{i\Delta}}{1 - \rho e^{i\Delta}} = \frac{\sin \Phi \tan \Phi}{\sqrt{\epsilon' - \sin^2 \Phi}}.$$

At normal incidence  $\Phi = 0$ ,  $\rho e^{i\Delta} = -1$ , i.e.  $\Delta = 0$  and  $\rho = -1$ , or the wave is reflected with a change of sign, but with no phase difference between the components; the light therefore remains plane polarized. The reflected waves form, by interference with the incident waves, a system of stationary waves, and since the reflection is accompanied by a change of sign we shall have a node at the reflecting surface.

At grazing incidence  $\phi = 90$ ,  $\rho e^{i\Delta} = 1$ , i.e.  $\Delta = 0$  and  $\rho = 1$ , or reflection occurs without change of sign and without elliptical polarization.

The ellipticity will be greatest for the angle of incidence for which  $\Delta = \frac{\pi}{2}$ . At this angle we have  $e^{i\Delta} = i$ , since  $e^{i\Delta} = \cos \Delta + i \sin \Delta = 0 + i$ .

This angle is termed the angle of principal incidence, and we will designate it by  $\Phi$ .

$$(9) \quad \frac{1 + i\rho}{1 - i\rho} = \frac{\sin \Phi \tan \Phi}{\sqrt{\epsilon' - \sin^2 \Phi}}.$$

Multiplying this equation by its complex conjugate

$$\frac{1 - i\bar{\rho}}{1 + i\bar{\rho}} = \frac{\sin \bar{\Phi} \tan \Phi}{\sqrt{\epsilon'' - \sin^2 \Phi}},$$

in which  $\epsilon''$  is the complex conjugate of  $\epsilon'$ .

The left-hand member thus becomes  $\frac{1 + \rho^2}{1 + \rho^2} = 1$ , and we have

$$\sin^4 \bar{\Phi} \tan^4 \Phi = (\epsilon' - \sin^2 \Phi)(\epsilon'' - \sin^2 \Phi),$$

or, substituting the value which we have found for  $\epsilon'$ ,

$$(10) \quad \sin^4 \bar{\Phi} \cdot \tan^4 \Phi = n^4(1 + \kappa^2)^2 - 2n^2(1 - \kappa^2)\sin^2 \bar{\Phi} + \sin^4 \Phi.$$

In the case of metals  $n^2(1 + \kappa^2)$  has a value much greater than unity (from 8 to 30), consequently it is sufficient if we take only the first term of the right-hand member ;

$$(11) \quad \therefore \sin \Phi \tan \Phi = n\sqrt{1 + \kappa^2}.$$

We can derive this expression from eq. (8) if we disregard  $\sin^2 \Phi$  in comparison to  $\epsilon'$ ,

$$\frac{1 + i\bar{\rho}}{1 - i\bar{\rho}} = \frac{\sin \bar{\Phi} \tan \Phi}{\sqrt{\epsilon'}}$$

in which

$$\sqrt{\epsilon'} = n(1 - i\kappa), \text{ see eq. (5).}$$

Multiplying this by its complex conjugate

$$1 = \frac{\sin^2 \bar{\Phi} \tan^2 \Phi}{(n - n i \kappa)(n + n i \kappa)} = \frac{\sin^2 \bar{\Phi} \tan^2 \Phi}{n^2 + n^2 \kappa^2},$$

$$\sin^2 \bar{\Phi} \tan^2 \Phi = n^2 + n^2 \kappa^2 = n^2(1 + \kappa^2);$$

$$\therefore \sin \bar{\Phi} \tan \Phi = n\sqrt{1 + \kappa^2}.$$

If by means of a Babinet compensator, which annuls the phase-difference introduced by the metallic reflection at any angle of incidence, we convert the elliptical vibration into a plane-polarized one, the plane of polarization will make an angle  $\Psi$  with the plane of incidence, and we have the relation  $\rho = \tan \Psi$ .

If we can establish equations connecting  $n$  and  $\kappa$  with  $\Phi$  and  $\Psi$ , it is clear that we can determine the refractive index  $n$  and the extinction coefficient of a metal by observations made on the state of polarization of the reflected light. Such methods are called katoptric methods in contrast to the dioptric methods employed in the case of transparent substances. They have been largely used in the determination of the optical properties of intensely opaque matter. They are obviously not as reliable as the dioptric methods, for, as we have seen, the ellipticity of the reflected light is profoundly affected by the presence of surface films, and we can seldom be certain that such films are not present in the case of metallic surfaces.

We will now establish relations between  $n$  and  $\kappa$ , and  $\Phi$ ,  $\Delta$  and  $\Psi$ .

We require first an expression for  $\frac{1 - \rho e^{i\Delta}}{1 + \rho e^{i\Delta}}$ , in terms of  $\Psi$  and  $\Delta$ , which expression we shall substitute in eq. (8).

Since  $\rho = \tan \Psi = \frac{\sin \Psi}{\cos \Psi}$  we may write the above expression,

$$\frac{1 - \tan \Psi (\cos \Delta + i \sin \Delta)}{1 + \tan \Psi (\cos \Delta + i \sin \Delta)} = 1 - \frac{\sin \Psi (\cos \Delta, \text{etc.})}{\cos \Psi (\cos \Delta, \text{etc.})} \\ = \frac{\cos \Psi - \sin \Psi (\cos \Delta, \text{etc.})}{\cos \Psi + \sin \Psi (\cos \Delta, \text{etc.})}.$$

Writing  $\cos \Psi = a$ ,  $\cos \Delta \sin \Psi = b$ ,  $i \sin \Delta \sin \Psi = c$ , the above expression is of the form

$$\frac{a - b - ic}{a + b + ic}.$$

Multiplying the numerator and denominator by  $a + b - ic$  gives us

$$\frac{a^2 - b^2 - c^2 - 2aib}{a^2 + b^2 + c^2 + 2ab},$$

in which we substitute the values for  $a$ ,  $b$  and  $c$ , and find (writing for  $\cos^2 \Psi - \sin^2 \Psi$  its equivalent  $\cos 2\Psi$ ), since  $\cos^2 \Delta = 1$ ,

$$\frac{1 - \rho e^{i\Delta}}{1 + \rho e^{i\Delta}} = \frac{\cos 2\Psi - i \sin 2\Psi \sin \Delta}{1 + \sin 2\Psi \cos \Delta} = \frac{n(1 - i\kappa)}{\sin \Phi \tan \Phi}, \text{ see eq. (5);}$$

$\therefore$  equating the real and imaginary parts,

$$\frac{n\kappa}{\sin \Phi \tan \Phi} = \frac{\sin \Delta \sin 2\Psi}{1 + \cos \Delta \sin 2\Psi}, \quad \frac{n}{\sin \Phi \tan \Phi} = \frac{\cos 2\Psi}{1 + \cos \Delta \sin 2\Psi},$$

and, dividing the first by the second,

$$\frac{n\kappa}{n} = \frac{\sin \Delta \sin 2\Psi}{\cos 2\Psi},$$

$$\kappa = \sin \Delta \tan 2\Psi, \quad n = \sin \Phi \tan \Phi \frac{\cos 2\Psi}{1 + \cos \Delta \sin 2\Psi},$$

$$n^2(1 + \kappa^2) = \sin^2 \Phi \tan^2 \Phi \frac{1 - \cos \Delta \sin 2\Psi}{1 + \cos \Delta \sin 2\Psi}.$$

If the light is incident at the angle of principal incidence  $\Phi$ , the corresponding angle  $\Psi$  is called the principal azimuth. In this case

$$\kappa = \tan 2\Psi \quad (\text{since } \sin \Delta = 1).$$

**Determination of Principal Incidence and Azimuth.**—Since reflection at the angle of principal incidence converts circularly-polarized into plane polarized light, we can easily determine the angle by reflecting a circular vibration, obtained by means of a quarter-wave plate or Fresnel rhomb, from the metallic surface, and determining the angle of incidence at which the reflected light can be quenched by a Nicol prism. The principal azimuth is the angle which the direction of the plane vibration (short diagonal of the Nicol) makes with the plane of incidence. Or we may start with plane-polarized light and employ a Babinet compensator (with its wedges set so as to displace the central fringe through a distance corresponding to a phase-difference of a quarter of a period) to analyze the reflected light, observing the angle

of incidence at which the central fringe returns to the central position. The principal azimuth is determined by observing the angle through which the second Nicol has to be rotated to make the central fringe black. Colored light, obtained by passing sunlight through colored glasses or solutions, should be employed, as the optical constants are a function of the wave-length.

**Determination of the Change of Phase by Reflection.**—The change of phase produced by perpendicular reflection at a silver surface is best determined by means of the Michelson interferometer, one of the back mirrors being coated over half its surface with a film of silver. The shift between the fringes on opposite sides of the dividing line, gives us the measure of the phase-difference between disturbances reflected from glass and silver, if we take into account the slight shift due to the shortening of one optical path by the material thickness of the film. The thickness can be determined by  $a$   $b$  silvering the plate in the manner shown in Fig. 258. The upper half  $a, b$  is first heavily silvered, the lower portion being covered with a glass plate. The left-hand portion  $a, c$  is then treated in a similar manner. The phase-change due to reflection will be the same at the surfaces  $a$  and  $b$ , the shift of the fringes at the boundary being  $c$  due to the difference in thickness, which is obviously the thickness of  $c$ . The shift at the boundary between  $c$  and  $d$  is next observed, attention being paid to the direction of shift in each case. From these data it is easy to calculate the phase-change produced by reflection, if we remember that reflection at a glass surface produces a change of half a period. (See Mann's *Manual of Advanced Optics*.)

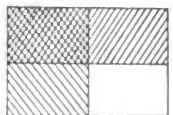


FIG. 258.

**Coefficient of Reflection.**—The ratio of the reflected to the incident intensity at normal incidence is called the coefficient of reflection. For the ratio of the amplitude we have in the case of transparent media (see page 295)

$$\frac{R_p}{E_p} = \frac{n-1}{n+1}.$$

In the present case we write for  $n$ , the square root of the complex dielectric constant  $\epsilon'$ , since  $\sqrt{\epsilon'} = n(1 - i\kappa)$ .

In the present case then we have

$$\frac{R_p}{E_p} = \frac{R_p e^{i\delta_p}}{E_p} = \frac{n(1 - i\kappa) - 1}{n(1 - i\kappa) + 1}.$$

Multiplying this equation by its complex conjugate, we obtain for the coefficient

$$R = \frac{n^2(1 + \kappa^2) + 1 - 2n}{n^2(1 + \kappa^2) + 1 + 2n}.$$

In the case of metals  $2n$  is small in comparison to  $n^2(1 + \kappa^2)$ . If we neglect it we get  $R=1$ . In the case of silver 95 % of the incident light is reflected, i.e.  $R=0.95$ . The reflecting power will be greatest for the wave-lengths for which  $\kappa$  has its greatest value, thus gold reflects red light much more powerfully than green. The best deter-

minations of the reflecting power of different metals in the visible and infra-red regions are those made by Rubens and Hagen (*Ann. der Physik*, i. 352, 1900). The values which they found for a number of the more common metals are given in the following table, which will be found useful for reference in all optical work involving the reflection of light from metallic surfaces :

REFLECTION OF METALS IN PERCENTAGES OF INCIDENT INTENSITY.

$\lambda =$	450	500	550	600	650	700	<i>R</i> calculated
<b>METAL.</b>							
Silver . . . .	90.6	91.8	92.5	93	93.6	94.6	95.3
Platinum . . .	55.8	58.4	61.1	64.2	66.3	70.1	70.1
Nickel . . . .	58.5	60.8	62.6	64.9	65.9	69.8	
Steel . . . . .	58.6	59.6	59.4	60	60.1	60.7	58.5
Gold . . . . .	36.8	47.3	74.7	85.6	88.2	92.3	85.1
Copper . . . .	48.8	53.3	59.5	83.5	89	90.7	73.2
Mercury . . . .	—	—	—	—	—	—	78.4
<b>SPRUCULUM METAL.</b>							
<i>Brashear.</i>							
68 % Cu + 31 Sn	61.9	63.3	64	64.4	65.4	68.5	
<i>Brondes Schuneman.</i>							
41 % Cu + 26 Ni + 24 Sn + 8 Fe + 1 Sb	49.1	49.3	48.3	47.5	49.7	54.9	
<i>Mach's Magnalium.</i>							
1 part Aluminum + 1.5 parts Magnesium	83.4	82.5	82.1	83.8	84.9	84.4	
Glass backed with Silver film	79.3	81	82	82	83	84	
	to 85.7	to 86	to 88	to 88	to 89	to 89	
Glass backed with Mercury	72.8	70.9	71.2	69.9	71.5	72.8	

It is apparent from the above table that the reflecting power of silver in contact with glass is somewhat less than that of silver in contact with air.

The same is true for mercury.

An easy way of exhibiting the loss of light by reflection from a metal is to half fill a test tube with mercury and plunge it in a jar of clean water ; the light reflected from the metal will appear quite dull in comparison with the light totally reflected at the glass-air surface.

In the above table there will be found in the last column the values of *R* calculated from observations by katoptric methods (yellow light). i.e. calculated from  $\Phi$ ,  $\Psi$ , and  $\Delta$ . They will be found in rough agreement with the values observed by Rubens.

In a subsequent paper (*Ann. der Physik*, 8, p. 1, 1902) by the same authors will be found a table similar to the above, but extending from the ultra-violet  $\lambda = 251$  to the infra-red  $1500 \mu\mu$ .

This table is too large to be given in full, but the values found for silver are of especial interest in showing the marked variations of the reflecting power.

## REFLECTING POWER OF SILVER.

$\lambda$	251	288	305	316	326	338	357	385	500	700	1000	1500
$R$	34	21.2	9.1	4.2	14.6	55.5	74	81	91	94	96.6	98.4

The minimum at  $\lambda = 316$  is very remarkable, the reflecting power of the metal for this wave-length being about that of glass for yellow light.

The highest reflecting power appears to be possessed by metallic sodium, for which  $R = 99.7$  according to Drude. This value was calculated however.

**Optical Constants of the Metals.**—These may be determined by various methods.  $n$  and  $\kappa$  may be determined directly by dioptric methods, accurate measurements involving great difficulties however, or they may be calculated from determinations of  $\Phi$  and  $\Psi$ , the angles of principal incidence and azimuth.

Kundt was the first to determine directly the refractive indices of the metals. He employed exceedingly acute prisms deposited on glass by means of the cathode discharge or chemical means, and actually measured the deviation of light produced by them. In some cases  $n$  was found to be less than unity; in other words, the light was propagated in the metal at a higher velocity than in vacuo.

Drude gives the following table for the constants of a number of the more common metals:

METAL	$nk$	$n$	$\Phi$	$\Psi$	$R$
Silver	3.67	.18	75°	43°	95 %
Gold	2.82	.37	72°	41°	85
Platinum	4.26	2.06	78°	32°	70
Copper	2.62	.64	71°	39°	73
Steel	3.40	2.41	77°	28°	58
Sodium	2.61	.005	71°	45°	99.7
Mercury	4.96	1.73	79°	35°	78.4

If we compare the values given in the above table with the equation  $\epsilon = n^2(1 - \kappa^2)$  we find that  $\epsilon$  is negative in every case, since  $\kappa = \tan 2\Psi$ , and  $2\Psi$  is greater than  $45^\circ$  in every case, *i.e.*  $\kappa > 1$ .

A negative dielectric constant has no meaning however. Neither does the relation  $n^2\kappa = \sigma T$  hold; for example, in the case of mercury  $\sigma T = 20$  and  $n^2\kappa = 8.6$ , while in the case of silver  $\sigma T$  is larger and  $n^2\kappa$  is smaller than in the case of mercury. As we shall see presently, these discrepancies are due to the fact that we have only taken the conducting electrons into account. The trouble is similar to the one which we experienced when we established the relation  $n = \sqrt{\epsilon}$  before taking the electrons into account at all.

**The Dispersion of Metals.**—We will now extend the treatment by considering that there are present in the metal two types of electrons, conducting and non-conducting, the motion of the latter being resisted by forces of restitution, as in the case of transparent substances.

For the conducting electrons we can put the constant  $\theta$  (eq. (1), page 333) equal to infinity, since  $\frac{1}{\theta}$  is proportional to the force of restitution.

The equation of motion of these electrons can be written

$$m \frac{\partial^2 \xi}{\partial t^2} = eX - re^2 \frac{\partial \xi}{\partial t}, \dots\dots\dots (1)$$

or if we call  $j_x = eN \frac{\partial \xi}{\partial t}$ , the current due to them,

$$\frac{m}{e^2 N} \frac{\partial j_x}{\partial t} + \frac{r}{N} j_x = X, \dots\dots\dots (2)$$

in which  $m$  is the mass of the electron,  $e$  the charge and  $n$  the number in unit volume

By eq. (2), page 364,  $\frac{\partial X}{\partial t} = \frac{i}{\tau} X$ , in which  $\tau = \frac{T}{2\pi}$ ;

$$\therefore X = -i\tau \frac{\partial X}{\partial t}, \text{ since } \frac{1}{i} = -i.$$

For periodic changes we can write  $j_x = -\tau i \frac{\partial j_x}{\partial t}$ , and if we substitute these values in eq. (2), we obtain

$$j_x \left\{ \frac{i}{\tau} \frac{m}{e^2 N} + \frac{r}{N} \right\} = -i\tau \frac{\partial X}{\partial t},$$

which equation can be easily brought into the form of eq. (4), page 334, by transposing the terms

$$j_x = \frac{1}{4\pi} \frac{\partial X}{\partial t} \left\{ -\frac{\tau e^2 N i \tau 4\pi}{im + e^2 \tau^2} \right\},$$

$$i_x = \frac{1}{4\pi} \frac{\partial X}{\partial t} \left\{ \frac{4\pi \tau N}{ir - \frac{m}{\tau e^2}} \right\},$$

the complex quantity in the brackets corresponding to the dielectric constant  $\epsilon$  of the earlier formula.

As we have seen in the chapter on Dispersion the dielectric constant of a medium containing non-conducting electrons is represented by

$$\epsilon' = 1 + \sum_h \frac{\theta_h'}{1 + i \frac{a_h}{\tau} - \frac{\theta_h}{\tau^2}}$$

The dielectric constant  $\epsilon''$  resulting from the presence of both types of electrons is therefore

$$\epsilon'' = 1 + \sum_k \frac{\theta_k'}{1 + i \frac{a_k}{\tau} - \frac{b_k}{\tau^2}} + 4\pi\tau \sum_k \frac{n}{ir + \frac{m}{r}},$$

in which

$$m' = \frac{m}{e^2}.$$

If we are dealing with vibrations, the periods of which are far removed from the free periods of the non-conducting electrons, so that  $a_k = 0$ , and if we write  $\epsilon'' = n^2(1 - \kappa^2 - 2i\kappa)$  we obtain by equating the real and imaginary parts of the above equation,

$$n^2(1 - \kappa^2) = 1 + \sum_k \frac{\theta_k'}{1 - \left(\frac{\tau_k}{\tau}\right)^2} - 4\pi \sum \frac{m'N}{r^2 + \left(\frac{m'}{\tau}\right)^2}$$

and

$$n^2\kappa = 2\pi\tau \sum \frac{r_k N}{r^2 + \left(\frac{m'}{\tau}\right)^2}.$$

Consider now the first of these two equations. In the case of transparent substances, where we neglect  $a_k$ ,  $\kappa = 0$ , and we have

$$n^2 = 1 + \sum \frac{\theta_k'}{1 - \left(\frac{\tau_k}{\tau}\right)^2}$$

for wave-lengths far removed from the natural period of the electrons.

In the present case we have a third term to take into account, so that even in the case of these wave-lengths we may have  $\kappa > 1$  (which means very heavy absorption), since the right-hand member can become negative as a result of the third term. This term will have large negative values for small values of  $r$ . Now  $r$  represents something which opposes the motion of the conducting electrons, which, in the chapter on Dispersion, we called for convenience "friction." Small values of  $r$  consequently represent high conductivity, and the most opaque metals will be those of the highest electrical conductivity. This point will be more clearly brought out in the section following.

The discrepancy between theory and experiment as represented by the equation deduced at the beginning of the chapter,  $n^2\kappa = \sigma T$ , is cleared up by the second of the two equations, if we write it in the form which it will take when we have infinitely long waves ( $\tau = \infty$ ). We must do this if we are to fit our optical equations to ordinary electrical measurements, made with very slow periods, or without any periodicity whatever.

For the conductivity we can write  $\sigma = \frac{N_1}{r_1} + \frac{N_2}{r_2}$ , since it is proportional to the number of conducting electrons and inversely as the frictional force which opposes their motion. Substituting these values we get at once

$$n^2\kappa = 2\pi\tau \sum_r \frac{N}{r} = \sigma T.$$

As has already been pointed out  $n^2\kappa = 8.6$  in the case of mercury, while  $\sigma T = 20$ .

If  $r$  is small, as in this case, and if we are dealing with small values of  $\tau$ , as when measuring  $n^2\kappa$  by optical methods, we cannot neglect  $\frac{m'}{\tau}$  in comparison to  $r$ : our second member will then be smaller than in the previous case, since the denominator is larger, and we shall have  $n^2\kappa < \sigma T$ . It is worthy of remark that the conductivity of electrolytes is not sufficient to cause appreciable absorption of light, the conductivity of the best electrolytes being only about  $\frac{1}{100000}$  that of mercury. For these  $\sigma$  will be of the order of magnitude  $7 \times 10^{11}$ , while for light-waves  $T = 2 \times 10^{-15}$ , therefore  $\sigma T = 14 \times 10^{-4} = .0014$ . By our formula  $n^2\kappa$  is never greater than  $\sigma T$ , generally much smaller, which shows us that  $\kappa$  must be very small in the present case.

**Relations between Optical and Electrical Properties of Metals.**—Very intimate relations have been established between the optical and electrical properties of metals by the recent work of Rubens and Hagen (*Ann. der Physik*, 11, 873, 1903). Previous to this important investigation many discrepancies existed between theory and experiment. The optical properties of metals could not be represented by Maxwell's theory in its original form. One of the most difficult things to account for was the comparatively great transparency of some metals for light waves. Moreover, the reflecting power appeared to stand in no definite relationship with the electrical conductivity. Kundt moreover found that the refractive indices of his metal prisms for red light arranged themselves in the order of the specific resistances of the metals, whereas on Maxwell's theory the reverse should hold true.

The work of Rubens and Hagen has shown conclusively that these discrepancies have resulted from the employment of too short waves. As soon as the optical work was carried on in the remote infra-red region of the spectrum between wave-lengths  $4\mu$  and  $12\mu$ , most perfect agreement with the theory was found.

They found, for example, that platinum and bismuth, which have a low conductivity, are much more opaque in the visible spectrum than gold and silver, which are much better conductors. This is contrary to what would be expected on Maxwell's theory. In the infra-red, however, it was found that they were much more transparent. This effect is not difficult to show. A thin quartz plate is lightly silvered chemically, the action being stopped when the film appears blue by transmitted light. On a similar plate a perfectly opaque film of bismuth is deposited by the cathode discharge. If the rays from a Welsbach light, without its chimney, are allowed to fall upon a sensitive thermopile, it will be found that the silver plate practically cuts off all of the long heat-waves, while the bismuth is fairly transparent for them.

Maxwell's theory gives for the reflecting power of a metal of conductivity  $\sigma$ , for electro-magnetic waves of period  $T$ , the formula  $R = 100 - \frac{200}{\sqrt{\sigma T}}$  (see Drude's *Physik des Aethers*, page 574). Introducing in place of the conductivity  $\sigma$  (measured in electrostatic units) the conductivity  $\kappa$ , i.e. the reciprocal of the resistance, measured in

ohms, which a conductor of 1 mm. cross section and 1 meter in length would have, and in place of  $T$  the wave-length  $\lambda$ , the formula takes the form

$$R = 100 - \frac{36.5}{\sqrt{x\lambda}}.$$

The intensity of the radiation which penetrates the surface is given by

$$100 - R = \frac{36.5}{\sqrt{x\lambda}},$$

$$\text{or } (100 - R)\sqrt{x} = \frac{36.5}{\lambda} = C_{\lambda}.$$

This equation shows that the product of  $(100 - R)\sqrt{x}$  is a constant for a given wave-length and is independent of the nature of the metal.

The reflecting powers were measured by means of the apparatus shown in Fig. 259. A Nernst lamp  $B$  was mounted on a turn table in such a position that its image  $A$ , formed by the concave mirror  $D$ , was symmetrically located with respect to the center of the turn table. By turning the table the filament of the lamp could be brought into the position previously occupied by its image. The surface of the mirror  $D$  was composed of the metal under

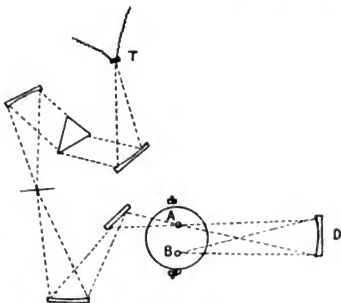


FIG. 259.

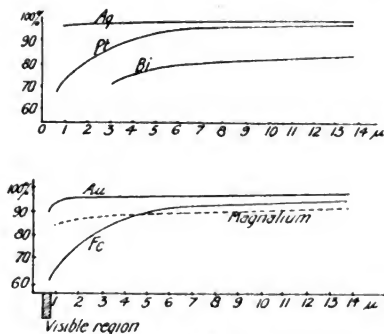


FIG. 260.

investigation. The rays from the filament or its image were focused upon the slit of a reflecting spectrometer, furnished with a fluorite prism, and the spectrum thrown upon the thermopile at  $T$ . It is clear that by this arrangement we have a means of comparing the incident with the reflected energy, since the loss by reflections from the surfaces

of the optical parts of the instrument is the same in each case. They investigated a large number of metals and alloys of known conductivity, obtaining curves of the type shown in Fig. 260.

It is worthy of note that magnalium, the most brilliantly reflecting alloy which we have, is surpassed by iron, in reflecting power, for wave-lengths greater than  $4\mu$ . These curves indicate that for infinitely long waves the metals would reflect 100 % of the incident intensity, as is the case with electro-magnetic waves of slow period.

According to theory the product  $(100 - R)\sqrt{x}$  should be a constant for a given wave-length.

This relation was verified in a remarkable manner as is shown by the following table, which is for  $\lambda = 12\mu$ :

	$\sqrt{x}$	$100 - R$	$(100 - R)\sqrt{x}$
Silver	7.85	1.15	9.0
Copper	7.56	1.6	12.1
Gold	6.43	2.15	13.8
Platinum	3.04	3.5	10.6
Nickel	2.92	4.1	12.0
Steel	2.24	4.9	11.0
Bismuth	.916	17.8	16.3

The mean value of the product  $(100 - R)\sqrt{x}$  was found to be 19.4 for  $\lambda = 4\mu$ , 13 for  $\lambda = 8\mu$  and 11 for  $\lambda = 12\mu$ . The theoretical values given by  $c = \frac{36.5}{\sqrt{\lambda}}$  for these same wave-lengths are 18.25, 12.9 and 10.54 respectively.

These results show that, in the case of metals, the resonance of electrons (which did not enter into Maxwell's theory) plays no part in the case of waves of length greater than  $4\mu$ , and that the discrepancies which occur when visible light is employed are in all probability due to the fact that, for these higher frequencies, the vibrations of the electrons are excited.

In the hope of finding still better agreement Rubens and Hagen then employed the "Rest-strahlen" from fluorite,  $\lambda = 25.5\mu$ , but in this case the reflecting power was so nearly 100 % for all metals, that it was not expedient to determine the coefficient of penetration  $(100 - R)$  from observations of the reflecting power.

On account of the relation between emission and absorption, expressed by Kirchhoff's law, we can determine  $(100 - R)$  by comparing the emission from the substance of waves of given length, with the corresponding emission of a perfect black body. Twenty different metals were investigated by this method. The radiations from the black body (see chapter on Laws of Radiation) and from the metal, both heated to the same temperature, were allowed to fall alternately upon a thermopile, after reflection from four fluorite surfaces, which eliminated all wave-lengths except  $25.5\mu$ . The temperature employed was  $170^\circ$ , maintained by an electrically heated bath of aniline, and the conductivity of the metal used in the calculations was of course the

conductivity for this temperature. The results appear in the following table :

TEMPERATURE 170°.

	$\sqrt{x}$	Emission $J = 100 - R$	$(100 - R)\sqrt{x}$
Silver	6.26	1.13	7.07
Copper	5.70	1.17	6.67
Gold	5.21	1.56	8.10
Aluminium	4.52	1.97	8.91
Zinc	3.19	2.27	7.24
Cadmium	2.86	2.55	7.29
Platinum	2.44	2.82	6.88
Nickel	2.29	3.20	7.33
Tin	2.24	3.27	7.32
Palladium	2.11	3.58	7.53
Steel	1.81	3.66	6.62
Mercury	.957	7.66	7.33
Bismuth	.716	25.6	18.3

It is at once clear that a further increase in the length of the wave has improved matters, as is shown by the more nearly constant value of the products in the last column. The mean value is 7.34, while that calculated is  $c_{25\mu} = \frac{36.5}{\sqrt{25.5}} = 7.23$ .

These results show that it is possible to determine the specific electrical conductivity of the metals by purely optical means, if we limit our observations to waves of sufficient length. As Lord Kelvin has remarked, it is very remarkable that the same laws hold for the electro-magnetic waves, which leave a magnet vibrating once a second and the waves of heat with a frequency of five billion per second.

## CHAPTER XVI.

### ROTATORY POLARIZATION.

WE have seen that, in general, when a ray of plane-polarized light is passed through a crystal in the direction of its optic axis, there is no double refraction, and the light emerges with its plane of polarization unchanged. The discovery was made by Arago in 1811 that a rotation of the plane occurs when light is transmitted through quartz in a direction parallel to the optic axis. If two Nicol prisms are placed in front of a sodium flame and so oriented as to completely extinguish the light, the introduction between the prisms of a quartz plate, cut perpendicular to the axis, causes the field to become bright again. On turning the analyzing Nicol through a certain angle the light can be completely extinguished, showing that it is still plane-polarized, but that the plane of polarization has been rotated through an angle, which is measured by the angle through which the Nicol has been turned.

If white light is used the field appears colored, the colors changing as the analyzing Nicol is rotated, the light never disappearing entirely, as was the case with the monochromatic sodium flame. This is due to the fact that the different colors are rotated by different amounts, the phenomenon being termed rotatory dispersion.

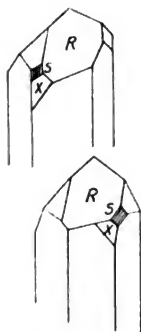


FIG. 261.

The amount of rotation was found to be proportional to the thickness of the crystal section, which shows that the action occurs within the medium and not at the surfaces. Moreover, some crystals were found to rotate the plane to the right, while others turned it to the left. The former are termed right-handed or dextrogyrate, the latter left-handed or laevogyrate. Simple inspection of the crystals is sufficient to determine their character. One can usually find small hemihedral planes which cut off the obtuse angles at the base of the hexagonal pyramid. The character of the crystal is determined by the relative positions of the surfaces *s* and *x*. If *s* and *x* lie to the right of *R*, the surface of the hexagonal pyramid which both touch, the crystal is dextrogyrate, if to the left, laevogyrate, as in the upper figure. The surface *x* is often absent, in which case we

can determine the nature of the crystal by the striae on the surface of  $s$ , which always run towards the position which  $x$  would occupy if present, as shown in the figures (261).

The direction of rotation of the plane is not affected by turning the plate around, consequently if the ray is reflected back through the crystal, the plane of polarization is turned back into its original position. This point is very important in connection with the magnetic rotation, which we shall discuss in a subsequent article, for in the case of substances which acquire rotatory power by being placed in a strong magnetic field, the direction of rotation depends on the direction of the magnetic field, and reflection back through the plate doubles the rotation.

It was subsequently found that many liquids and solutions possessed like quartz the power of rotating the plane of polarization, though in a much less degree. The rotation of the red rays by a quartz plate 1 mm. thick amounts to about  $18^\circ$ , while that due to a layer of turpentine of equal thickness is about one quarter of a degree.

**Rotatory Dispersion.**—The phenomenon of rotatory dispersion was investigated by Biot, who found that the rotation was nearly, though not exactly proportional to the inverse square of the wave-length. Still more accurate measurements were made by Brock, who obtained the following values for a plate 1 mm. thick, for wave-lengths corresponding to those of the principal Fraunhofer lines :

	<i>B</i>	<i>C</i>	<i>D</i>	<i>E</i>	<i>F</i>	<i>G</i>
$\rho$ ,	$17^\circ 30'$	$17^\circ 24'$	$21^\circ 67'$	$27^\circ 46'$	$32^\circ 50'$	$42^\circ 20'$
$\rho\lambda^2$ ,	7238	7249	7511	7596	7622	7841

The values of  $\rho\lambda^2$  are seen to increase with decreasing wave-length, which shows that Biot's law is only approximately followed. Brock's experiment, which is well worth repeating, consisted in passing sunlight through two Nicols, between which a quartz plate was mounted, and then analyzing the light with a spectroscope. On turning the analyzing Nicol, a dark band entered the spectrum from the red end, and passed slowly down towards the violet as the Nicol was rotated. By setting the center of the band on a Fraunhofer line and reading the position of the Nicol, the rotation for the wave-length of the line was determined.

Stefan calculated from Brock's measurements the following empirical formula  $\phi = -1.581 + \frac{8.0403}{\lambda^2 10}$ , of which the first member represents the departure from Biot's law. If the law held rigorously it would be equal to zero. Cauchy's formula for ordinary dispersion is

$$n = A + \frac{B}{\lambda^2},$$

in which, however,  $A$  is always positive.

The rotatory power of quartz must be kept in mind in all experiments with polarized ultra-violet light. If quartz plates or lenses are used, the planes of polarization of the different lines in the spectrum will be in every conceivable direction owing to the enormous rotatory dispersion. The difficulty can be overcome by employing equal

optical paths of right and left-handed quartz. Lenses can be made which do not rotate the plane, by combining two plano-convex lenses, one of right the other of left-handed quartz. Optical apparatus is frequently designed without due reference being paid to this phenomenon. The following table will be found extremely useful in all polarization work involving the use of quartz plates or lenses :

ROTATION OF THE PLANE OF POLARIZATION BY A QUARTZ PLATE  
1 mm. THICK. For Fraunhofer and Cadmium lines.

$\lambda$	$\delta$	$\lambda$	$\delta$	$\lambda$	$\delta$
A 7600	12° 668	h	47° 481	P	74° 571
a	14° 304	H	51° 193	Q	78° 579
B 6870	15° 746	K	52° 155	Cd 12. 3252	80° 459
C 6563	17° 318	L	55° 625	R	84° 972
D <sub>1</sub> 5890	21° 684	M	58° 894	Cd 17. 2748	121° 052
D <sub>2</sub> 5896	21° 727	Cd 9. 3612	63° 628	Cd 18. 2572	143° 266
E 5270	27° 543	N	64° 459	Cd 23. 2314	190° 426
F 4861.5	32° 773	Cd 10. 3466	69° 454	Cd 24. 2266	201° 824
G 4340.6	42° 604	O	70° 587	Cd 25. 2195	220° 731
		Cd 11. 3403	72° 448	Cd 26. 2147	235° 972

If we employ a plate a centimeter or more in thickness the spectrum will be found to be crossed with a number of dark bands which are very nearly equidistant, and which move down the spectrum as the Nicol is turned. The angular rotation for wave-lengths corresponding to two adjoining bands differs of course by 180°.

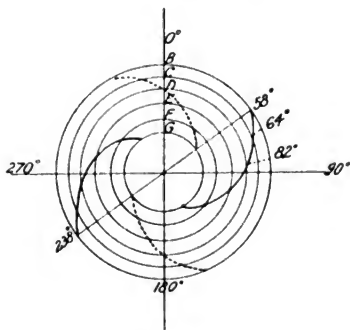


FIG. 262.

We can represent by a geometrical construction the position in the spectrum of the maxima and minima for any position of the analyzing Nicol. Describe around a point (Fig. 262), which we may consider as the source of light seen through the Nicols, a series of concentric circles with radii proportional to the wave-lengths of the principal Fraunhofer lines, the system representing a circular spectrum, such as would be seen by viewing the source of light through

a diffraction grating revolving rapidly in its own plane. We will now represent the case of a quartz plate 3.75 mm. thick, for which the rotations for the specified wave-lengths are as follows, if  $\Phi$  is the rotation for a 1 mm. plate :

B	C	D	E	F	G	H
3.75 $\Phi$ , 58° 1'	64° 5'	81° 7'	104° 1'	123° 9'	159° 6'	191° 8'

Let the diameter  $0^\circ - 180^\circ$  represent the direction of the vibration of the light after it has passed the first Nicol. The quartz plate will rotate the red ( $B$ ) through an angle of  $58^\circ 1'$ ; consequently we mark two points on the  $B$  circle at  $58^\circ$  and  $238^\circ$ , this diameter representing the position of the vibration plane of the analyzing Nicol when the red is most copiously transmitted. Doing the same for the other colors and connecting each set of points by a line, we obtain two spirals, which represent graphically the rotations of the different colors. The loci of planes perpendicular to these planes are given by joining points  $90^\circ$  away from the first point. We obtain in this way two other spirals (dotted in diagram) rotated through an angle of  $90^\circ$  with respect to the first. For thinner plates the spirals will have less curvature and be shorter, while for very thick plates they will have one or more complete convolutions.

These maximum and minimum spirals can be shown experimentally by means of a very ingenious device due to Mach. A brilliant point source of light, a Nicol and the quartz plate, are set up in line, followed by a Nicol and a nearly direct vision prism or transmission grating, mounted in a tube which can be set in rapid rotation by means of a pulley (Fig. 263). The revolving prism spreads out the source of light into the circular spectrum, which we represented in our geometrical construction, from which the colors disappear in different regions as the Nicol rotates with the prism. The circular spectrum is seen traversed by a pair of intensely black spirals, the direction of rotation of which depends on whether we use a plate of dextro or laevo-rotatory quartz. This extremely beautiful experiment can be projected, though the colors are naturally much more vivid when viewed subjectively.

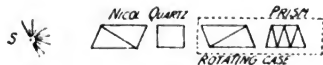


FIG. 263.

**Fresnel's Explanation of the Rotation**—A theory was formulated by Fresnel to account for rotatory polarization, which was based upon the fact that a ray of circularly-polarized light is propagated without change through a quartz crystal in a direction parallel to the optic axis. A linear vibration can be regarded as the component of two oppositely-polarized circular vibrations, and Fresnel made the assumption that the plane-polarized light, upon entering the crystal, was decomposed into two oppositely-polarized circular vibrations, which were propagated with unequal velocities. This inequality in the velocity of propagation will produce a rotation of the resultant, which will amount to  $90^\circ$  after a thickness has been traversed such that one circular disturbance is half a wave-length ahead of the other. Let two points moving in opposite directions around a circle represent the two circular vibrations. The resultant linear vibration will be represented by the line joining the two points at which the moving points pass each other. Let  $x$  and  $y$



FIG. 264.

be the disturbances which in the upper diagram have the resultant  $AB$  (Fig. 264). In the lower diagram the disturbance  $y$  is half a period ahead of  $x$ , and the resultant is  $A'B'$ .

The existence of these two circular components was shown experimentally by Fresnel, who reasoned that if there existed in reality two circularly-polarized disturbances which travelled with different velocities, they should be refracted by different amounts on emerging into the air through an oblique surface; in other words, quartz ought to show feeble double refraction in a direction parallel to the optic axis, and the two images produced thereby should be circularly polarized. Fresnel first tried a single surface, but the effect was too slight to be noticed. By the ingenious device of building up a compound prism composed of alternate prisms of right and left-handed



FIG. 265.

quartz the effect was found (Fig. 265). This method permitted the use of very oblique refracting surfaces, since only the small differences in the velocities came into play, while in the case of a single prism we have the refractive index with respect to air to consider, and if the prism angle be too large, light cannot be made to traverse it, owing to total reflection. By Fresnel's device the separation is increased at each surface, as will be seen from the following consideration. The difference between right and left-handed quartz lies in the fact that the right-handed circular component is the faster in the former, the slower in the latter, that is, the  $R$  prisms in Fig. 265 act as the rarer, the  $L$  prisms as the denser media with respect to this component. The reverse holds true for the left-handed component, the  $R$  prisms being the denser in this case. The former component is therefore bent down and the latter up, the angular separation increasing at each surface. If the two images of the source seen through the compound prism are examined with a Nicol, they are seen to remain unaltered when the Nicol is rotated. This means that the compound crystal prism has either depolarized the light completely or transformed it into circularly-polarized light.

The introduction of a quarter-wave plate of mica causes the images to disappear in succession as the Nicol is rotated, which proves the two images to be circularly polarized, and in opposite directions, the mica plate transforming them into plane-polarized images, the planes of polarization being at right angles.

Cornu has recently shown that with a single  $60^\circ$  prism of quartz the separation of the rays can be shown, the angle amounting to  $27''$  for sodium light. This means that even a quartz prism so cut that the rays travel along the optic axis yields double images. To remedy this defect Cornu devised a prism consisting of two right-angle prisms, the one of right, the other of left-handed quartz (Fig. 266). It is apparent that the optical path parallel to the base of the prism is the same for the two circular components; consequently the double refraction, which would otherwise cause a doubling of the spectrum lines, is eliminated.



FIG. 266.

Prisms of this type are used in practically all of the quartz spectrographs constructed at the present time.

**Babinet's Experiment.**—The difference between the velocities of light and left-handed circularly-polarized rays in quartz was shown in a different way by Babinet, who made use of the fringe system produced by a pair of Fresnel mirrors. The incident light, which was monochromatic and plane polarized, passed, before falling on the mirror, through a quartz plate which was covered with two  $\frac{\lambda}{4}$  plates of mica, one of which produced right, the other left-handed circular polarization. The two oppositely-polarized circular disturbances passed through the same quartz plate and fell, the one upon the mirror, the other upon its neighbor. Since, however, a path-difference between two interfering rays which are circularly polarized in opposite directions does not affect the intensity of the illumination, but only the direction of the resultant plane-polarized vibration into which they unite, no maxima and minima fringes are produced; there exists, however, a fringe system differentiated not by intensity but by the position of the plane of polarization. If the path-difference is zero the plane coincides with that of the light originally; if there is a path-difference of  $\frac{\lambda}{2}$  the plane is rotated through  $90^\circ$ . The system when viewed through a Nicol prism becomes visible, owing to the extinguishing of the light in those regions where the plane of vibration is crossed with the plane of vibration of the analyzer. If, now, the beams of right-handed and left-handed light traverse the crystal with equal velocities, the fringes should occupy the same position as in the original Fresnel experiment. If, however, one travels faster than the other there will be a path-difference, and a corresponding shift of the fringes. This was found to be the case, for when the plane of the incident light was rotated through  $90^\circ$ , the two quarter-wave plates exchanged properties, and the fringes shifted in position. In this experiment we have the equivalent of two circularly-vibrating sources, of the same period, but with opposite directions of rotation. Changing the plane of the incident light by  $90^\circ$  is equivalent to reversing the revolutions of the sources.

**Unequal Absorption of the Circular Components.**—The existence of the two circular components was shown in another way by Dove,<sup>1</sup> who found that in colored crystals of quartz (amethyst), the two disturbances were absorbed in different amounts. A similar phenomenon has been more recently observed by Cotton, who found that the strongly colored solutions of certain tartarates absorb right and left-handed circular vibrations in unequal amounts. These cases will be referred to again when we consider the theory of rotatory polarization.

**Other Rotatory Crystals.**—Descloizeaux found that crystals of cinnabar have a rotatory power similar to quartz, but fifteen times stronger, some crystals being right handed, others left. Sulphate of strychnia and sulphate of aethylendiamin have the same property.

<sup>1</sup> Pogg. Ann., cx.

The curious discovery was made by Marbach that crystals of sodium chlorate show the phenomenon of rotatory polarization in all directions, sections 1 mm. thick turning the plane (yellow light) 3.7 degrees, no matter how they are cut from the crystal.

Sulphate of strychnia has the power of rotating both in the crystalline state and in solution. As we shall see presently, many solutions exhibit the phenomenon of rotatory polarization, but in general the substances crystallize with two optic axes, the double refraction which occurs in all directions masking any rotatory power which may be present. If, however, the substance can be obtained as an amorphous solid the rotatory power is preserved. Quartz in an amorphous condition (fused, for example) does not have the rotatory power, neither have solutions of quartz in potash. This makes it quite certain that in the case of quartz and other active crystals, the optical activity depends on the arrangement of the molecules in the crystal, while in the case of substances which are active in the dissolved state the property depends on the arrangement of the atoms making up the molecule.

**Rotatory Polarization in Convergent Light.**—The behavior of quartz plates cut perpendicular to the optic axis was investigated both theoretically and experimentally by Airy in 1831.<sup>1</sup> Inasmuch as only circularly-polarized rays are propagated without change parallel to the axis, and plane-polarized rays perpendicular to the axis, Airy made the hypothesis that in any other direction the only form of vibration capable of being propagated without change was an elliptical one, assuming that a plane-polarized ray incident in a direction inclined

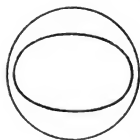


FIG. 267.

to the axis was decomposed into two oppositely-polarized elliptical vibrations which travelled through the crystal with different velocities. Airy further assumed that the ellipses were similar, and that the path-difference between the two elliptically-polarized rays was the same as in ordinary non-rotatory uniaxial crystals, *increased* by an amount which was independent of direction, and inversely proportional to the square of the wave-length. This means that in active crystals the two wave-surfaces are completely separated, the very small distance between them and the points where they intersect the optic axis being the increment referred to above. The wave-surfaces are shown in Fig. 267.

**Rotatory Polarization of Liquids.**—The rotation of the plane of polarization by liquids was accidentally discovered by Biot in 1815, while experimenting upon the effect of the surrounding medium upon the colors of thin crystalline plates. The changes which he observed in the colors of plates immersed in oil of turpentine, led him to try the effect of the oil alone on polarized light. He found that the turpentine behaved in a manner similar to a quartz plate cut perpendicular to the axis, turning the polarization plane to the left by an amount proportional to the thickness of the fluid traversed, and approximately inversely proportional to the square of the wave-length.

On examining other substances for this property he found that a

<sup>1</sup> *Camb. Trans.*, iv., part i., pages 79-198.

large number of fluid organic compounds behaved in a similar way, some turning the plane to the left, others to the right. The rotatory power was, however, very much less than that of quartz, a column of turpentine 100 mm. long turning the plane of sodium light  $37^\circ$ . An equal length of a solution of santonid or parasantonid in a mixture of chloroform and bisulphide of carbon, which is the most active liquid substance known, gives a rotation of  $446^\circ$ , while a quartz plate of equal thickness would turn the plane  $2167^\circ$ , or six complete revolutions.

Biot found further that a mixture of an active substance with an inactive one, chemically inert towards it, had a rotatory power proportional to the amount of active substance present; in other words, the rotation was simply proportional to the number of active molecules in the path of the light, and wholly independent of the proximity of similar or different molecules.

The neutral effect of dilution with an inactive substance made the investigation of solid substances possible by bringing them into solution in some optically inactive liquid.

In this way the number of active organic substances was enormously increased, many sugars, gums, albumens, alkaloids, fruit-acids, etc., being added to the list. We shall see presently, however, that in some cases the solvent, even if optically inactive, is not without influence on the rotatory power.

**Rotatory Power and Change of State.**—Biot found that substances which were active in solution, preserved their property in the solid state, provided they were prevented from crystallizing. Sugar and tartaric acid can be obtained as amorphous solids, and are optically active in this state. If the substance crystallizes, the double refraction completely masks the rotatory polarization.

Bringing the substance into the vaporous state does not affect its activity, as Biot ascertained in 1818 by means of a tube 15 meters long filled with the vapor of turpentine. His apparatus took fire before the completion of the experiment and was destroyed, and it remained for Gerney to make careful measurements of the rotatory power of vapors. He found that the specific activity was the same in the vaporous state as in the liquid, *i.e.* a long column of vapor has the same rotatory power as the short column of liquid into which it condenses (the cross sections being, of course, equal).

**Rotatory Dispersion of Liquids.**—The rotatory dispersion of liquids was found by Biot to be similar to that of quartz. That the increase in rotation is not strictly proportional to the inverse square of the wave-length he showed by filling a tube with a mixture of dextro-rotatory and laevo-rotatory liquids in such proportion that perfect compensation was secured for a single color. If the law of the inverse squares of  $\lambda$  was strictly followed, the compensation would be perfect for all wave-lengths, which was found not to be the case, the light appearing colored through the analyzer. The departure from the law is different for different liquids, being very small for the oils and very large in the case of a solution of camphor in alcohol.

**Molecular Rotation and the Influence of the Solvent on the Rotatory Power.**—The fact that the rotation produced by a given substance appeared to be proportional to the number of molecules in

the path of the light, gave rise to the opinion that the rotatory power was inherent in the molecule, and led to the term molecular rotatory power. This we may define as the amount of rotation produced by a column of the solution 1 dm. in length containing 1 gram of the substance per cubic centimeter. If we dissolve  $p$  grams of the substance in  $q$  grams of the solvent, the density being  $\delta$ , then  $\frac{p}{p+q} \delta$  is the amount of the substance contained in unit volume of the solution, and if we fill with this solution a tube of length  $l$  and observe a rotation for some particular wave-length, we have

$$\rho = [\rho] \frac{p}{p+q} l \delta,$$

in which  $[\rho]$  is a constant for the substance, and is defined as the molecular rotatory power. This constant may also be defined as the rotation produced by a thickness of 1 dm. of the pure substance divided by the density of the substance. Biot found that the rotation was not strictly proportional to the amount of dissolved substance, and that it varied, moreover, with the nature of the solvent. If the change in the rotatory power with changing concentration is continuous, a formula may be deduced by which we may determine the molecular rotatory power of the pure substance from observations of solutions.

This matter was very carefully investigated by Landolt, who worked with fluid substances, so that the activity of the pure substance could be directly determined, and then compared these values with the values calculated from observations made with the substance dissolved in various inactive solvents.

He found that the molecular rotatory power could be expressed as a function of the quantity of active substance contained in solution. If  $q$  represents the weight of the solvent in 100 parts by weight of the solution, an equation of the following form could be applied:

$$(\rho) = A + Bq + cq^2.$$

The constants could be determined by making observations with solutions of various concentration, the constant  $A$  being the molecular rotation of the pure substance ( $q = 0$ ).

In this way both the effects of the nature of the solvent and the degree of concentration are eliminated.

For example, the value obtained with sodium light for pure oil of turpentine was  $37^\circ.01$ , the equations obtained with alcohol, benzol and acetic acid as solvents being

1. *Alcohol.*

$$[\rho]_0 = 36^\circ.974 + .004816q + .000133q^2.$$

2. *Benzol.*

$$[\rho]_0 = 36^\circ.970 + .02153q + .00006673q^2.$$

3. *Acetic Acid.*

$$[\rho]_0 = 36^\circ.894 + .02455q + .0001369q^2.$$

In some cases concentrated solutions are dextro-rotatory; dilute,

laevo-rotatory. Such is the case with malic acid, which is represented by the equation

$$[\rho]_0 = 5^{\circ}891 - 08959q,$$

right-handed rotation being regarded as positive.

For  $q = \frac{5.891}{.0896} = 65.7$  we have an optically inactive solution.

**Anomalous Rotatory Dispersion.**—In the case of solutions of tartaric acid in water, the dispersion at first increases with decreasing wavelength, reaches a maximum, and then decreases. Measurements made by Arndtsen<sup>1</sup> for 50 parts of crystallized acid in 50 parts of water gave the following values :

<i>C</i>	<i>D</i>	<i>E</i>	<i>b</i>	<i>F</i>	<i>e</i>
11°·9	13°	14°	13°·7	13°·3	10°·3

The formula for this substance is  $[\rho] = A + Be$ , in which  $e$  is the percentage of water, the value for  $A$  and  $B$  for the various colors being

<i>C</i>	<i>D</i>	<i>E</i>	<i>b</i>	<i>F</i>	<i>e</i>
$A = +2^{\circ}748$	$1^{\circ}95$	$0^{\circ}153$	$-0832^{\circ}$	$-3^{\circ}598$	$-9^{\circ}657$
$B = +9^{\circ}446$	$13^{\circ}03$	$17^{\circ}514$	$19^{\circ}147$	$23^{\circ}977$	$31^{\circ}437$

From this it is apparent that the pure substance in an amorphous state is dextro-rotatory for all wave-lengths on the red side of a point in the spectrum a little below the *E* line, and laevo-rotatory for wave-lengths below this point, a circumstance which had already been noticed by Biot in the case of amorphous plates of the acid, made by fusing the crystals and pouring the liquid on a glass plate.

In the case of active substances which show strong selective absorption, we may have true anomalous rotatory dispersion when we cross the absorption band. Such cases are not to be confused with the one just mentioned, which in all probability owes its peculiarity, as we shall see when we take up the theory of the rotation, to the presence of both a dextro and laevo-rotatory system within the molecule. We have a somewhat remote analogy in the achromatic prism. Considered as a single dispersing system it may be said to show anomalous dispersion, the outstanding colors due to imperfect compensation being arranged in anomalous order, as we have seen.

**Double Refraction of Active Liquids.**—The division of the plane-polarized ray into two oppositely-polarized circular disturbances was shown experimentally by E. v. Fleischl<sup>2</sup> by a method identical with the one employed by Fresnel in the case of quartz. A long narrow trough was divided into 22 prismatic compartments by means of oblique partitions of plane-parallel glass (Fig. 268). These compartments were filled alternately with dextro and laevo-rotatory liquids of the same refractive index, the best results having been obtained with oil of orange and a mixture of the oils of turpentine and ricinus. A small source of polarized-sodium light appeared doubled when viewed through the compound-fluid prism, and examination with a quarter-wave plate



FIG. 268.

<sup>1</sup> *Pogg. Ann.*, Bd. cv.

<sup>2</sup> *Ber. Wien. Akad.*, 1884.

and Nicol prism showed the two images to be circularly polarized in opposite directions.

**Theory of Rotatory Polarization.**—The first attempt to bring the phenomenon of rotatory polarization within the range of mathematical analysis was made by M'Cullagh in 1836. Making no assumptions regarding the ultimate physical structure of media which had the power of rotating the plane of polarization, he investigated the changes which were required in the equations of wave-motion in doubly refracting substances, to make them include the phenomenon of rotation. His treatment would be out of place here, as it is purely mathematical, and is of no assistance in forming an idea of the possible cause of the rotation. He found, in brief, that if the introduction of a third derivative into the equations of wave-motion was made, the equations broke up into expressions representing circular vibrations which were propagated with different velocities.

The equations of wave-motion in doubly refracting media are

$$\frac{d^2\xi}{dt^2} = b^2 \frac{d^2\xi}{dz^2}, \quad \frac{d^2\eta}{dt^2} = b^2 \frac{d^2\eta}{dz^2}$$

for polarized vibrations propagated along the optic axis, which coincides with the  $z$  axis of coordinates.

M'Cullagh modified these equations as follows :

$$\frac{d^2\xi}{dt^2} = b^2 \frac{d^2\xi}{dz^2} + c \frac{d^3\eta}{dz^3}, \quad \frac{d^2\eta}{dt^2} = b^2 \frac{d^2\eta}{dz^2} - c \frac{d^3\xi}{dz^3},$$

the introduction of the third derivatives being purely arbitrary.

Solving these equations, he found that they represented right and left-handed circular vibrations, the former being propagated faster or slower according as the constant  $c$  was taken positive or negative. The resultant plane-polarized vibration was rotated by an amount

$$\rho = \frac{2\pi^2 c V^2}{b^2 \lambda^2},$$

in which  $\lambda$  and  $V$  represent wave-length and velocity of propagation in the crystal. If we disregard dispersion, i.e. consider  $V$  independent of  $\lambda$ , the expression shows that the rotation varies as the inverse square of the wave-length, as Biot originally believed it to be. M'Cullagh introduced the same arbitrary modifications into the expressions representing rays propagated in other directions than that of the optic axis, and showed that they represented elliptical vibrations, as had been imagined by Airy in his treatment of the subject. His treatment of the whole subject is given in Verdet's *Optics*, together with a theoretical investigation of the rotatory power of liquids, based upon the somewhat fanciful supposition that they are made up of, or contain molecules, which, separately considered, act like crystal films, breaking a linear vibration up into two elliptical vibrations.

**Physical Explanation of the Rotation.**—In the case of rotation by crystals we can refer the phenomenon to the crystalline structure. Ewell<sup>1</sup> has shown that twisted gelatine cylinders show rotatory

<sup>1</sup> *American Journal of Science*, 8, 89, 1899.

polarization. If we imagine a bar of elliptical cross-section which has been twisted torsionally, we have a rough analogy, which may help us to understand how a spiral arrangement of the axes of maximum and minimum elasticity may account for the rotation of a plane-polarized vibration.

In the case of liquids and solutions, however, we must necessarily refer the rotatory power to the structure of the molecule. All rotatory liquids contain carbon, and their power has been ascribed by Le Bel and Van't Hoff to the quadrivalence of this element.

If the four atoms or radicals, which are in combination with the carbon atom, form the corners of a regular tetrahedron, we can arrange them in two different ways, such that one is the looking-glass image of the other, and yet no amount of turning enables them to be brought into coincidence. One of these we may consider dextro, the other laevo-rotatory. Right and left-handed spirals have similar geometrical properties.

The earliest attempts to explain rotatory polarization were based upon an experiment made by Reusch, who found that, if thin mica plates were superposed, each plate having its principal section turned through a definite angle either to the right or left, with reference to the principal section of the plate below, the combination imitated the behavior of a quartz plate cut perpendicular to the axis, rotating plane-polarized light to the right or left according as the pile of plates were built up clockwise or counter clockwise. The thinner the plates and the greater their number, the more nearly the pile imitated a rotatory crystal. It was quite natural, in view of this very suggestive experiment, to ascribe a somewhat similar structure to quartz, but the efforts to explain the rotation of liquids on the assumption of molecules having a laminated structure was pushing the analogy too far. The modern ionic theory is capable of explaining both the natural rotation of liquids and solids, and the magnetic rotation discovered by Faraday.

We will begin with the case of rotatory liquids, following the admirable treatment given by Drude.

We assume that the electrons, which are set in vibration by the light-waves, are forced to move back and forth over a spiral path instead of along straight lines. The force of restitution is supposed to act along the axis of the spiral, and to vary with the displacement, precisely as we assumed in the case of the electrons which caused dispersion. It is clear that the electron will be urged along its spiral path, the axis of the spiral being parallel to  $z$ , not only by the force  $X$ , but also by the forces  $Y$  and  $Z$ . If the spiral is oriented as in Fig. 269 a positive electric force  $X$  will displace the electron to the right, regardless of its position. The  $y$  component will aid or oppose the  $x$  component according as the electron is on the lower or upper side of the spiral. If it is at  $B$ , a positive  $Y$  will carry it to the right, if at  $A$  to the left. If  $Y$  has the same value at both points, there will be compensation, but if  $Y$  varies with  $z$ , there will be an outstanding effect for each revolution which the electron makes, which will be represented by

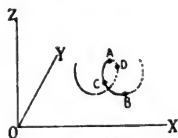


FIG. 269.

$-\frac{\partial Y}{\partial z}$ , if  $Y$  increases with  $z$ . In the same way the  $z$  component, which aids  $X$  when the electron is at  $D$  and opposes  $X$  when it is at  $C$ , exerts an outstanding effect represented by  $\frac{\partial Z}{\partial y}$ . The necessary modification which we must make in our previous conception of the electron's motion is therefore that it moves not only as the result of the force  $X$  at the point which it occupies, but also as the result of the values which the  $y$  and  $z$  components have in its immediate vicinity. With this modification our equation for the motion of the electron becomes

$$m \frac{\partial^2 \xi}{\partial t^2} = e \left( X + f' \left( \frac{\partial Y}{\partial z} - \frac{\partial Z}{\partial y} \right) - \frac{4\pi e^2}{\theta} \xi - v e^2 \frac{\partial \xi}{\partial t} \right).$$

For the condition shown in Fig. 269,  $f'$  is of course negative, and its value depends on the diameter of the spiral path and its pitch. If the pitch is small, the electron is obliged to make a larger number of revolutions in travelling a given distance along the  $x$  axis, and the resultant effect of the  $y$  and  $z$  components of the electric force is greater than when the pitch is small. If the pitch is infinite, the spiral degenerates into a straight line and  $f' = 0$ .

The electric convection current due to the motion of the electrons along the  $x$  axis we will designate, as before, by

$$(j_x)_1 = e_1 N \frac{\partial \xi}{\partial t}.$$

For period changes we have  $\xi = A e^{i \frac{t}{\tau}}$ , from which we get, as before,

$$e_1 \xi = \frac{1}{4\pi} X' \frac{\theta_1}{1 + \frac{i}{\tau} a_1 - \frac{b}{\tau^2}}, \text{ in which } X' = X + f' \left( \frac{\partial Y}{\partial z} - \frac{\partial Z}{\partial y} \right).$$

$$\text{This gives us } (j_x)_1 = \frac{\theta_1 N}{4\pi \left( 1 + \frac{ia}{\tau} - \frac{b}{\tau^2} \right)} \frac{\partial}{\partial t} \left( X + f' \left[ \frac{\partial Y}{\partial z} - \frac{\partial Z}{\partial y} \right] \right).$$

For periodicities not too near the free period of the electron we neglect the friction term  $\frac{a}{\tau}$ . Adding to  $(j_x)_1$  the quantity  $(j_x)_0 = \frac{1}{4\pi} \frac{\partial X}{\partial t}$ , which represents the displacement current in the ether, we get for the total current

$$j_x = \frac{1}{4\pi} \frac{\partial X}{\partial t} + \frac{\theta_1 N}{4\pi \left( 1 - \frac{\tau_h}{\tau} \right)^2} \frac{\partial}{\partial t} \left( X + f' \left( \frac{\partial Y}{\partial z} - \frac{\partial Z}{\partial y} \right) \right), \dots\dots\dots(1)$$

which, if we consider that we have electrons of different free periods, and write, for abbreviation,

$$\epsilon = 1 + \sum \frac{\theta_h N_h}{1 - \left( \frac{\tau_h}{\tau} \right)^2}, \quad f = \sum \frac{\theta_h f'_h N_h}{1 - \left( \frac{\tau_h}{\tau} \right)^2},$$

becomes

$$j_x = \frac{1}{4\pi} \frac{\partial}{\partial t} \left\{ \epsilon X + f \left( \frac{\partial Y}{\partial z} - \frac{\partial Z}{\partial y} \right) \right\}.$$

The fundamental Maxwell equations now take the form,

$$\left. \begin{aligned} \frac{1}{c} \frac{\partial}{\partial t} \left( \epsilon X + f \left[ \frac{\partial Y}{\partial z} - \frac{\partial Z}{\partial y} \right] \right) &= \frac{\partial \gamma}{\partial y} - \frac{\partial \beta}{\partial z}, \\ \frac{1}{c} \frac{\partial}{\partial t} \left( \epsilon Y + f \left[ \frac{\partial Z}{\partial x} - \frac{\partial X}{\partial z} \right] \right) &= \frac{\partial \alpha}{\partial z} - \frac{\partial \gamma}{\partial x}, \\ \frac{1}{c} \frac{\partial}{\partial t} \left( \epsilon Z + f \left[ \frac{\partial X}{\partial y} - \frac{\partial Y}{\partial x} \right] \right) &= \frac{\partial \beta}{\partial x} - \frac{\partial \alpha}{\partial y}, \end{aligned} \right\} \dots\dots\dots (2)$$

which, if we differentiate successively with respect to  $x$ ,  $y$  and  $z$  and add, give

$$\frac{\partial}{\partial t} \left( \frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y} + \frac{\partial Z}{\partial z} \right) = 0.$$

The magnetic equations remain unchanged :

$$\frac{1}{c} \frac{\partial \alpha}{\partial t} = \frac{\partial Y}{\partial z} - \frac{\partial Z}{\partial y}, \quad \frac{1}{c} \frac{\partial \beta}{\partial t} = \frac{\partial Z}{\partial x} - \frac{\partial X}{\partial z}, \quad \frac{1}{c} \frac{\partial \gamma}{\partial t} = \frac{\partial X}{\partial y} - \frac{\partial Y}{\partial x}. \dots\dots\dots (3)$$

By elimination of  $\alpha$ ,  $\beta$ ,  $\gamma$  from the fundamental equations (2) and (3) we get

$$(10) \quad \left\{ \begin{aligned} \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \left( \epsilon X + f \left[ \frac{\partial Y}{\partial z} - \frac{\partial Z}{\partial y} \right] \right) &= \Delta X, \\ \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \left( \epsilon Y + f \left[ \frac{\partial Z}{\partial x} - \frac{\partial X}{\partial z} \right] \right) &= \Delta Y, \\ \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \left( \epsilon Z + f \left[ \frac{\partial X}{\partial y} - \frac{\partial Y}{\partial x} \right] \right) &= \Delta Z. \end{aligned} \right\} \dots\dots\dots (4)$$

If we are dealing with plane-waves propagated along the  $z$  axis, we can write

$$(11) \quad X = M e^{\frac{i}{\tau}(t - pz)}, \quad Y = N e^{\frac{i}{\tau}(t - pz)}, \quad Z = 0,$$

in which  $p$  represents the reciprocal of the wave-velocity.

Differentiating and multiplying by  $\epsilon$  gives us

$$\epsilon \frac{\partial^2 X}{\partial t^2} = -\frac{\epsilon M}{\tau^2} e^{\frac{i}{\tau}(t - pz)}, \text{ a similar expression for } \frac{\partial^2 Y}{\partial t^2} \text{ and } \epsilon \frac{\partial^2 Z}{\partial t^2} = 0,$$

$$\frac{\partial Y}{\partial z} = -\frac{i}{\tau} N p e^{\frac{i}{\tau}(t - pz)},$$

$$\frac{\partial Z}{\partial y} = 0, \quad \frac{\partial^2}{\partial t} \left( \frac{\partial Y}{\partial z} \right) = \frac{N p i}{\tau^2} e^{\frac{i}{\tau}(t - pz)}.$$

Substituting in the first of equations (4),

$$\frac{1}{c^2} \left[ -\frac{\epsilon M}{\tau^2} e^{\frac{i}{\tau}(t - pz)} + f \frac{i N p}{\tau^2} e^{\frac{i}{\tau}(t - pz)} \right] = \Delta X$$

$$\text{Now} \quad \frac{\partial^2 X}{\partial x^2} = 0, \quad \frac{\partial^2 Y}{\partial y^2} = 0, \quad \frac{\partial^2 X}{\partial z^2} = -\frac{M}{\tau^2} p^2 e^{\frac{i}{\tau}(t - pz)};$$

$$\therefore e^{\frac{i}{\tau}(t-pz)} \frac{1}{c^2} \left[ -\frac{eM}{\tau^2} + f \frac{iNp}{\tau^3} \right] = -\frac{M}{\tau^2} p^2 e^{\frac{i}{\tau}(t-pz)},$$

$$\frac{1}{c^2} \left[ \epsilon M - f \frac{iNp}{\tau} \right] = Mp^2,$$

or

$$\epsilon M - \frac{i}{\tau} fpN = Mp^2c^2,$$

$$\epsilon N + \frac{i}{\tau} fpM = Np^2c^2.$$

Multiplying the first equation by  $N$ , and the second by  $M$ , gives us

$$-iN^2 = iM^2 \quad \text{or} \quad M^2 = -N^2 \quad \text{or} \quad M = iN, \quad M = -iN.$$

Substituting these values in  $\epsilon - p^2c^2 = \frac{i}{\tau} fp \frac{M}{N}$  gives

$$\epsilon - p^2c^2 = \frac{fp}{\tau}, \quad \epsilon - p^2c^2 = -\frac{fp}{\tau},$$

equations which show us that we have two different values of  $p$ , i.e. that we are dealing with two waves of different velocities. To find these velocities, we solve the two quadratic equations by completing the squares, and get at once

$$p' = \frac{1}{V'} = -\frac{f}{2\tau c^2} + \frac{1}{c} \sqrt{\frac{f^2}{4\tau^2 c^2} + \epsilon}, \quad p'' = \frac{1}{V''} = +\frac{f}{2\tau c^2} + \frac{1}{c} \sqrt{\frac{f^2}{4\tau^2 c^2} + \epsilon}.$$

The nature of these waves we find from the equations connecting  $N$  and  $M$ , the amplitudes. In  $M = iN$ , if  $M$  is real,  $N$  must be imaginary, i.e. if the wave has a real amplitude along the  $x$  axis, it has an imaginary amplitude along  $y$ . The physical significance of this can be at once found, if we remember that, as before, only the real part of equation (11) is to be taken.

Writing now equations (11) in the form

$$X = M \cos \frac{1}{\tau}(t - pz) + Mi \sin \frac{1}{\tau}(t - pz),$$

$$Y = \cos \frac{1}{\tau}(t - pz) + Ni \sin \frac{1}{\tau}(t - pz),$$

and substituting for  $N$  its equivalents  $\frac{M}{i}$  and  $-\frac{M}{i}$ , we get, if we confine ourselves to the real parts only,

$$X = M \cos \frac{1}{\tau}(t - pz), \quad Y = M \sin \frac{1}{\tau}(t - pz) \quad (\text{for } iN = M),$$

and  $X = M \cos \frac{1}{\tau}(t - pz), \quad Y = -M \sin \frac{1}{\tau}(t - pz) \quad (\text{for } iN = -M),$

which represent circularly-polarized waves, which are propagated with the velocities given by the expressions which we have already deduced for  $p'$  and  $p''$ .

The amount of the rotation computed from the above equations is

$$\delta = \frac{z}{\tau} \frac{p'' - p'}{2} = \frac{f}{2\tau^2 c^2} z = 2\pi^2 \frac{f}{\lambda^2} z,$$

in which  $\lambda$  is the wave-length of the light (in vacuum).

This expression shows us in the first place that the amount of rotation is proportional to  $f$  which we have written for  $\sum \frac{\theta_a f_a N_a}{1 - \left(\frac{\tau_a}{\tau}\right)^2}$ . The

sign of  $f$  depends on whether  $\tau$  is larger or smaller than  $\tau_a$ , consequently we should expect the rotation to have its largest value when  $\tau$  is very nearly equal to  $\tau_a$  and to change its sign when we cross the region of the spectrum defined by  $\tau_a = \frac{T_a}{2\pi}$ , that is the center of the absorption band caused by the electrons in question.

This is in perfect agreement with the experiments of Cotton, who found anomalous rotatory dispersion in the case of certain strongly absorbing active substances.

It will be remembered that certain cases of anomalous rotatory dispersion have been cited, which did not appear to be due to the presence of an absorption band.

We are now in a position to explain this curious phenomenon.

**Real and Spurious Anomalous Rotatory Dispersion.**—The rotation of the plane of polarization as a function of the wave-length of the light is given by the formula

$$\delta = \frac{k}{\lambda^2} \sum \frac{\theta_a f_a N_a}{1 - \left(\frac{\tau_a}{\tau}\right)^2} = \sum \frac{k_a}{\lambda^2 - \lambda_a^2},$$

the summation being restricted to the electrons which produce optical activity.

Now the sign of  $k_a$  depends on whether the electron for which it stands is dextro or laevo-rotatory, consequently if we have an infra-red dextro-rotatory electron and an ultra-violet laevo-rotatory one, the sign of the rotation for all wave-lengths comprised between the two will be the same, since  $\lambda^2 - \lambda_a^2$  is negative if  $\lambda_a > \lambda$  and positive if  $\lambda > \lambda_a$ . In this case the rotation will have a minimum

value somewhere near the middle of the visible spectrum. If the sign of  $k_a$  is the same for both electrons, the rotation will be zero near the middle of the visible spectrum, and will be of opposite sign on either side of this point, increasing as the absorption bands are approached. This is the condition in the case of solid tartaric

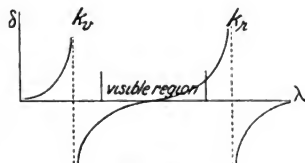
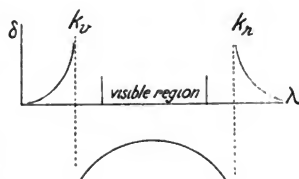


FIG. 270.

acid, as we have seen (page 387). The rotatory dispersion curves for the two cases just considered are shown in Fig. 270.

**Rotatory Dispersion.**—The variation of the rotation with the wave-length is given by the formula

$$\delta = \frac{k}{\lambda^2} \sum \frac{\theta_h f'_h N_h}{1 - \left(\frac{\tau_h}{\tau}\right)^2}.$$

If  $\left(\frac{\tau_h}{\tau}\right)^2$  is small in comparison to 1, as will be the case when the period of the electrons is small in comparison to that of the waves (ultra-violet electrons), the terms summed will give us a constant, and we can write our formula  $\delta = \frac{k'}{\lambda^2}$ , which is identical with Biot's empirical formula.

For a limited range of the spectrum this formula often represents the dispersion with a fair degree of accuracy. If, however, we extend our observations over a wide range of wave-lengths, we must use the complete formula.

If we apply the formula

$$\delta = \sum \frac{k_h}{\lambda^2 - \lambda_h^2}$$

to the observations which have been made with quartz, we shall find that much light is thrown upon the nature of the different electrons which give rise to absorption and dispersion. These observations cover a range extending from wave-length  $2.14\mu$  in the infra-red to  $.219$  in the ultra-violet. As we have seen in the chapter on Dispersion, we have three absorption bands which have to be taken into account, two in the infra-red at  $\lambda_2 = 8.5\mu$  and  $\lambda_3 = 21\mu$ , and one in the ultra-violet at  $\lambda_1 = .1\mu$ . There are in addition other bands further down in the ultra-violet, for which  $\lambda_h$  is small in comparison to  $\lambda$ . These give us the term  $\frac{k'}{\lambda^2}$  and we write our formula

$$\delta = \frac{k_1}{\lambda^2 - \lambda_1^2} + \frac{k_2}{\lambda^2 - \lambda_2^2} + \frac{k_3}{\lambda^2 - \lambda_3^2} + \frac{k'}{\lambda^2},$$

(Ultra-violet)    (Inf.-red)    (Inf.-red)    (Remote ult.-violet)

If now we calculate the constants from the observations of the rotations for various wave-lengths, we find that both  $k_2$  and  $k_3$  are equal to zero, which shows us that the infra-red electrons do not contribute to the rotatory dispersion, *i.e.* they are inactive. We can therefore write the formula

$$\delta = \frac{k_1}{\lambda^2 - \lambda_1^2} + \frac{k'}{\lambda^2},$$

in which  $k_1 = 12.2$ ,  $k' = -5.046$ .

The accuracy with which this formula represents the rotatory dispersion of quartz can be seen from the following table:

$\lambda$ .	$\delta$ (obs.).	$\delta$ (calculated).
2.14 $\mu$	1.60	1.57
1.45	3.43	3.43
Red .67	16.54	16.56
Green .51	29.72	29.67
Violet .40	48.93	48.85
.27	121.06	121.34
.22	220.7	220.57

**Rotatory Dispersion of Absorbing Media.**—In the case of absorbing media, *i.e.* media which have absorption bands in the region under investigation, we cannot neglect the friction coefficient  $a_h$ , and both  $\epsilon$  and  $f$  of equations (4) become complex :

$$\epsilon = 1 + \sum \frac{\theta_h N_h}{1 + i \frac{a_h}{\tau} - \frac{b_h}{\tau^2}}, \quad f = \sum \frac{\theta_h f_h' N_h}{1 + i \frac{a_h}{\tau} - \frac{b_h}{\tau^2}}.$$

In the chapter on Absorption (equation (4), page 364) we have seen that if  $\epsilon$  is complex, then  $p$  in the equation  $X = Me^{\frac{i}{\tau}(t - p\tau)}$ , (eq. (11), page 391) must be complex as well, which, as we have seen, means absorption.

Bearing in mind that  $p$  is the reciprocal of the velocity in the medium, and writing as before  $p = \frac{1 - i\kappa}{V}$ , it becomes at once evident that, since we have two different values for  $p$ , corresponding to right and left-handed circularly-polarized rays, we must also have two values of  $\kappa$ , the extinction coefficient. In other words, for a given wave-length of circularly-polarized light, the absorbing power of the medium will depend on the direction of revolution of the luminous vibration.

This effect has been observed by Cotton (*Comptes Rendus*, 120, pages 989, 1044) in the case of solutions of copper tartarate and chromium tartarate in potash. The chromium salt has an absorption band in the yellow, transmitting red and green. Cotton found that if circularly-polarized sodium light of unit intensity was passed through 1 cm. of his solution, the emergent light had an intensity of .0077 if the vibration was left handed, while in the case of a right-handed vibration it was .0059. The difference is very marked, though the strong absorption indicates that the original light must be very intense. The effect can be very easily shown by preparing a quarter wave-plate of mica and cutting it in two along one of the directions of vibration ; the two halves are to be mounted with their edges in contact, one plate being turned through an angle of 90° with respect to the other. If plane-polarized light is passed through this plate (the plane making an angle of 45° with the directions of vibration in the mica), we shall have two emergent beams of oppositely-polarized circular light. The polarizing system is illuminated with a brilliant sodium flame, and the field examined through the absorbing solution, when one half will be found to be much darker than the other. If white light is employed in place of the sodium flame the two halves of the field appear differently colored.

This difference of absorbing power for oppositely circularly-polarized vibrations leads us to a somewhat startling conclusion, which was foreseen by Cotton and verified by experiment. Ordinary unpolarized light can be regarded as containing equal amounts of oppositely-polarized circular vibrations. (See chapter on Natural Light.) It ought therefore by mere passage through the solution to exhibit traces of circular polarization. This was found to be the case.

If a suitable medium could be found it might be possible to obtain in this way circularly-polarized light just as plane-polarized light is obtained by means of a tourmaline plate. The plane vibrations in the natural light would give no trouble, for, as we know, they are decomposed into circular vibrations, which traverse the medium with different velocities.

**Elliptical Polarization produced by Absorbing Active Media.**—In the case of transparent active media, the plane-polarized light remains plane polarized during transmission, emerging with its plane rotated through a certain angle. The emergent plane vibration is the resultant of the two equal circular vibrations. As we have seen, in absorbing media, one of these may be reduced in intensity more than the other, and the resultant of two circular vibrations of different amplitude is not a plane vibration but an elliptical one. Cotton found that the tartarate solutions above mentioned transformed plane into elliptically-polarized light, it being impossible to completely extinguish the emergent light with a Nicol. The ellipticity was found to be greatest in the region of the spectrum where the difference of absorbing power was greatest.

**Possible Production of an Optically-Active Substance from an Inactive, by Circularly-Polarized Light.**—Since the absorption of light is often accompanied by chemical change, it is possible that a solution containing equal numbers of dextro and laevo-rotatory molecules, and consequently inactive, might acquire rotatory power by the action of circular light. If the molecules were unstable and easily decomposed by light, the effect of a circular vibration would be to break down one set of molecules and leave the others unaffected. Both Le Bel and Cotton have pointed out the possibility of effecting unique chemical transformations by the action of circularly-polarized light.

**Rotatory Dispersion in Absorbing Media.**—One formula for the rotatory dispersion shows us that on crossing an absorption band the sign of the rotation may change, or we may have a high positive value on one side and a low positive value on the other. This amounts to saying that anomalous dispersion of the rotation is to be expected in absorbing media. Cotton found that this was the case.

In the case of the chromium tartarate the rotations were as follows :

$\lambda$ .	$\delta$ .
657	+ 1° 26'
589	+ 2° 30'
581	+ 1° 46'
562	- 1° 21'
522	- 2° 50'
473	- 1° 52'

## CHAPTER XVII.

### MAGNETO-OPTICS.

**The Faraday Effect: Magnetic Rotation of the Plane of Polarization.**—The discovery was made by Faraday that a transparent isotropic medium, when placed in a powerful magnetic field, acquires the property of rotating the plane of polarization, when the light traverses the medium in the direction of the lines of magnetic force. The phenomenon differs, however, from natural rotation, in that the direction in which the plane of the vibration turns, depends upon whether the light rays are passing through the medium from the north pole of the magnet towards the south, or in the reverse direction. The rotation is therefore doubled if the light is reflected back through the medium, instead of being annihilated as in the case of quartz and other active substances. The effect is most pronounced with media having a high refractive index, such as bisulphide of carbon or dense flint glass. With a powerful Ruhmkorff magnet, the poles and cores of which are bored out to allow of the passage of light rays along the lines of force, the rotation can be easily observed with a thick piece of ordinary plate-glass. Sun or arc light is passed through a Nicol prism, the hollow magnet cores and the glass block between the poles. A second Nicol is placed in such a position as to extinguish the emergent beam. On throwing the current into the magnet, the field immediately becomes brilliantly illuminated, and by turning the analyzing Nicol until darkness is again produced the amount and direction of the rotation can be determined.

Owing to the rotatory dispersion this position will vary with the color, and the field will appear blue, purple and red in succession, as will be readily understood from Fig. 271, in which the dotted arrow represents the original direction of the vibration, and the other arrows the rotated red, green, and blue vibrations. The analyzing Nicol in its original position is indicated by *N*. The rotation in this case is clockwise, and all of the colors are transmitted with more or less freedom, consequently the field appears nearly white. On turning the

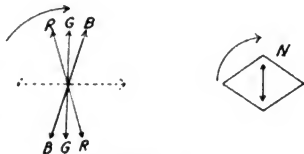


FIG. 271.

Nicol in the same direction it will extinguish the red first, leaving an outstanding color of a bluish-green; the green goes next, leaving a purple field made up of the transmitted red and blue, and finally the blue disappears, leaving the field orange red. If we examine the light through a spectroscope, as we turn the Nicol, we shall see a dark band enter the spectrum on the red side and leave it on the violet side.

**Explanation of the Magnetic Rotation.**—The explanation of the natural rotation in active substances which Fresnel gave, was that the plane vibration was decomposed into two oppositely polarized circular vibrations, which were propagated with different velocities. The same explanation will do for the magnetic rotation, provided that it can be shown that the refractive index of a medium in a magnetic field for circularly-polarized light depends upon the direction of revolution. The matter was attacked experimentally by Righi and Becquerel independently, and both investigators found that the interference fringes, formed by two streams of circularly-polarized light, one of which had traversed a block of glass placed between the poles of a magnet, were displaced when the magnetic field was formed. The direction of the displacement depended on whether right or left-handed circular light was used, which showed that the effect of the field was to increase the refractive index for one type of vibration and diminish it for the other. It remained only to show that the actual decomposition of the plane vibrations into circular ones actually occurred by some experiment analogous to the one which Fresnel made with his battery of quartz prisms built of right and left-handed crystals in alternation.

**Resolution into Circular Components.**—The experimental resolution of the light into its two circular components in the Faraday effect is a much more difficult problem than the one which confronted Fresnel, since we do not have at our disposal two liquids of the same index of refraction and of opposite magnetic rotation, with which hollow prisms might be filled, in the manner adopted by Fleischl in the case of natural rotation. The problem has, however, been attacked and solved in a very beautiful manner by Brace.<sup>1</sup>

It will be remembered that in Fresnel's arrangement of right and left-handed quartz prisms, the clockwise circular component which travelled at the higher velocity in one prism, travelled at the slower velocity in the following prism. The very ingenious idea occurred to Brace to reverse the direction of revolution of the circular vibrations at the boundary surface between the two prisms, which can be done with a half-wave plate of mica. By this artifice the same thing is accomplished as by employing prisms of dextro and laevo-rotatory media in succession. A double prism of extra dense flint-glass was employed with a half-wave plate cemented between the two components. The experimental difficulties were, however, found to be too great and no conclusive results were obtained. It subsequently occurred to Brace to make use of reflection instead of refraction, and look for evidences of a division of the ray into two circularly-polarized rays. This at first sight seems to antagonize the law of reflection, but it must be remembered that the law of equality between the angles of incidence and

<sup>1</sup> *Wied. Ann.*, xxvi., page 576, 1885; *Phil. Mag.* (6), i., page 464, 1901.

reflection is based upon the fact that the velocity is the same before and after reflection. If we apply the Huygens construction for reflection to a case in which the velocity is less after reflection than before, we shall find that the angle of reflection is *less than* the angle of incidence. Suppose now that our two circular components in the magnetized medium encounter a reflecting surface which reverses the direction of revolution of each. The fast component now becomes the slow, and *vice versa*, and we have a division of the ray. The reversal of the circular vibrations was accomplished by a reflection at  $72^\circ$  at a glass-silver surface, but the separation of the two rays was too small to be detected.

Under these conditions a division of the ray into three rays is to be expected, for the silver-surface transforms the circular vibrations into plane ones, i.e. it acts like a  $\frac{\lambda}{4}$  plate. Each of the resulting plane

vibrations is again broken up into two circular vibrations by the medium. Of these one pair have the same direction of revolution as the components from which they were originally derived, and these will be reflected under the condition  $i=r$ . Of the other two, one is a right-handed revolution derived from the original left-handed component, which we will assume was the "fast" one, and the reflection is under the condition  $i>r$ . The opposite is true for the other component, or  $r>i$ . For the block of glass used the angular separation between the central ray and one of the deviated rays was calculated to be less than  $3''$  or arc, which was less than the instrument was capable of resolving.

The difficulty was finally overcome by making use of multiple reflections. A rectangular block of glass was made by cementing two right-angled prisms ( $n=1.903$ ) together, with a half-wave plate of mica between them to reverse the direction of the circular vibrations (Fig. 272). The incident light enters the prism normally through a small auxiliary prism *A*, traverses the  $\frac{\lambda}{2}$  plate, which turns its plane of

polarization through  $90^\circ$ . The light is travelling perpendicular to the lines of force, consequently the magnetic field does not affect it. As soon, however, as it suffers total reflection, it travels along the lines of force, and is consequently broken up into circular components, one of which travels faster, the other slower than the original disturbance before reflection. A division therefore results, and we have two reflected rays. Total reflection transforms a plane vibration into an elliptical one, while two total reflections give approximately a circular vibration. This makes it a little harder to study out just what happens than in the previous case. The arrangement figured so alters the vibrations that the change is always from fast to slow and from slow to fast, the angular separation increasing each time the rays

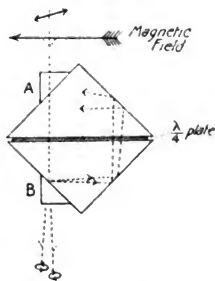


FIG. 272.

are reflected along the lines of force. The light travelled around the prism five times, undergoing twenty internal reflections, and emerged through a second auxiliary prism *B* at the top of the block. The source was a vertical slit powerfully illuminated with an oxy-hydrogen flame fed with sodium. On turning on the magnetic field the image in the telescope was seen distinctly doubled, and on examination with a Nicol prism the two lines were found to be nearly plane-polarized, due to the fact that the last reflection, combined with passage through the  $\frac{\lambda}{2}$  plate, introduced a phase-difference of  $\frac{3}{8}\lambda$  between the components of the circular vibrations, by which they were changed into ellipses of considerable eccentricity. The positions of the major axes of these elliptical vibrations (they were at  $90^\circ$  to one another) showed that the circular vibration, in which the direction of revolution was the same as that of the Amperian currents, was accelerated, while the other was retarded by the magnetic field.

The velocity of right and left-handed circularly-polarized light in a magnetized medium was measured by Mills (*Phys. Rev.*, Feb. 1904) by means of a Michelson interferometer. By means of a Brevais double plate, one half of the field was illuminated with right, the other half with left-handed circularly-polarized light, the fringes crossing the field in a direction perpendicular to the dividing line between the two halves of the plate. On exciting the magnet the fringes on one side moved up, while those on the other side moved down. The accelerated ray was found to be the one in which the direction of the circular vibration was the same as that of the Amperian currents.

**Direction of the Magnetic Rotation.**—The results of the earlier experiments upon magnetic rotation appeared to indicate that all substances rotated the plane of polarization in the same direction, when placed in a magnetic field, but continued investigations showed that this was not the case, many diamagnetic substances being found which gave a rotation in the opposite direction. A generalization cannot, however, be made, for dextro and laevo rotation are to be found in both paramagnetic and diamagnetic substances, as is shown by the following table:

DIAMAGNETIC.		PARAMAGNETIC.	
Rotation Positive.	Rotation Negative.	Rotation Positive.	Rotation Negative.
Ferrocyanide of Potassium. Water. Lead Borate (Faday glass). Almost all other solids, liquids and gases.	Chloride of Titanium.	Iron, cobalt and nickel. Oxygen. Salts of cobalt, nickel, manganese and copper.	Ferro and ferri salts of iron. Ferricyanide of Potassium. Chromate and bichromate of potassium. Salts of cerium, lanthanum and didymium.

**Relation between the Field Strength and Rotation.**—The angular rotation increases in general in proportion to the strength of the field, but this rule is not strictly followed, the most marked exceptions being shown by iron, nickel and cobalt. In the case of iron the relation between the field strength  $H$  and the rotation  $\delta$  is shown in the following table. If the rotation was proportional to  $H$ , the figures of the last column would be approximately the same :

$H$ .	$\delta$ .	$\frac{10^6 H}{\delta}$ .
4420	1.72°	39
8060	3.47°	43
14100	4.41°	31
18500	4.45°	24
30100	4.36°	14

If instead of  $H$  we take the *magnetization*, we shall find that the rotation is proportional to this quantity. In the case of iron the magnetization increases with the field strength, but not at the same rate, finally becoming "saturated," beyond which point a further increase of field produces no increase in the magnetization. The rotation also attains a maximum value at the same point.

**Time required for the Faraday Effect: Rotation by Oscillatory Discharge.**—An experiment was performed by Villari which was interpreted as showing that a certain amount of time was required for the development of the Faraday effect. A block of heavy glass was spun between the poles of an electro-magnet, and the rotation of the plane of polarization was observed to diminish when the speed of rotation exceeded 100 turns per second, becoming practically zero at 200 revolutions. Subsequent experiments made by Bichat, Blondlot<sup>1</sup> and Lodge<sup>2</sup> showed that some other cause must be found for the effect observed. If a Leyden jar was discharged through a helix of insulated wire surrounding a tube filled with carbon bisulphide, the plane of polarization was found to turn with each impulsive rush of the oscillatory discharge, being waved to and fro at the rate of some 70,000 times per second.

To find out whether any time was required for the development of the effect, Bichat and Blondlot illuminated the upper portion of a slit with the light of the spark and the lower portion with the light restored by the  $CS_2$  tube, and examined it in a revolving mirror. The illuminated slit was seen spread out into a serrated band, but no discontinuity was found between the two illuminated halves, showing that the effect is practically instantaneous.

A tentative explanation of the effect of spinning the glass block in Villari's experiment is given by Lodge in his paper, based upon the assumption that it is due to centrifugal strains induced in the glass by the high speed.

**Magnetic Rotation and the Second Law of Thermodynamics.**—Lord Rayleigh has called attention to a curious disposition of apparatus which permits the passage of light in one direction, while refusing absolutely to transmit it in the reverse direction. Two Nicol prisms are mounted with their directions of vibration at an angle of 45°.

<sup>1</sup> *Comptes Rendus*, xciv., p. 1590, 1882.  
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<sup>2</sup> *Phil. Mag.*, April 1899.

Between them consider a medium in a magnetic field of sufficient strength to produce a rotation of the plane of polarization of  $45^\circ$ . The polarized light passing through one Nicol will be rotated into such a plane that the second Nicol will stop it. If we reverse the direction of the light the polarized vibration will be brought into the plane of vibration of the second Nicol and be freely transmitted, since the magnetized medium rotates the plane of polarization in the same absolute direction (*i.e.* considered without regard to the direction of propagation) whichever way the light be travelling.

It is thus possible to construct an apparatus through which we can see without being seen, provided we limit ourselves to monochromatic light. It might be supposed, at first sight, that, by making use of this contrivance, an exception to the second law of thermodynamics might be realized, a radiating body throwing more energy into a second body than it received in exchange and thus elevating its temperature. Wien considered this case in his treatment of the theory of radiation presented at the International Congress of Physics, held in Paris (1900), and offered certain ways of escape out of the difficulty. Lord Rayleigh shows, however (*Nature*, lxiv.), that, after all, no difficulty really exists.

**The Kerr Effect.**—The important discovery was made by Kerr<sup>1</sup> that plane-polarized light becomes elliptically polarized when reflected from the polished pole of an electro-magnet. The incident light must be polarized either in, or perpendicular to, the plane of incidence, otherwise elliptical polarization results from the metallic reflection. On setting a Nicol prism in such a position as to completely extinguish the reflected light, and exciting the magnet, the light instantly reappeared, and could not be extinguished by further rotation of the Nicol, except by the introduction of a quarter-wave plate. The ellipticity is not very great, and we can regard the effect as a rotation of the plane of polarization.

Intimately connected with the Kerr effect is the rotation of the plane of polarization by thin films of iron in a magnetic field, which was studied by Kundt. Films with a thickness of about one third of the wave-length of the light employed, when placed in a strong magnetic field, gave a rotation of over  $4^\circ$ . An iron plate 1 mm. thick, if it were possible to get any light through it, would rotate the plane through  $20,000^\circ$  or 66 complete turns, while a quartz plate of equal thickness gives a rotation of about  $20^\circ$ . If now we consider that the light penetrates the surface of the magnet pole in the act of reflection, the rotation is at once accounted for. A more complete explanation will be given when we come to the Theory of Magnetic Rotation, and the simple explanation just given is quite inadequate, for it has been shown that the effects are considerably modified by surface-films.

**Magnetic Rotatory Dispersion.**—The angular rotation of the plane of polarization in a magnetic field of given strength increases as the wave-length decreases, provided we limit ourselves to a region of the spectrum within which the substance shows absorption bands. In the case of absorbing media anomalous rotatory dispersion may occur

<sup>1</sup> *Phil. Mag.*, May 1877 and March 1888.

when we cross the absorption band, provided that the absorption and rotation are both due to the same electron, as in the case of substances which show natural rotation.

The behavior of alcoholic solutions of the aniline dyes in a magnetic field has been studied by Schmauss,<sup>1</sup> who claimed that the effect of the dye was to increase the rotation of the alcohol on the red side of the absorption band and decrease it on the violet side. If this is the case we should expect the solid dye to rotate the plane of polarization for waves on opposite sides of the absorption band in opposite directions. The author has, however, been unable to detect any trace whatever of rotatory polarization in a film of solid cyanine so thick that nothing but red light was transmitted. Saturated solutions of cyanine between plates of very thin glass also showed no trace of the phenomenon, and it seems possible that the results obtained by Schmauss were due to experimental errors.

Bates (*Ann. der Physik*, 12, page 1901 (1903)), making use of a much more sensitive method, failed to detect any anomalies, and came to the same conclusion as just stated.

A solution of a didymium salt, which has a strong absorption band in the yellow, has been carefully examined by the author (*Phil. Mag.*, May 1904), and found to exhibit unmistakable evidences of anomalous rotatory dispersion at the edge of the absorption band. The curve obtained is shown in Fig. 273a.

Contrary results have been obtained in the case of sodium vapor, in which the rotation has a high value on both sides of the *D* lines, the direction of rotation, however, being the same, as shown in Fig. 273b.

These cases will be more fully considered in the section on the Theory of the Magnetic Rotation.

**The Zeeman Effect.**—A search for a possible effect of a powerful magnetic field upon a source of monochromatic vibrations was first made by Faraday. Placing a sodium flame between the pole pieces of a large electro-magnet, he examined the appearance of the *D* lines when the field was "on" and "off." He was unable, however, to convince himself that any change resulted in the appearance of the lines, a circumstance which we now know resulted from the insufficient power of his spectroscope. This was in 1862. Twenty-three years later the same experiment was tried by Fizeau, who observed changes which appeared to be the result of the magnetic field. If the lines were single before the current was turned into the magnet, they appeared with dark reversals down their centers as soon as the magnet was excited. These observations do not appear to have attracted much attention, and they were not followed up. It is questionable even whether the phenomenon

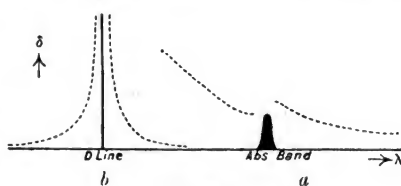


FIG. 273.

<sup>1</sup> *Ann. der Physik*, 1902-1903.

observed by Fizee was in reality the effect that we are about to study.

In 1896 the fact was established beyond all doubt by Zeeman, that the periodic time of vibration of a source of monochromatic radiations is altered when the source is placed in a magnetic field.

In his first announcement he described the effect as a simple broadening of the lines, but his knowledge of the probable cause of the effect that he was searching for, led him to believe that the edges of the line should appear circularly polarized when the light was emitted in the directions of the lines of force. To test this he employed a magnet with perforated pole pieces, and passed the light through a quarter wave plate, which as we have seen transforms a circular vibration into a plane one. He found that the lines now shifted their position in the spectrum when the direction of the current in the magnet was reversed, which indicated that the edges of the line were circularly polarized in opposite directions, exactly as theory showed that they should be.

When the light was examined in a direction perpendicular to the lines of force, the edges were found to be plane polarized, from which Zeeman drew the conclusion that, with a sufficiently strong field, the line would appear triplet, a prediction which was speedily verified by experiment, the two outer components being polarized with their vibration directions perpendicular to the lines of force, while the central component was polarized along the lines of force.

Soon after Zeeman's announcement Michelson made a very complete study of the influence of magnetism upon radiations by means of the interferometer. From the visibility curves obtained with this instrument, he concluded that the lines were doubled when the light was examined in a direction perpendicular to the lines of force. This was subsequently found to be due to the fact that the light of the central component was polarized in such a direction that the oblique interferometer plate refused to reflect it.

The phenomena of radiation in a magnetic field are easily explained on the electron theory, at least the simple cases just described. A further study has shown, however, that many lines are broken up into quadruplets and sextuplets, the  $D$  lines of sodium belonging to this class. In a powerful field  $D_1$  is seen to be a quadruple line, the inner components being polarized with their vibrations along the lines of force, the outer components perpendicular to them.  $D_2$  is a sextuplet, with its four outer components vibrating perpendicular to the field. A remarkable case is that of certain iron lines which appear as triplets with the directions of polarization the exact reverse of the usual order of things.

No simple explanation can be given for these more complicated cases. They will be considered in the section on the Theory of Magneto-Optical Phenomena.

We will now consider the formation of the circularly-polarized doublet seen in the direction of the field, and the plane-polarized triplet seen perpendicularly to it.

Consider the source of monochromatic vibrations as a swarm of atoms, the charged electrons of which rotate in circular or elliptical orbits, or vibrate back and forth along straight lines. On either

hypothesis we can account for the altered appearance of the line in a magnetic field, by considering the force which a charged particle in rapid motion experiences in a magnetic field. This force acts in a direction perpendicular both to the direction of motion of the electron, and to the direction of the lines of force, and is equal to zero only when the direction of motion coincides with the direction of the field.

We will first consider electrons revolving in circular orbits, the planes of which are oriented in all possible positions. Those electrons which are moving in orbits perpendicular to the lines of force are subjected to a force which is directed towards or away from the center of rotation, according to the direction of revolution. The ones which are revolving in such a direction that the force acts towards the center, are drawn in and have their period of revolution accelerated, while those rotating in opposite directions have their centripetal force diminished and their periods slowed down. The force which acts on the electron as a result of the field, either increases or diminishes

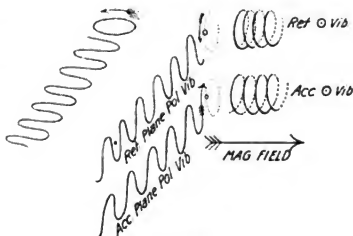


FIG. 274.

the centripetal force which holds the electron in its orbit. An apple whirled around on a string is a useful mechanical analogy. If we increase the tension on the string, we pull in the apple and decrease the time of its revolution. If we relax the string, the orbit opens out and the time of revolution becomes greater.

Consider now the types of waves sent out by the two types of revolving electrons which we have considered. In a direction parallel to the lines of force, the whirling electrons will radiate circularly-polarized rays, one ray polarized clockwise, the other counter clockwise. The periodic time of one is accelerated, while that of the other is retarded; consequently we get two lines in the spectrum, circularly polarized in opposite directions.

In a direction perpendicular to the lines of force the two electrons give off vibrations plane polarized in the plane of their orbits (Fig. 274) of periods similar to those of the circular vibrations. These rays give us the outer components of the normal Zeeman triplet.

In the case of an electron revolving in an orbit the plane of which is parallel to the lines of force, the force exerted on

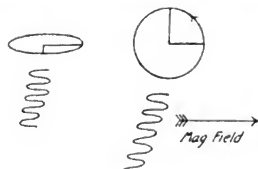


FIG. 275.

the electron will be perpendicular to the plane of the orbit. The circular vibration can be resolved into two rectilinear vibrations at right angles to one another. Consider two orbits oriented as shown in Fig. 275. The component parallel to field will be unaffected, and we shall have plane-polarized vibrations of unchanged period emitted

in a direction perpendicular to the field. This component sends out no radiation in the direction of the lines of force however. The other component must be regarded as affected in the manner about to be described. After considering the effect of the field upon an electron moving in a straight line, it will be found profitable to apply the same reasoning to the present case. It will then appear that right and left-handed circular vibrations are given out along the lines of force and plane-polarized vibrations of accelerated and retarded frequencies perpendicular to them.

We will now examine the effect of a magnetic field on electrons which normally vibrate back and forth along straight lines, instead of moving in circular orbits. They will be subjected to a force which acts

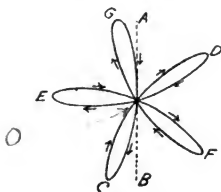


FIG. 276.

in a direction perpendicular to the field and their direction of motion. Let  $AB$  (Fig. 276) represent the normal path of the electron, and consider the lines of force as perpendicular to the paper. If the electron is at  $O$ , and moving down at the moment when the field is thrown on, it will be deflected to  $C$ , moving in a curve, the convex side of which is towards the right. On its return trip it will move along a curve, the convex side of which is to the left, since the force acting on it is now in the opposite direction, owing to the reversal of its direction of motion. It will thus pass in succession

through the points  $D, E, F, G$ , the curve reminding us of the type of vibration resulting from the passage of a plane-polarized ray through a rapidly revolving Nicol prism. (See Light-Beats, page 280.) As we have seen, a vibration of this type is the equivalent of two oppositely-polarized circular vibrations of different periods; consequently the rays sent out in the direction of the field, *i.e.* perpendicular to the paper, are resolved by the spectroscope into two rays circularly polarized in opposite directions.

The rays sent out by the electron perpendicular to the lines of force, *i.e.* in the plane of the paper, will be plane polarized, and at first sight would appear to be monochromatic. If we remember, however, that the electron cannot give out transverse waves in the direction of its motion, we shall see at once that the *amplitude* of the waves sent out in any given direction, say  $AB$ , will depend upon the position of the electron in its star-shaped orbit, attaining its maximum value when the loop  $E$  is being traversed, and its minimum (practically zero) when the vibration is along  $GC$ . At any given point in the plane of the paper, the illumination due to this electron will fluctuate between zero and, say, unity. Now a precisely similar condition of affairs results when two periodic disturbances of slightly different periods pass through the same point. At a given instant the phases will be the same, and we shall have increased illumination, the next instant the phases will be opposed and we shall have darkness. We have the same phenomenon in acoustics, two tuning-forks of nearly the same pitch producing beats, or periodic fluctuations of intensity. We shall thus have a type of radiation emitted perpendicular to the

lines of force, peculiar in that the intensity suffers extremely rapid fluctuations, which the spectroscope separates into two lines, plane polarized in a direction perpendicular to the field. The period of the beats will be the period of revolution of the orbit, which we shall presently see is about  $\frac{1}{20,000,000,000}$  of a second. As Lord Rayleigh has pointed out in his *Theory of Sound*, if we should interrupt a train of sound-waves of frequency  $N$ ,  $n$  times per second the resultant disturbance would contain three frequencies  $(N+n)$ ,  $N$  and  $(N-n)$ . It would be extremely interesting if we could interrupt a beam of monochromatic light rapidly enough to cause the line to become triple in the spectroscope. The thing might possibly be done by means of a revolving diffraction grating, fine lines parallel to the radii being ruled through a silver film deposited around the edge of a mica disc. The elements of the beam would in this case be interrupted in succession by the bars of the grating. If 20,000 lines to the inch were ruled on a 10-inch disc, driven at a speed of 1000 revolutions per second (which is possible), a modern interferometer, *e.g.* the one of Fabry and Perot, should easily separate the components corresponding to the frequencies  $(N+n)$  and  $(N-n)$ .

An apparent difficulty may occur to the reader in connection with the manner in which we have considered the spectroscopic analysis of the "beats" coming from the electron moving in the star-shaped orbit. Beats result when two continuous trains of uniform amplitude, and slightly different frequencies, pass through a point simultaneously. The components, therefore, into which the spectroscope analyzes the disturbance must be continuous in time, *i.e.* must show no fluctuations in intensity. How now is it possible to have continuous illumination in the spectroscope when there are moments at which the slit is in darkness? This question has nothing to do with the persistence of vision: the illumination must be continuous regardless of any physiological peculiarities of the eye. The difficulty is only apparent, as can be seen by the following considerations. An effect at the slit occupying an infinitesimal of time, is by the action of the grating or prism spread out over a finite interval of time when it reaches the eye. This will be better understood after reading the chapter on the Nature of White Light. In the case of the grating there is no difficulty, since the disturbances from the different grating elements, resulting from a *single* disturbance at the slit, reach the eye in succession. This being true it is obvious that what occurs at the eye at a given *instant* is the result of disturbances lasting for a finite time at the slit. There is thus no trouble about having a continuous disturbance at the eye, where there is a discontinuous disturbance at the slit.

From our knowledge of the magnitude of the separation of the components of the Zeeman doublet, we can calculate how many to-and-fro excursions the electron makes while the path makes one complete revolution. The separation amounts at best to only  $\frac{1}{12}$  of the distance between  $D_1$  and  $D_2$ , which means that each component only moves  $\frac{1}{24}$  of this distance from the original position of the line. Taking the frequency differences for the  $D$  lines, and dividing this by 24, gives us the frequency difference corresponding to the shift. If  $n$  is the normal frequency of the light, and the plane of polarization

turns  $N$  times per second, the two component circular vibrations, into which it can be decomposed, have frequencies  $(n + N)$  and  $(n - N)$  (compare Righi's Experiment, page 280). The frequency difference in the above case turns out to be roughly 20,000,000,000, and if we divide this number into the original frequency, we shall obtain a number representing the number of to-and-fro excursions made by the electron while its path turns through one complete revolution. This number turns out to be 30,000, which gives us the number of loops in the star-shaped orbit previously figured.

**Different Types of the Zeeman Effect.**—The circularly-polarized doublet seen along the lines of magnetic force, and the plane-polarized triplet seen in a direction perpendicular to the lines of force, may be regarded as the normal types. Many lines, however, behave quite differently. The  $D$  lines of sodium, when examined at right angles to the field, were found by Cornu, and independently by Preston, to act in a curious manner.  $D_1$  gave instead of a triplet a quadruplet, the two central components being polarized in the same plane as the central line of the normal triplet, while  $D_2$  gave a sextet, the two central components of which were plane polarized in a similar manner, while the four outer components were plane polarized like the outer lines of a triplet. Preston was at first inclined to regard the effect as due to reversal, for the sodium lines reverse, *i.e.* show dark lines down their centers, upon the slightest provocation.

The fact, however, that the components separate as the intensity of the magnetic field is increased, combined with the fact that the general appearance differs markedly from that observed when the lines reverse, caused him to abandon this theory and adopt the hypothesis that the orbit of the electrons is subjected to perturbations, the origin of which is not very expressly stated. If the orbit suffers in addition to a precessional motion around the lines of force, an apsidal motion, or rotation in its own plane, this second motion will result in the emission of two different wave-lengths. It is possible also that the magnetic field exerts a force upon the electrons which tends to bring the orbits into the same plane, the inertia of the system causing it to vibrate, like a compass needle, with small oscillations about a position of equilibrium. This oscillatory motion combined with the precession of the orbit will give rise to the quartet observed in the case of  $D_1$ .

A most singular observation was made by Becquerel and Deslandres, and subsequently by Ames, Earhart and Reese, in the case of some of the iron lines. These lines broke up into triplets, which, however, exhibited reversed polarization, *i.e.* the vibrations in the central component were perpendicular to the lines of magnetic force, while those of the outer components were parallel to it. Preston has attempted to explain this by assuming it to be a case of abnormal separation of the two central components of a quartet, which have passed over and beyond the outer components. This explanation is hardly satisfactory, but no better one has been offered up to the present time.

**Magnitude of the Separation.**—Spectroscopic recognition of the Zeeman phenomenon requires an instrument of high resolving power. The most convenient form is undoubtedly the echelon grating devised by Michelson; it can be adjusted in a few minutes, gives a large amount

of light and is fairly compact. The author has seen the effect without difficulty with an echelon improvised from four interferometer plates which were mounted on the table of a spectrometer with a step-width of 1 mm. A small screen of card-board with a rectangular opening 5 mms. wide and 2 cms. high limited the beam coming from the collimator. This was mounted in such a position as to have a strip 1 mm. wide along the edge of the grating, five interfering beams, with high relative retardations being obtained in this way. The spectrometer was illuminated with the light from a mercury vacuum tube placed between the poles of an electro-magnet and a small direct vision prism placed between the echelon and the telescope to separate the mercury lines. The splitting up of the green line on exciting the magnet was easily seen with this improvised apparatus. A helium tube is equally satisfactory and less troublesome, as it requires no heating. The sodium flame is unsatisfactory, owing to the small distance between the  $D$  lines.

The results obtained with the echelon are not, however, easy to interpret, and the proximity of the spectra of other orders gives trouble, when anything more than a qualitative experiment is to be made. For accurate quantitative work the concave grating is undoubtedly the best type of instrument to use.

The magnitude of the separation, even for very intense fields, is very slight. Zeeman concluded from measurements of photographs that for a field-strength of 10,000 c.g.s. units the distance between the outer components of the  $D_1$  quadruplet amounted to  $\frac{1}{13}$  of the distance between  $D_1$  and  $D_2$ . The same separation will of course be found in the case of the two circular components seen along the lines of force.

The magnetic separation of the different lines in the spectrum of a given substance is by no means the same. This is of course to be expected, for the effect of the field on the motion of the electron will depend upon the ratio of its charge to its mass. Certain lines in the spectrum are, however, supposed to have a common origin, and theory shows that in this case the magnetic separation will decrease with the wave-length. If we represent the separation by  $d\lambda$  we should have the relation  $\frac{d\lambda}{\lambda^2} = \text{const.}$ , which has been verified for a number of lines

by Preston, Reese, Kent and others. This relation holds, however, only for lines which belong to the same series.

We should expect the magnitude of the separation to vary directly with the field-strength  $H$ . Any departure would be difficult to reconcile with the theory of the phenomenon. Kent and Reese were of the opinion that the separation did not increase in proportion to the field-strength, the linear relation only holding up to values of  $H$  in the neighborhood of 15,000 c.g.s. units. Above this point the separation becomes less than the required amount. The departure is very small, however, and more recent work by Runge and Paschen make it appear probable that if high enough resolving powers are used, the separation is strictly proportional to the strength of the field.

**Study of the Zeeman Effect without a Spectroscope.**—A very convenient and simple method of showing the change in the wave-

length is that employed by Cotton<sup>1</sup> and Koenig.<sup>2</sup> The spectroscope is dispensed with entirely, and an absorbing flame put in its place. As is well known the sodium flame has the power of absorbing strongly, radiations of the same wave-length as those which it emits. If we place a bright sodium flame between the poles of an electro-magnet and in front of it a second sodium flame, preferably a less luminous flame, such as can be obtained by burning a jet of illuminating gas at the tip of a piece of soft glass tubing drawn down to a point, the light which is emitted by the first flame will be partially absorbed by the second, which will appear dark in consequence. If the second flame contains much sodium, it may happen that its edges only appear dark.

If now the magnet is excited, the wave-lengths emitted by the first flame are changed, and the second flame, no longer able to absorb them, brightens up in consequence, or, to be more exact, it no longer appears darker than the background. It is best to try the experiment first along the lines of force, using perforated pole-pieces, and placing the second flame close to one end of the magnet. The phenomenon is less marked in a direction perpendicular to the lines of force, since in this case the emission line breaks up into a triplet, and the second flame is capable of absorbing completely the middle component, the period of which is unchanged. This central component is, however, plane polarized and can be cut out by means of a Nicol prism, under which conditions the brightening up of the flame is quite as conspicuous as in the direction of the lines of force. We may modify the experiment by placing the absorbing flame in the magnetic field. In this case the vertical vibrations will predominate in the beam coming from the two flames, for the flame in the field absorbs only the horizontal vibrations coming from the other flame. The presence of polarized light can be easily recognized by means of a Savart plate. This experiment was performed by Lorentz in connection with a study of an effect found by Egoroff and Georgiewsky, which will be next considered.

**Partial Polarization of the Light emitted by a Flame in a Magnetic Field.**—We have seen that the light emitted in a direction perpendicular



FIG. 277.

to the lines of force consists of three different sets of vibrations, one polarized parallel, the other perpendicular to the field. If the total amount of light in the two outer components of the triplet is greater or less than the amount in the central component, the light should exhibit traces of plane polarization. We may regard the revolving electrons as circular or elliptical convection currents, each one accompanied by its own magnetic field, and it would seem therefore as if the external field might well exercise a directive force upon the orbits, orienting them in the same manner as the hypothetical molecular currents are supposed to be oriented in Ampere's theory of magnetism. If this were the case we should expect the light emitted in a direction perpendicular to

<sup>1</sup> *Comptes Rendus*, cxxv., p. 865.

<sup>2</sup> *Wied. Ann.*, 63, p. 268.

the lines of force to be more or less completely plane polarized, as will be readily understood by reference to Fig. 277, in which the electronic orbits have all been brought into the same plane. Though traces of polarization have been detected by Egoroff and Georgiewsky it is very doubtful if the phenomenon is to be referred to this action. In the first place an orientation such as we have assumed would result in an emission of circularly-polarized light of a single type and wavelength along the lines of force, instead of the two equal components of different periodicities and opposite directions of revolution. How then are we to explain the presence of plane-polarized light in the beam emitted perpendicularly to the field? The experiments above referred to showed that fully 11% of the emitted light was plane polarized with its vibration direction perpendicular to the field, just as it should be if a partial orientation, as assumed above, had taken place. But it was found that the phenomenon only appeared in the case of luminous vapors which showed strong absorption, in other words it only occurred in the case of lines easily reversed. This appeared to indicate that absorption played some part in the production of the polarization.

As we have seen, we may have plane polarization produced when we have two flames, one in the magnetic field, the other outside. The same thing can take place if we have a single flame in a non-homogeneous field. Cotton, however, found that even in homogeneous fields the emitted light was partially plane-polarized. The complete explanation was given by Lorentz,<sup>1</sup> who showed that if the intensity of the outer components of the triplet is only one-half that of the inner (which is to be expected if no orientation has taken place), then the absorption co-efficient for the vertical vibrations is only half as great as for the horizontal. If the original intensity of the central component is  $I_1$  and the intensities of the outer components are  $I_2$  and  $I_3$ , we have

$$I_2 + I_3 = I_1.$$

The outer absorbing mantle of the flame is in the magnetic field also, and as the absorbing power is proportional to the emissive power, it will absorb the central component (horizontal vibrations) more powerfully than the outer components (vertical vibrations.) We have, therefore, after absorption

$$I_2 + I_3 > I_1,$$

or, since  $I_2$  and  $I_3$  consist of vertical vibrations, we are able to detect traces of polarization.

Egoroff's experiment therefore cannot be regarded as evidence of an orientation of the orbits. If polarization could be found in the case of some gas which does not exhibit the phenomenon of reversal, for example helium or hydrogen, it would be pretty certain evidence that the orbits of the electrons were brought to a greater or less extent into a plane perpendicular to the lines of force.

**Theory of Magnetic Rotation.**—The theory of magnetic-rotatory dispersion has not been as completely worked out as the theory of ordinary dispersion, and very few experimental verifications of the formulae have been made.

<sup>1</sup> *Rapp. près au Congrès Intern. de Phys.* (Paris, 1900), vol. iii., page 29.

Drude in his *Lehrbuch der Optik* has deduced two different formulae expressing the relation between wave-length and magnetic rotation in terms of the wave-length corresponding to the free period of the electron and certain constants. Two different hypotheses are made to account for the rotation. The first assumes the existence of molecular currents, as conceived by Ampere and Weber, to explain magnetism and diamagnetism. In paramagnetic substances these currents are already in existence, the action of a magnetic field merely orienting them so that their lines of magnetic force are superposed on the exciting field. In paramagnetic substances the currents are induced within the molecule (Weber's theory) as soon as the substance is brought into a magnetic field. These currents will persist as long as the body remains in the field, for the molecular circuits are assumed to be devoid of resistance, and they will be in such a direction as to be repelled by the pole from which spring the lines of force which have brought them into existence.

We can best think of these currents as revolving electrons, which have, in the case of diamagnetic substances, been set in motion by the inductive action of the field, and which will continue to rotate until the field is destroyed, when the opposite inductive action brings them to rest. When now these revolving electrons are acted upon by the periodic electric forces of the light-waves, the points around which they rotate will suffer periodic displacements, and the lines of magnetic force resulting from the whirling electrons will be moved back and forth with the molecular currents.

The result of this is that the moving electron not only contributes to the electric current density  $j_x$  in the fundamental equation (as is the case in ordinary dispersion), but also to the density of the magnetic current  $s_x$ . The difference is seen to lie in the fact that ordinarily the electron is not accompanied by a magnetic field, whereas in the present case it is.

The assumption of these molecular currents is a natural hypothesis to make as a basis for a theory of magnetic rotation, since it has been found useful in explaining the phenomena of magnetism and diamagnetism of substances, and the magneto-optical properties of matter are without doubt closely related to their magnetic properties. The hypothesis, however, leads us to equations which, while they account for the rotation of the plane of polarization, call for rotations of opposite sign on opposite sides of an absorption band, that is the magnetic dispersion curve expressed by the final formula, has the same general form as the ordinary dispersion curve, with oppositely-directed branches at the edge of the absorption band. While very little is known regarding the magnetic-rotatory dispersion in the vicinity of absorption bands, it is certain that, in one case at least, the direction of the rotation is the same on opposite sides of the band. This one case, namely that of sodium vapor, shows us that our hypothesis of molecular currents will not account for the rotation in all media. We must make some hypothesis which leads us to equations calling for rotations of the same sign on opposite sides of the absorption band.

The second hypothesis is that of the "Hall effect."

An electric current or a moving electron, is subjected in a magnetic

field to a deflecting force which is at right angles to the direction of the current and the lines of force. In a magnetic field, then, an electron which is thrown into vibration by light-waves will experience a force which will be proportional to the velocity with which the electron is moving, and this force must be added to the forces which we have already considered in forming the differential equation which expresses its motion. As we shall see, this method of attacking the problem leads us to a rotatory-dispersion formula which agrees with the results found in the case of sodium vapor.

While it is possible or even probable that, in the majority of cases, both of the above mentioned causes may be operative, it is best for the sake of simplicity to treat them separately, and we will begin with the hypothesis of the molecular currents. If  $e$  is the charge of the electron, and  $T$  its time of revolution around a point  $P$ , the intensity of the molecular current (convection current) will be  $i = \frac{e}{T}$ . The periodic electrical forces of the light-waves acting on the revolving electron will shift the point  $P$  around which it revolves, waving it back and forth in the same way that we considered the electron moved in the treatment of dispersion, and we shall consider its motion represented by the same differential equation, viz.:

$$m \frac{\partial^2 \xi}{\partial t^2} = eX - \frac{4\pi e^2}{\theta} \xi - re^2 \frac{\partial \xi}{\partial t}.$$

The revolving electron is accompanied by its own magnetic field, and its two-and-fro motion therefore contributes to the magnetic current (change in density of lines of force) as well as to the electric current.

If the orbits of the electrons are all perpendicular to the lines of force of the field, and if  $N$  be the number in unit volume, the number of lines of force due to the molecular currents per unit surface can be easily shown to be  $M = 4\pi iq \frac{N}{c}$ , in which  $q$  is the cross-section of the orbit of the electron, and  $i$  the strength of the convection current due to its motion, defined electro-statically. The components of  $M$  along the three axes of coordinates are

$$\alpha_1 = \frac{4\pi}{c} iqN \cos(Kx), \quad \beta_1 = \frac{4\pi}{c} iqN \cos(Ky), \quad \gamma_1 = \frac{4\pi}{c} iqN \cos(Kz).$$

The fundamental Maxwell equations are

$$\frac{4\pi}{c} j_x = \frac{\partial \gamma}{\partial y} - \frac{\partial \beta}{\partial z}, \text{ etc.}, \quad \frac{4\pi}{c} s_x = \frac{\partial Y}{\partial z} - \frac{\partial Z}{\partial y}, \text{ etc.}$$

In our treatment of dispersion it will be remembered that to our  $x$  component of current  $j_x = \frac{1}{4\pi} \frac{\partial X}{\partial t}$  (displacement current in the ether) we added the components of convection currents represented by the motion of the electrons, the complete expression being

$$j_x = \frac{1}{4\pi} \frac{\partial X}{\partial t} + eN \frac{\partial \xi}{\partial t},$$

if but a single type of electron is considered. The expression for the magnetic current  $s_z$  in the previous cases required no modification. In the present case, however, we must take into account the change in the density of the magnetic lines of force which results from the oscillatory motion of the revolving electron. We require now an expression for  $4\pi s_z$ .

In a small rectangle  $dydz$  the change in the density of the magnetic lines of force due to the light-wave is represented by  $dydz \frac{\partial a}{\partial t}$ . The change due to the moving-field of the electron can be found as follows:

Consider first the lines  $a_1$  parallel to the  $x$  axis ( $x$  components). Let  $\xi, \eta, \zeta$  be the components of displacement of  $P$ , the center of the electron's orbit.

In unit time the number of lines of force which enter the rectangle through the side  $a$  is given by  $\left(a_1 \frac{\partial \eta}{\partial t}\right)_a dz$ , while the number which leave it through the side  $c$  is  $\left(a_1 \frac{\partial \eta}{\partial t}\right)_c dz$ , the suffixes  $a$  and  $c$  indicating that we are to assign to the terms  $a_1 \frac{\partial \eta}{\partial t}$  the values which they have at the two sides  $a$  and  $c$ .

$$\text{Now} \quad \left(a_1 \frac{\partial \eta}{\partial t}\right)_c = \left(a_1 \frac{\partial \eta}{\partial t}\right)_a + \frac{\partial}{\partial y} \left(a_1 \frac{\partial \eta}{\partial t}\right) \cdot dy.$$

The increase in the number of lines of force will be represented by the difference between the number which enter through side  $a$ , and the number which leave through side  $c$ , i.e.  $-dydz \frac{\partial}{\partial y} \left(a_1 \frac{\partial \eta}{\partial t}\right)$ . This formula holds for non-homogeneous media also. In the case of homogeneous media  $a_1, \beta_1$  and  $\gamma_1$  are constant. Similarly, the  $a_1$  components entering and leaving through the sides  $b$  and  $d$  contribute the part

$$-dydz \frac{\partial}{\partial z} \left(a_1 \frac{\partial \xi}{\partial t}\right).$$

The lines  $\beta_1$  parallel to the  $y$  axis can only enter and leave the rectangle through the sides  $a$  and  $c$ . If more pass through  $c$  than pass through  $a$  by a motion  $\xi$  of the point  $P$ , it will be the equivalent of a rotation of the  $\beta_1$  lines about the  $z$  axis, the rotation being positive when the  $+y$  direction rotates into the position of the  $+x$  direction. The effect of this rotation will be to alter the number of lines cutting through the rectangle, and the amount will be given by

$$\left(\beta_1 \frac{\partial \xi}{\partial t}\right)_c dz - \left(\beta_1 \frac{\partial \xi}{\partial t}\right)_a dz,$$

or, since  $\left(\beta_1 \frac{\partial \xi}{\partial t}\right)_c = \left(\beta_1 \frac{\partial \xi}{\partial t}\right)_a + dy \frac{\partial}{\partial y} \left(\beta_1 \frac{\partial \xi}{\partial t}\right)$ , the part of the magnetic flux contributed by the rotation of  $\beta$ , will be  $+dydz \frac{\partial}{\partial y} \left(\beta_1 \frac{\partial \xi}{\partial t}\right)$ .

Similarly, the  $\gamma_1$  components contribute the part  $+dydz \frac{\partial}{\partial z} \left(\gamma_1 \frac{\partial \xi}{\partial t}\right)$ .

Adding up the partial effects gives us

$$dydz \left\{ \frac{\partial \alpha}{\partial t} - \frac{\partial}{\partial y} \left( \alpha_1 \frac{\partial \eta}{\partial t} \right) - \frac{\partial}{\partial z} \left( \alpha_1 \frac{\partial \xi}{\partial t} \right) + \frac{\partial}{\partial y} \left( \beta_1 \frac{\partial \xi}{\partial t} \right) + \frac{\partial}{\partial z} \left( \gamma_1 \frac{\partial \xi}{\partial t} \right) \right\}.$$

The change in the number of the lines of force which cut unit area (normal to  $x$  axis) in unit time is therefore

$$4\pi s_x = \frac{\partial}{\partial t} \left\{ \alpha + \frac{\partial}{\partial z} (\gamma_1 \xi - \alpha_1 \xi) - \frac{\partial}{\partial y} (\alpha_1 \eta - \beta_1 \xi) \right\},$$

since for a constant external field  $\alpha_1, \beta_1, \gamma_1$  do not vary with  $t$ .

Strictly speaking, the current density will be modified in a very complicated manner by the to-and-fro motions of the revolving electron. If, however, the time of rotation bears no fixed ratio to the period of the light-wave, the average effect can be obtained by considering the  $\xi, \eta, \xi$  components of the motion of the point  $P$ , *i.e.* the motion of revolution can be disregarded.

For the current density, we have as before

$$(5) \quad j_x = \frac{1}{4\pi} \frac{\partial X}{\partial t} + eN \frac{\partial \xi}{\partial t},$$

and for the equation of motion of the point  $P$ ,

$$(6) \quad m \frac{\partial^2 \xi}{\partial t^2} = eX - \frac{4\pi e^2}{\theta} \xi - re^2 \frac{\partial \xi}{\partial t},$$

if  $P$  has a position of equilibrium about which it vibrates (as is the case in insulating media).

If the ions are free to move indefinitely under the action of a steady electric force, as is the case in metals, we must use the equation which we formed when considering the absorption and dispersion of metals, namely:

$$(7) \quad m \frac{\partial^2 \xi}{\partial t^2} = eX - re^2 \frac{\partial \xi}{\partial t}.$$

For periodic disturbances, in which  $X$  and  $\xi$  are proportional to  $e^{i\frac{t}{\tau}}$ , we have, as before, remembering that  $\frac{\partial \xi}{\partial t} = \frac{i}{\tau} \xi$  and  $X = -i\tau \frac{\partial X}{\partial t}$  (see pages 333, 364),

$$(8) \quad \text{from (6), } e \frac{\partial \xi}{\partial t} \left\{ 1 + i \frac{r\theta}{4\pi\tau} - \frac{m\theta}{4\pi e^2} \cdot \frac{1}{\tau^2} \right\} = \frac{\theta}{4\pi} \frac{\partial X}{\partial t},$$

$$(9) \quad \text{from (7), } e \frac{\partial \xi}{\partial t} \left( r + \frac{i}{\tau} \frac{m}{e^2} \right) = X = -i\tau \frac{\partial X}{\partial t}.$$

Writing, as before,  $\frac{r\theta}{4\pi} = a$ ,  $\frac{m\theta}{4\pi e^2} = b = \tau_1^2$ , and  $\frac{m}{e^2} = m'$ , we have, from (5), for insulators,

$$(11) \quad j_x = \frac{1}{4\pi} \frac{\partial X}{\partial t} \left\{ 1 + \frac{N\theta}{1 + i \frac{a}{\tau} - \frac{b}{\tau^2}} \right\},$$

and for metals, (12)  $j_x = \frac{1}{4\pi} \frac{\partial X}{\partial t} \left\{ 1 + \frac{4\pi\tau N}{ir - \frac{m'}{\tau}} \right\}.$

In either case we can write

$$(13) \quad j_x = \frac{\epsilon'}{4\pi} \frac{\partial X}{\partial t}, \quad j_y = \frac{\epsilon'}{4\pi} \frac{\partial Y}{\partial t}, \quad j_z = \frac{\epsilon'}{4\pi} \frac{\partial Z}{\partial t},$$

in which  $\epsilon'$  stands for the complex dielectric constant, depending on  $\tau$ , i.e. the quantity in the brackets.

We are now in a position to deduce expressions for  $\gamma_1 \xi$ ,  $a_1 \xi$ , etc., which occur in equation (4).

From equations (1) and (2) we have

$$\gamma_1 = 4\pi e N \frac{q}{cT} \cos(Kz), \quad \gamma_1 \xi = 4\pi e N \frac{q}{cT} \cos(Kz) \frac{\partial \xi}{\partial t} \frac{\tau}{i}, \quad \text{since } \xi = \frac{\partial \xi}{\partial t} \frac{\tau}{i}.$$

$$\text{By equations (5), (11)} \quad \frac{1}{4\pi} \frac{\partial X}{\partial t} \frac{N\theta}{1 + i \frac{a}{\tau} - \frac{b}{\tau^2}} = e N \frac{\partial \xi}{\partial t}.$$

$$\text{Substituting} \quad \gamma_1 \xi = \frac{q}{cT} \cos(Kz) \frac{\partial X}{\partial t} \frac{\tau}{i} \frac{N\theta}{1 + i \frac{a}{\tau} - \frac{b}{\tau^2}},$$

$$(14) \text{ (for insulators)} \quad \gamma_1 \xi = \frac{N\theta}{1 + i \frac{a}{\tau} - \frac{b}{\tau^2}} \frac{q}{cT} \cos(Kz) X, \quad \text{since } X = -i\tau \frac{\partial X}{\partial t}.$$

Similarly, (15)

$$\text{(for conductors)} \quad \gamma_1 \xi = \frac{4\pi\tau N}{ir - \frac{m}{\tau}} \frac{q}{cT} \cos(Kz) X.$$

In both cases we can write

$$(16) \quad \gamma_1 \xi = v \cos(Kz) X,$$

in which  $v$  is a complex quantity depending on  $\tau$ .

Writing  $v \cos(Kx) = v_x$ ,  $v \cos(Ky) = v_y$ , etc., we have the fundamental equations  $\frac{4\pi}{c} j_x = \frac{\partial \gamma}{\partial y} - \frac{\partial \beta}{\partial z}$ , etc., and  $\frac{4\pi}{c} s_x = \frac{\partial Y}{\partial z} - \frac{\partial Z}{\partial y}$ , etc., in the following form:

$$(17) \quad \frac{\epsilon'}{c} \frac{\partial X}{\partial t} = \frac{\partial \gamma}{\partial y} - \frac{\partial \beta}{\partial z}, \quad \frac{\epsilon'}{c} = \frac{\partial a}{\partial z} - \frac{\partial \gamma}{\partial x}, \quad \frac{\epsilon'}{c} = \frac{\partial \beta}{\partial x} - \frac{\partial a}{\partial y}.$$

$$(18) \quad \begin{cases} \frac{1}{c} \frac{\partial}{\partial t} \left\{ a + \frac{\partial}{\partial z} (v_x X - v_z Z) - \frac{\partial}{\partial y} (v_x Y - v_y Z) \right\} = \frac{\partial Y}{\partial z} - \frac{\partial Z}{\partial y}, \\ \frac{1}{c} \frac{\partial}{\partial t} \left\{ \beta + \frac{\partial}{\partial x} (v_x Y - v_y X) - \frac{\partial}{\partial z} (v_y Z - v_z Y) \right\} = \frac{\partial Z}{\partial x} - \frac{\partial X}{\partial z}, \\ \frac{1}{c} \frac{\partial}{\partial t} \left\{ \gamma + \frac{\partial}{\partial y} (v_x Z - v_z Y) - \frac{\partial}{\partial x} (v_x X - v_z Z) \right\} = \frac{\partial X}{\partial y} - \frac{\partial Y}{\partial x}. \end{cases}$$

If both conducting and non-conducting electrons are present, we have

$$(19) \quad \epsilon' = 1 + \underbrace{\frac{N\theta}{1 + i \frac{a}{\tau} - \frac{b}{\tau^2}}}_{\text{Non-conducting}} + 4\pi\tau \underbrace{\frac{N}{ir - \frac{m}{\tau}}}_{\text{Conducting}}.$$

$$(20) \quad \nu = \frac{1}{c} \frac{N\theta}{1 + i \frac{a}{\tau} - \frac{b}{\tau^2}} \frac{q}{T} + \frac{4\pi\tau}{c} \frac{N}{ir - \frac{m}{\tau}} \frac{q}{T}.$$

Non-conducting. Conducting.

**The Magnetic Rotation of the Plane of Polarization.**—Let the  $z$  axis be parallel to lines of force in the magnetic field, and consider plane waves travelling parallel to it. Then  $X$ ,  $Y$ ,  $\alpha$  and  $\beta$  depend only on  $z$  and  $t$ , and  $Z = \gamma = 0$ .

Furthermore,  $v_x = v_y = 0$ ,  $v_z = v$ .

Our fundamental equations (17), (18) now reduce to

$$(22) \quad \frac{\epsilon'}{c} \frac{\partial X}{\partial t} = -\frac{\partial \beta}{\partial z}, \quad \frac{\epsilon'}{c} \frac{\partial Y}{\partial t} = \frac{\partial \alpha}{\partial z}$$

and

$$(23) \quad \begin{aligned} \left\{ \frac{1}{c} \frac{\partial}{\partial t} \left( \alpha + v \frac{\partial X}{\partial z} \right) \right\} &= \frac{\partial Y}{\partial z} \\ \left\{ \frac{1}{c} \frac{\partial}{\partial t} \left( \beta + v \frac{\partial Y}{\partial z} \right) \right\} &= -\frac{\partial X}{\partial z}. \end{aligned}$$

Differentiating (22) with respect to  $t$  and (23) with respect to  $z$ , and substituting in the former the values of  $\frac{\partial^2 \alpha}{\partial t \partial z}$  given by the latter, we obtain at once,

$$(24) \quad \begin{cases} \frac{\epsilon'}{c^2} \frac{\partial^2 X}{\partial t^2} = \frac{\partial^2 X}{\partial z^2} + \frac{v}{c} \frac{\partial^3 Y}{\partial t \partial z^2} \\ \frac{\epsilon'}{c^2} \frac{\partial^2 Y}{\partial t^2} = \frac{\partial^2 Y}{\partial z^2} - \frac{v}{c} \frac{\partial^3 X}{\partial t \partial z^2} \end{cases}$$

To integrate these we write, as in the case of Nat. rotation (page 391),

$$X = M e^{\frac{i}{\tau}(t - \rho z)}, \quad Y = N e^{\frac{i}{\tau}(t - \rho z)}.$$

Substituting these values in (24) gives us (compare with treatment of natural rotation),

$$\epsilon' M = p^2 c^2 \left( M + i \frac{v}{c\tau} N \right),$$

$$\epsilon' N = p^2 c^2 \left( N - i \frac{v}{c\tau} M \right),$$

which can be satisfied by

$$(26) \quad p^2 c^2 \left( 1 + \frac{v}{c\tau} \right) = \epsilon', \quad M' = iN$$

and

$$(27) \quad p^2 c^2 \left( 1 + \frac{v}{c\tau} \right) = \epsilon', \quad M = -iN.$$

We obtain the physical interpretation of these equations in the same manner as in the case of natural rotation.

Two circularly-polarized waves are indicated, the first left handed, with  $p$  given by

$$(28) \quad p'c = \sqrt{\frac{\epsilon'}{1 + \frac{v}{c\tau}}}$$

the second right handed, with  $p$  given by

$$(29) \quad p''c = \sqrt{\frac{\epsilon'}{1 - \frac{v}{c\tau}}}$$

$p'$  and  $p''$  being the reciprocals of the velocities of propagation.

The superposition of the two waves gives us, if we consider  $\epsilon'$  and  $v$  (also  $p'$  and  $p''$ ) real, a plane-polarized vibration, the plane rotating as the disturbance travels along the  $z$  axis, through an angle given by

$$(30) \quad \delta = \frac{z}{l} \frac{p'' - p}{2}.$$

If  $\frac{v}{c\tau}$  is small in comparison to 1, as is, in general, the case, we can write for the above

$$(31) \quad \delta = \frac{v\sqrt{\epsilon'}}{2c^2\tau^2} z.$$

**Rotatory Dispersion.**—This formula gives us the means of expressing the rotation in terms of the wave-length and certain constants of the medium. In the chapter on Dispersion (page 336), we have seen that we can write

$$(32) \quad \epsilon' = 1 + \sum \frac{\theta N}{1 - \left(\frac{\lambda_1}{\lambda}\right)^2}.$$

We have written  $v$  as an abbreviation for

$$\frac{\theta N}{1 + \frac{ia}{\tau} - \frac{b}{\lambda^2}} \cdot \frac{q}{cT} \quad \text{or} \quad \frac{\theta N}{1 - \left(\frac{\lambda_1}{\lambda}\right)^2} \cdot \frac{q}{cT}$$

for transparent media, in which case we neglect  $i\frac{a}{\tau}$ .

In the case of transparent media we have as our dispersion formula

$$(33) \quad n^2 = a + \frac{b\lambda^2}{\lambda^2 - \lambda_1^2},$$

and since  $v$  only differs from  $n^2$  by a constant quantity  $\frac{q}{cT}$ , we can write

$$(34) \quad v = a' + \frac{b'\lambda^2}{\lambda^2 - \lambda_1^2}.$$

Substituting in our formula for  $\delta$  the value  $\lambda = Tc$ , and remembering that  $\tau = \frac{2\pi}{T}$ , we have for the rotation in terms of the wave-length of the light and the refractive index  $n$ ,

$$\delta = \frac{2\pi^2\nu\sqrt{\epsilon'}}{\lambda^2} z = \frac{2\pi^2\nu n}{\lambda^2} z.$$

Substituting in this equation our value for  $\nu$  given by eq. (34) (letting  $2\pi^2z=1$ ), we find

$$\delta = n \left( \frac{a'}{\lambda^2} + \frac{b'}{\lambda^2 - \lambda_1^2} \right).$$

This formula differs from the dispersion formula for substances which show natural rotation in the occurrence of the factor  $n$ . While the formula represents fairly well the magnetic rotation of bisulphide of carbon and creosote in the visible region, it can scarcely be said that experimental proof of the correctness of the fundamental hypothesis has been given, for a totally different formula deduced from the Hall-effect hypothesis represents the rotatory dispersion of these substances nearly as well. Moreover, since the term  $\frac{b'}{\lambda^2 - \lambda_1^2}$  changes sign as we cross the absorption band, we must have a high value of the rotation on the red side of the band and a low value on the violet. In other words, anomalous rotatory dispersion is a necessary consequence of the formula. The results of Schmauss, which have been alluded to, appear to be in accord with the theory, and though serious objections have been raised against the work, which have never been answered, confirmatory results obtained by the author in the case of a salt of didymium, make it seem probable that the effect exists. We know, however, that in the case of sodium vapor at least, the rotations have high values and similar signs on opposite sides of the absorption band.

**Hypothesis of the Hall-Effect**—If an electron is set in motion by light-waves in a magnetic field it will experience a force which acts at right angles to its direction of motion and the direction of the magnetic lines of force. This force will be proportional to the velocity with which the electron is moving, and to the strength of the field. It will be zero when the electron reaches its turning point, and will attain its maximum value at the moment when the position of equilibrium is passed. If the charge of the electron is  $e$ , and if it moves a distance  $d\eta$  in time  $dt$  along the  $y$  axis the force acting on it in the direction of the  $x$  axis will be represented by

$$K_x = \frac{e}{c} \frac{\partial \eta}{\partial t} h_z,$$

in which  $h_z$  is the strength of the magnetic field, which we will assume parallel to the  $z$  axis. If the magnetic field is parallel to the  $y$  axis, and the electron moves along the  $z$  axis, the force will be

$$K_x = -\frac{e}{c} \frac{\partial \xi}{\partial t} h_y.$$

These forces are to be added to the right-hand member of the equation of motion of the electron,

$$(42) \quad m^2 \frac{\partial^2 \xi}{\partial t^2} = eX - \frac{4\pi e^2}{\theta} \xi - re^2 \frac{\partial \xi}{\partial t} + \frac{e}{c} \left( \frac{\partial \eta}{\partial t} h_z - \frac{\partial \xi}{\partial t} h_y \right).$$

The equations for the electric and magnetic current-densities are as before (Dispersion Theory)

$$4\pi j_x = \frac{\partial}{\partial t} (X + 4\pi \Sigma_e N \xi), \text{ etc.},$$

$$4\pi s_x = \frac{\partial a}{\partial t}, \text{ etc.}$$

For periodic disturbances we write (42) (see page 334)

$$e\xi \left( 1 + i \frac{a}{\tau} - \frac{b}{\tau^2} \right) - \frac{i\theta}{4\pi c\tau} (\eta h_x - \zeta h_y) = \frac{\theta}{4\pi} X.$$

If we take the  $z$ -axis parallel to the lines of force,  $h_x = h_y = 0$ , and write

$$1 + i \frac{a}{\tau} - \frac{b}{\tau^2} = \Theta, \quad \frac{\theta}{4\pi c\tau e} h = \Phi,$$

we have

$$e\xi\Theta - ie\eta\Phi = \frac{\theta}{4\pi} X,$$

$$e\eta\Theta + ie\xi\Phi = \frac{\theta}{4\pi} Y,$$

$$e\xi\Theta = \frac{\theta}{4\pi} Z.$$

Multiplying the first two equations first by  $\Theta$  and then by  $\Phi$ , and then multiplying the second pair of equations thus obtained by  $i$ , and solving for  $\xi$ ,  $\eta$  and  $\zeta$ , gives us

$$4\pi e\xi(\Theta^2 - \Phi^2) = \theta(\Theta X + i\Phi Y),$$

$$4\pi e\eta(\Theta^2 - \Phi^2) = \theta(\Theta Y - i\Phi X),$$

$$4\pi e\zeta\Theta = \theta Z,$$

which by differentiation and substitution in (44) give

$$4\pi j_x = \frac{\partial X}{\partial t} \left( 1 + \sum \frac{\theta N \Theta}{\Theta^2 - \Phi^2} \right) + i \frac{\partial Y}{\partial t} \sum \frac{\theta N \Phi}{\Theta^2 - \Phi^2},$$

$$4\pi j_y = \frac{\partial Y}{\partial t} \left( 1 + \sum \frac{\theta N \Theta}{\Theta^2 - \Phi^2} \right) - i \frac{\partial X}{\partial t} \sum \frac{\theta N \Phi}{\Theta^2 - \Phi^2},$$

$$4\pi j_z = \frac{\partial Z}{\partial t} \left( 1 + \sum \frac{\theta N}{\Theta} \right),$$

which can be abbreviated by writing  $\epsilon''$  for the term in the parenthesis and  $\nu$  for the second summation :

$$4\pi j_x = \epsilon'' \frac{\partial X}{\partial t} + i\nu \frac{\partial Y}{\partial t},$$

$$4\pi j_y = \epsilon'' \frac{\partial Y}{\partial t} - i\nu \frac{\partial X}{\partial t},$$

$$4\pi j_z = \epsilon' \frac{\partial Z}{\partial t}.$$

**Light Rays Parallel to the Magnetic Field.**—In this case  $\alpha$ ,  $\beta$ ,  $X$  and  $Y$  depend only on  $z$  and  $t$ , and substitution of the above values in the fundamental equations gives us

$$\frac{1}{c} \left( \epsilon'' \frac{\partial X}{\partial t} + i\nu \frac{\partial Y}{\partial t} \right) = -\frac{\partial \beta}{\partial z}, \quad \frac{1}{c} \left( \epsilon'' \frac{\partial Y}{\partial t} - i\nu \frac{\partial X}{\partial t} \right) = \frac{\partial \alpha}{\partial z},$$

$$\frac{1}{c} \frac{\partial \alpha}{\partial t} = \frac{\partial Y}{\partial z}, \quad \frac{1}{c} \frac{\partial \beta}{\partial t} = -\frac{\partial X}{\partial z}, \quad Y = Z = 0.$$

Eliminating  $\alpha$  and  $\beta$  by differentiating the first pair of equations with respect to  $t$  and the second pair with respect to  $z$ ,

$$\epsilon'' \frac{\partial^2 X}{\partial t^2} = \frac{\partial^2 X}{\partial z^2} - \frac{i\nu}{c^2} \frac{\partial^2 Y}{\partial t^2},$$

$$\epsilon'' \frac{\partial^2 Y}{\partial t^2} = \frac{\partial^2 Y}{\partial z^2} + \frac{i\nu}{c^2} \frac{\partial^2 X}{\partial t^2}.$$

Integrating as before by writing

$$X = M e^{i(t-pz)}, \quad Y = N e^{i(t-pz)}$$

gives us  $\epsilon'' M = p^2 c^2 M - i\nu N$ ,  $\epsilon'' N = p^2 c^2 N + i\nu M$ .

Multiplying the first equation by  $N$  and the second by  $M$  gives us  $M = \pm iN$ , and by substitution of these values in the equations, gives

$$p^2 c^2 = \epsilon'' + \nu \quad \text{and} \quad p^2 c^2 = \epsilon'' - \nu.$$

We have seen in the chapter on Optics of Metals that when  $p$  is complex, we can write  $p = \frac{1-i\kappa}{V}$ , in which  $V$  is the velocity in the medium.

In the present case we can therefore write

$$p^2 c^2 = \frac{(1-i\kappa)^2 c^2}{V^2} = n'^2 (1-i\kappa')^2 = \epsilon'' + \nu,$$

$$p^2 c^2 = n''^2 (1-i\kappa'')^2 = \epsilon'' - \nu,$$

in which  $n'$  and  $\kappa'$  represent the refractive index and extinction coefficient for left-handed circular vibrations,  $n''$  and  $\kappa''$  for right-handed vibrations.

Substituting for  $\epsilon''$  and  $\nu$  their equivalents

$$\sum \left( 1 + \frac{\theta N \Theta}{\Theta^2 - \Phi^2} \right) \quad \text{and} \quad \sum \frac{\theta N \Phi}{\Theta^2 - \Phi^2}$$

$$n'^2 (1-i\kappa')^2 = 1 + \sum \frac{\theta N}{\Theta - \Phi}, \quad n''^2 (1-i\kappa'')^2 = 1 + \sum \frac{\theta N}{\Theta + \Phi}.$$

If we limit ourselves to a region of the spectrum outside of the absorption band, we can neglect  $\frac{i\alpha}{\tau}$  and write  $\kappa' = \kappa'' = 0$ , and since  $\Phi$  is small in comparison to  $\Theta$ ,

$$n'^2 = 1 + \sum \frac{\theta N}{\Theta} \left( 1 + \frac{\Phi}{\Theta} \right), \quad n''^2 = 1 + \sum \frac{\theta N}{\Theta} \left( 1 - \frac{\Phi}{\Theta} \right).$$

In the chapter on Natural Rotation we have seen that the rotation of the plane of polarization in terms of the refractive indices for the two circular vibrations is given by

$$\delta = z \frac{\pi}{\lambda} (n'' - n') = z \frac{\pi}{\lambda} \frac{n''^2 - n'^2}{n'' + n'},$$

in which we can write  $2n$  for  $n'' + n'$ ,  $n$  being the mean refractive index for the circular vibrations.

Substituting in  $\delta = z \frac{\pi}{\lambda} \frac{n''^2 - n'^2}{2n}$  the values for  $n''^2$  and  $n'^2$

gives us 
$$\delta = - \frac{\pi}{n} \frac{z}{\lambda} \sum \frac{\theta N \Phi}{\Theta^2} \text{ for the rotation}$$

and 
$$n^2 = 1 + \sum \frac{\theta N}{\Theta} \text{ for the refractive index.}$$

**Magnetic Rotatory Dispersion.**—Substitution of the values of  $\Theta$  and  $\Phi$  in the above equation gives us

$$\delta = - \frac{\pi}{2n} \frac{z}{\lambda^2} h \sum \frac{\theta N}{\left(1 - \frac{b}{\tau^2}\right)^2} \cdot \frac{\theta}{c}.$$

Let  $A = \frac{\pi z}{2} h$ , then 
$$\delta = \frac{A}{n \lambda^2} \sum \frac{\theta^2 N^2}{\left(1 - \frac{b}{\tau^2}\right)^2} \cdot \frac{1}{c N},$$

and remembering that  $b = \frac{m \theta}{4 \pi e^2} = \tau_h^2$  ( $\tau_h$  being the free period of the electron, see page 334), and that  $\theta N = \theta_h'$ ,

$$\delta = \frac{A}{n \lambda^2} \sum \left( \frac{\theta_h'}{1 - \left(\frac{\tau_h}{\tau}\right)^2} \right)^2 \cdot \frac{1}{c N}.$$

Consider two types of electrons to be present, then

$$\delta = \frac{A}{n \lambda^2} \left( \frac{\theta_h'}{1 - \left(\frac{\tau_h}{\tau}\right)^2} \right)^2 \cdot \frac{1}{e_1 N_1} + \left( \frac{\theta_h''}{1 - \left(\frac{\tau_h}{\tau}\right)^2} \right)^2 \cdot \frac{1}{e_2 N_2},$$

and since  $\frac{1}{e_1 N_1} = - \frac{1}{e_2 N_2}$ , there being no free charge, we can combine  $\frac{1}{e N}$  with our constant, if we change the sign of the second parenthesis,

$$\delta = \frac{A}{n \lambda^2} \left\{ \left( \frac{\theta_h'}{1 - \left(\frac{\tau_h}{\tau}\right)^2} \right)^2 - \left( \frac{\theta_h''}{1 - \left(\frac{\tau_h}{\tau}\right)^2} \right)^2 \right\}.$$

If we now consider the second parenthesis as representing the effect of remote ultra-violet electrons, for which  $\tau_h'$  is very small in comparison

to  $\tau$ , the term reduces to  $(\theta_a'')^2$ , as we showed in the chapter on Dispersion ;

$$\therefore \delta = \frac{A_1}{n\lambda^2} \left\{ \left( \frac{\theta_a'}{1 - \left(\frac{\tau}{\lambda}\right)^2} \right)^2 - (\theta_a'')^2 \right\},$$

and substituting  $\lambda$  for  $\tau$ ,

$$\delta = \frac{A_1}{n\lambda^2} \left\{ \left( \frac{\theta_a'\lambda^2}{\lambda^2 - \lambda_1^2} \right)^2 - (\theta_a'')^2 \right\}.$$

Squaring the parentheses, dividing  $\lambda^2$ , and writing  $B = A_1(\theta_a')^2$  and  $C = A_1(\theta_a'')^2$ , we get

$$\delta = \frac{1}{n} \left( \frac{-C}{\lambda^2} + \frac{B\lambda^2}{(\lambda^2 - \lambda_1^2)^2} \right).$$

In this formula the first term in the parenthesis represents the effect of an absorption band so far down in the ultra-violet that it can be regarded as contributing a certain rotatory power which varies inversely with the square of the wave-length. In the dispersion formula the corresponding term contributes a certain fixed amount to the refractive index, independent of wave-length, while in the present case we have  $\lambda^2$  occurring in the denominator. The second term is the more interesting, for we have  $(\lambda^2 - \lambda_1^2)^2$  in the denominator. The term will have large values for values of  $\lambda$  very near  $\lambda_1$ , but the sign will not change when we cross the absorption band, since the square of the minus quantity, which we have when  $\lambda_1 > \lambda$ , is a positive quantity. This shows us that the sign of the rotation is the same on opposite sides of the absorption band, the rotation decreasing, however, as we recede from the band in either direction.

**Proof of the Rotatory Dispersion Formula.**—The experiments of Macaluso and Corbino<sup>1</sup> have shown that the rotation is in the same direction on opposite sides of the absorption band in the case of sodium vapor. A small sodium flame, placed between the poles of an electromagnet, was traversed by a beam of polarized white light, in the direction of the lines of magnetic force. A Nicol prism was oriented so as to completely extinguish the light when the current was not traversing the coils. On forming the magnetic field a brilliant yellow light was found to be transmitted by the Nicol, which the spectroscope showed to consist of narrow bands symmetrically placed on each side of the  $D$  lines. By turning the Nicol first in one direction and then in the other it was easy to see that the rotation was of the same sign on opposite sides of the band. The formula thus applies qualitatively to the magnetic rotation exhibited by a sodium flame. To test it quantitatively the rotation must be observed over a wide range of wave-lengths on opposite sides of the absorption band. In the case of sodium vapor  $n$  varies but little from unity, except within a small fraction of an Ångström unit of the  $D$  lines. Moreover, we are concerned only with the effect of the electrons which cause the  $D$  lines, for since those of

<sup>1</sup> *Compt. Rend.*, cxxviii., p. 548.

shorter period exert no appreciable influence on the refraction, as we have seen, it is justifiable to assume their effect on the rotatory power as negligible. We can consequently neglect the term  $\frac{-e}{\lambda^2}$ , and write the formula

$$\delta = \frac{B\lambda^2}{(\lambda^2 - \lambda_m^2)^2}.$$

If the formula is to be used in the immediate vicinity of, or between the *D* lines, we must make use of two terms and write,

$$\delta = \frac{A\lambda^2}{(\lambda^2 - \lambda_{D_1}^2)^2} + \frac{B\lambda^2}{(\lambda^2 - \lambda_{D_2}^2)^2}.$$

This formula has been verified by the experiments of Wood,<sup>1</sup> at least for the region of the spectrum not comprised between the *D* lines. Preliminary experiments in collaboration with H. W. Springsteen<sup>2</sup> showed the feasibility of testing the formula by measuring the rotation of the vapor of metallic sodium, formed in exhausted tubes in a powerful magnetic field, and established the fact that the numerous absorption lines of the vapor in the red and green-blue region exercised powerful rotatory effects.

In this preliminary work glass tubes were used, which were exhausted and sealed off from the pump. It was subsequently found that the hydrogen liberated from the sodium interfered greatly with the rotatory effects, and in the subsequent work the tubes were kept in connection with the pump. As the phenomena exhibited by the vapor are extremely beautiful, and very easily shown, the apparatus in its final form will be described in detail.

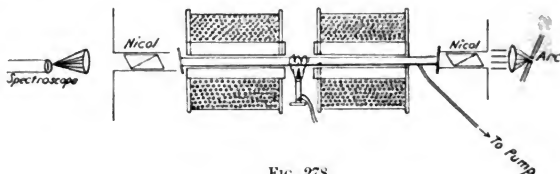


FIG. 278.

A piece of thin, seamless steel tubing (bicycle tubing) of such diameter as to permit of its being slipped easily through the hollow cores of the electro-magnet is procured. A short piece of small brass tubing is brazed into one end, through which the tube is exhausted. The ends are closed with small pieces of plate glass cemented on with sealing-wax. The arrangement of the apparatus is shown in Fig. 278. A piece of sodium the size of a walnut is rolled out into a cylinder and inserted into the tube just before the second end-plate is cemented on. The tube is at once placed in position in the magnet and exhausted. If a piston pump is used for the exhaustion, a glass stop cock should be

<sup>1</sup> "Magneto-Optics of Sodium Vapor and the Rotatory Dispersion Formula," *Phil. Mag.*, Oct. 1905.

<sup>2</sup> *Phys. Rev.*, June 1905.

put between the pump and the tube to prevent back leakage of air. Care must be taken to have the lump of sodium midway between the poles of the magnet. The steel tube is now heated by means of a Bunsen burner, and the pump worked to remove the hydrogen liberated from the sodium, after which the burner is removed and the tube allowed to cool.

Light from a heliostat, or an arc lamp, is now passed in succession through a Nicol prism, the steel tube, a second Nicol, and then concentrated on the slit of a spectroscope. If the instrument has a large dispersion (a 14-foot concave grating was used in the present case) all of the phenomena now to be described can be seen.

The Nicols are set for complete extinction and a small flame placed beneath the tube. As soon as the vapor begins to form two very

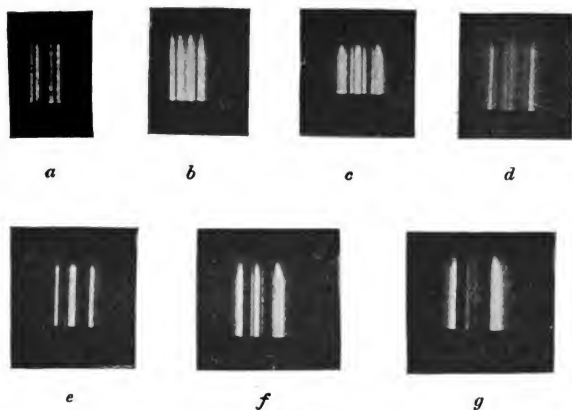


FIG. 279.

bright lines will appear in the position of the *D* lines the moment the magnet is excited. These lines represent the constituents of the white light, which are rotated through  $90^\circ$  by vapor and passed by the analyzing Nicol. The lines are in reality double, though their duplicity cannot be made out when they first appear. As the density of the vapor increases the components separate, four lines being distinctly visible. This condition is shown in Fig. 279, *a*, which is from a photograph. The lines continue to separate, and presently a second pair appears between them for which the rotation is  $270^\circ$ , the dark regions between representing rotations of  $180^\circ$ . This stage is shown in Fig. 279, *b* and *c*.

In the former the two inner  $90^\circ$  lines are beginning to fuse together, the center being partially dark however; in the latter the fusion is complete and the center of the system of lines is bright. With a further increase in the vapor density the outer lines ( $90^\circ$ ) separate still further, and widen out into broad flares of light, other lines appearing

between them corresponding to larger rotations, the system resembling a set of diffraction fringes, as shown in Fig. 280. These bright lines represent rotations of  $270^\circ$ ,  $450^\circ$ ,  $630^\circ$ , etc., and by measuring their positions with an eye-piece micrometer, the wave-lengths corresponding to these rotations were determined. The center of the system, as we may designate a point mid-way between  $D_1$  and  $D_2$ , becomes bright and dark in succession, as many as eight complete alternations having been observed in some instances. This corresponds to a rotation of  $1440^\circ$ . If the burner is removed the changes take place very rapidly, the center "winking" bright and dark almost as rapidly as one can count.

The results obtained from measurements made with the micrometer are shown in the form of a curve, observed values being represented by circles. Values of the constants  $A$  and  $B$  in the formula were calculated from two observed values of  $\delta$ , and the values of  $\delta$  for various wave-lengths calculated. These calculated values are represented by crosses on the plate, and will be found to fall almost exactly upon the experimental curve. The value of the constant  $B$ , which is associated with the absorption line  $D_2$ , was about double that of  $A$ , which belongs to  $D_1$ . Tables of rotations for various vapor

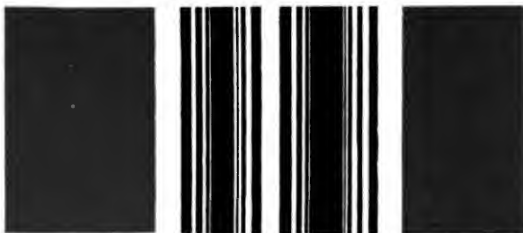
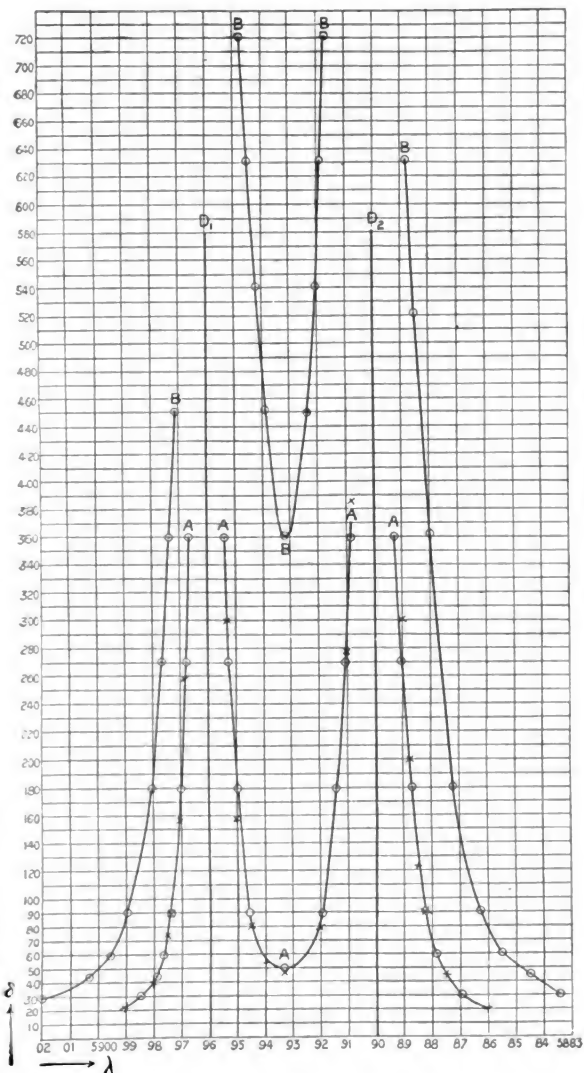


FIG. 280.

densities were made, and the formula tested under various conditions. With fairly dense vapors the observed value of  $\delta$  midway between  $D_1$  and  $D_2$  was usually larger than the calculated. A fuller discussion of the results will be found in the original paper.

With vapor of considerable density the rotation is measured in a different manner. The Nicols are set in a parallel position, and the entire spectrum appears with the exception of the broad absorption band at the  $D$  lines. On each side of this absorption band a dark  $90^\circ$  rotation band appears. As we turn the Nicol these dark bands move, the one up, the other down the spectrum. By noting their positions we determine the values of  $\lambda$  corresponding to the rotation of the Nicol in degrees measured from the position of extinction.

The spectrum of the light transmitted, with the Nicols in various positions, was photographed, and the positions of the dark bands measured subsequently. Owing to the great density of the vapor it was found that  $D_1$  and  $D_2$  could be considered as forming a single absorption band, and  $\lambda_m$  was given an intermediate value 5893. For a particular density and length of vapor column, the constant  $B$  was



Magnetic rotatory dispersion of sodium vapor. Curve "A," rare vapor; "B," denser vapor.

To face p. 426.

Mag. Rotation.  
Abs. Spectrum.

Portion of bright line rotation spectrum and absorption spectrum of sodium vapor, photographed with 14-foot concave grating.

found from a single observation of  $\delta$ . The observed and calculated values are given in the following table :

VALUE OF CONSTANT  $B = 135600$ .

$\lambda$ .	$\delta$ (obs.).	$\delta$ (cal.).
5980	5°	4°·47
5950	10°	10°·4
5933	20°	23°·2
5923	40°	38°
5917	66°	58°·9
5912·5	90°	89°·2
5874	90°	93°·1
5869	43°	43°
5864	40°	40°·6
5852	20°	20°·2
5833	10°	9°·2
5814	5°	5°·2

This table shows that with very dense vapor the rotatory dispersion is well represented by a single term formula, the observations being limited to a region not very near  $D_1$  or  $D_2$ .

A series of photographic records obtained in this manner is shown in Fig. 281, which, taken collectively, exhibit the general form of the rotatory dispersion curve.

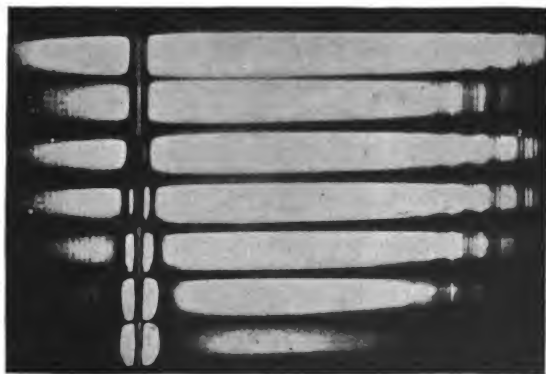


FIG. 281.

**The Bright Line Rotation Spectrum of Absorbing Vapors.**—With the apparatus described in the previous section, a very remarkable phenomenon appears when the vapor has considerable density. With the Nicols crossed and the magnet excited, the transmitted light, when

examined with a prism spectroscope, will be found to form a most beautiful bright line spectrum, the general appearance of which is shown in Fig. 3 of the colored frontispiece. Over a hundred lines can be counted in the red, and about the same number in the blue-green region. A photograph of a portion of the spectrum is shown in Fig. 281a, while Fig. 3 of the colored frontispiece gives a good idea of the general appearance of the entire spectrum. These lines have been photographed with a 14-foot concave grating, and found to coincide with absorption lines, the significant fact being, however, that comparatively few of the absorption lines are represented in the rotation spectrum. Just why this is so is not apparent. The rotatory power of an electron is probably inversely proportional to its mass. The lighter the electron in proportion to its charge, the greater will be the perturbations in its orbit produced by the magnetic field. It is possible that the absorption lines which exercise rotatory power result from the negative electrons of small mass, while the other absorption lines are due to heavier corpuscles, perhaps carrying positive charges. The fact that the bright lines of the fluorescent spectrum appear to coincide with those of the magnetic rotation spectrum, favors this hypothesis, for we should expect the lighter electrons to be set in more violent vibration by the light-waves than the heavier ones. A further study of the phenomenon will doubtless throw more light on the subject. Iodine vapor also gives a very beautiful bright line



Red.                      Or.                       $D_1 D_2$                       Yellow.

FIG. 281a.

spectrum. A few crystals are introduced into a small glass bulb which is highly exhausted and sealed off from the pump. This bulb, when placed between the perforated conical pole pieces of a Ruhmkorff magnet (Nicols crossed) and gently warmed, restores light of a most beautiful emerald green color, which the spectroscope shows to be discontinuous. It is instructive to prepare two bulbs, one exhausted, the other at atmospheric pressure. The latter shows no effect whatever.

**Magnetic Rotation within an Absorption Band: Experiments of Zeeman.**—It was established theoretically by Voigt (*Wied. Ann.*, 67, 359, 1899) that, in the case of an absorption line separated into a magnetic doublet, the rotation of the plane of polarization was positive for all periods lying outside of the components of the doublet and negative for all periods between the components, the light traversing the medium in the direction of the lines of force.

This was verified by Zeeman (*Proc. Amsterdam Acad.*, June 1902), who made use of a method similar to the one which had been previously employed by Voigt in demonstrating magnetic double-refraction.

The light of an arc-lamp, after passage through a Nicol, was focused upon the slit of a grating spectrometer, in front of the slit of which was placed a system of right and left-handed quartz prisms, similar to the arrangement employed by Fresnel in his experiment on the division of a plane-polarized ray into two circular components by rotatory media. Between the slit and the grating a second Nicol was mounted which cut off the vibrations which had been rotated into its plane of extinction by the quartz wedges. The amount of rotation at each point of the slit depended on the difference between the thicknesses of the right and left-handed quartz elements at the point in question, and the spectrum was found to be traversed by a system of dark bands

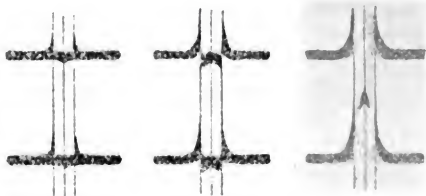


FIG. 282.

parallel to its length. Between the first Nicol and the spectrometer the absorbing flame of sodium was mounted in a magnetic field, any rotation produced by it adding itself to that produced by the quartz wedges. A rotation impressed upon any wave-length by the flame thus caused a vertical deviation of the dark band at the corresponding point of the spectrum, a shift equal to the width of a complete fringe corresponding to a rotation of  $180^\circ$ .

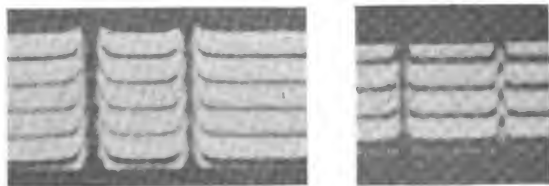


FIG. 282a.

With a field of 15,000 C.G.S. units the dark absorption line was distinctly resolved into a doublet, and on increasing the amount of sodium in the flame the dark bands outside of the components curved upwards, while the portion of the band between them slid down in the opposite direction, as shown in Fig. 282, in which the appearances of the bands for different densities of the absorbing flame are shown. Photographs of the phenomenon are reproduced in Fig. 282a.

Increasing the strength of the field caused the portion of the band between the components to move back towards its original position, which was in agreement with Voigt's prediction that the negative rotation within the band decreased with increasing field strength. This is, of course, true only with fairly strong fields; in other words, for a given density of vapor the negative rotation between the components reaches a maximum value for a certain strength of field. In the case of the positive rotation which occurs outside of the lines, no such turning-point is found.

**Rays Perpendicular to the Magnetic Field.**—On the hypothesis of molecular currents we should expect no effect to be produced by the magnetization of the medium when the rays of light are perpendicular to the lines of force. The Hall-effect hypothesis, however, calls for an effect in this case, which, though small, is perhaps not impossible of detection. Consider the waves as propagated along the  $x$  axis, instead of along the lines of force ( $z$  axis). In this case we have the relations

$$\begin{aligned}\epsilon'' \frac{\partial X}{\partial t} + iv \frac{\partial Y}{\partial t} &= 0, \text{ since } j_x = 0, \\ \frac{1}{c} \left( \epsilon'' \frac{\partial Y}{\partial t} - iv \frac{\partial X}{\partial t} \right) &= -\frac{\partial \gamma}{\partial x}, \quad \frac{\epsilon'}{c} \frac{\partial Z}{\partial t} = \frac{\partial \beta}{\partial x}, \\ \alpha &= 0 \quad \frac{1}{c} \frac{\partial \beta}{\partial t} = \frac{\partial Z}{\partial x}, \quad \frac{1}{c} \frac{\partial \gamma}{\partial t} = -\frac{\partial Y}{\partial x}.\end{aligned}$$

Eliminating  $\beta$  and  $\gamma$  gives

$$\begin{aligned}\epsilon'' X + iv Y &= 0, \\ \frac{\epsilon''}{c^2} \frac{\partial^2 Y}{\partial t^2} &= \frac{\partial^2 Y}{\partial x^2} + i \frac{v}{c^2} \frac{\partial^2 X}{\partial t^2}, \quad \frac{\epsilon'}{c^2} \frac{\partial^2 Z}{\partial t^2} = \frac{\partial^2 Z}{\partial x^2}.\end{aligned}$$

Elimination of  $X$  from the first two equations gives

$$\left( \epsilon'' - \frac{v^2}{\epsilon''} \right) \frac{\partial^2 Y}{\partial t^2} = c^2 \frac{\partial^2 Y}{\partial x^2}.$$

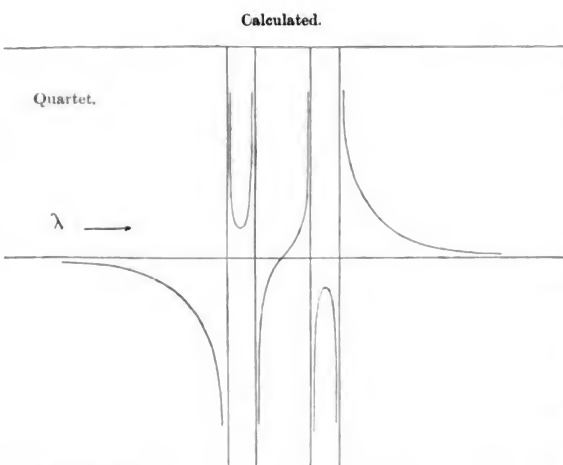
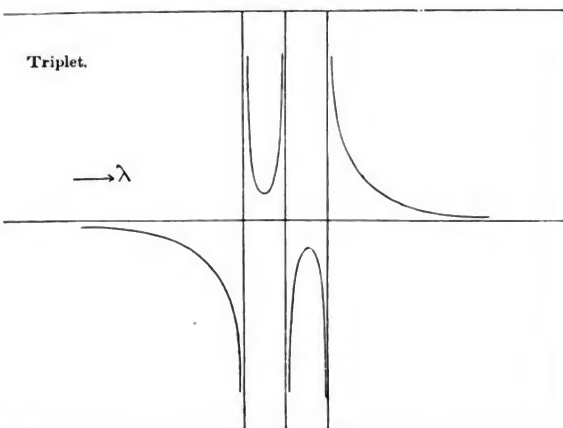
Writing as before,  $X = Me^{\frac{i}{\tau}(t-p'x)}$ ,  $Y = Ne^{\frac{i}{\tau}(t-p'x)}$ ,  $Z = \Omega e^{\frac{i}{\tau}(t-p'x)}$ .

The velocities of the  $x$  and  $y$  components are obviously the same, since they are symmetrical about the lines of force. The  $z$  component may, however, be propagated with a different velocity, hence we are obliged to distinguish between  $p'$  and  $p$ .

By differentiation and substitution we get

$$\epsilon'' - \frac{v^2}{\epsilon''} = p'^2 c^2, \quad \epsilon' = p^2 c^2, \quad n = -\frac{iv}{\epsilon''} N.$$

This equation shows us that the  $z$  and  $y$  components are propagated with different velocities, and that the vibration, if originally a plane one, making an angle of  $45^\circ$  with the lines of force, becomes transformed into an elliptical one, in other words the medium is doubly refracting. The velocities of the  $z$  and  $y$  components are given by substituting the values of  $\epsilon'$  and  $\epsilon''$  in the above equations.



Observed.



FIG. 283.

$$p^2 c^2 = n^2 (1 - i\kappa')^2 = \epsilon' = 1 + \sum \frac{\theta N}{\Theta}$$

gives the velocity of a vibration parallel to the field, while

$$n'^2 (1 - i\kappa')^2 = 1 + \sum \frac{\theta N \Theta}{\Theta^2 - \Phi^2} - \frac{\left( \sum \frac{\theta N \Phi}{\Theta^2 - \Phi^2} \right)^2}{1 + \sum \frac{\theta N \Theta}{\Theta^2 - \Phi^2}}$$

represents the velocity of the component perpendicular to it.

The difference between  $n$  and  $n'$  will be very small unless  $\Theta$  is very small, which only occurs in the immediate vicinity of an absorption band.

The above formulae were derived by Voigt, who writes  $\left(\frac{v}{O_y}\right)^2$  for  $n'^2 (1 - i\kappa')^2$ ,  $v$  representing the velocity of light in space and  $O_y$  the complex amplitude of the  $y$  component. The field strength  $h$  occurs in  $\Phi$  only, consequently the velocity of the  $z$  component is uninfluenced by the magnetic field.

An attempt was made by Voigt to detect the double refraction by placing a block of glass between the poles of a powerful electro-magnet, and causing the light to traverse it a large number of times in a direction perpendicular to the lines of force. The ends of the block were silvered, with the exception of two narrow strips for the entrance and exit of the beam. A Babinet compensator was used to detect a possible change of phase between the two components, and though a very slight shift of the fringes was noted when the field was thrown on, it was too small to be regarded as established.

The double refraction was, however, found by Voigt and Weichert in the case of sodium vapor. They placed a small sodium flame between the poles of the magnet and passed through it a beam of white light, polarized at an angle of  $45^\circ$  to the lines of force. The interference bands seen in the compensator were found to be curved in opposite directions, on opposite sides of the absorption band, the appearance being similar to the anomalous dispersion curve exhibited by the vapor. This experiment was subsequently enlarged upon and improved by Zeeman and Geest (*Proceed. Amsterdam Acad.*, Jan. 25, 1905).

**Double Refraction near the Components of Absorption Lines magnetically split into Several Components.**—Zeeman and Geest obtained some extremely interesting results by applying the same method of study to light which had traversed an absorbing flame in a direction perpendicular to the magnetic field. In this case  $D_1$  is split up into a quartet and  $D_2$  into a sextet. A formula was deduced which expressed the phase-difference between the vibrations parallel and normal to the field, and the deformations of the horizontal fringes, which resulted from the phase-differences, were calculated and recorded graphically. These curves were then verified by experiment, the agreement being most excellent. The calculated curves and observed fringe contortions are shown in Fig. 283. The calculated curves are for a single dark band. The observed show two adjacent bands, it being impossible to limit the observation to a single one.

## CHAPTER XVIII.

### TRANSFORMATION OF ABSORBED RADIATION. FLUORESCENCE AND PHOSPHORESCENCE.

IN the case of all substances which absorb light, *i.e.* in cases when the sum of the transmitted and reflected energy is not equal to the incident energy, we have a transformation of the radiant energy into energy of some other form. The commonest case is that in which the luminous radiations are spent in warming the body, the absorption of the light increasing the kinetic energy of molecules. Just how the transformation takes place we do not know, though it is not difficult to make assumptions. In the case of selective absorption, where we refer the phenomenon to resonance, or the vibration of electrons of the same free period as the absorbed vibrations, we assume something akin to friction, which impedes the free motion of the electron and produces a rise of temperature. In some way the energy taken up by the electron is spent in increasing the velocity of the molecule, which makes it seem as if the action were an action between the molecules, and not something going on within the molecule, for we cannot speak of a molecule as rising in temperature. As the temperature of the substance increases, it emits more and more energy in the form of long heat-waves, and it is this re-emission of energy which prevents the temperature from rising indefinitely. If the substance absorbs strongly waves of all lengths, it is possible to throw radiant energy into it at such a rate that its temperature rises to the point of incandescence, as was shown by Tyndall. The incident energy may consist wholly of invisible heat-rays, in which case we have a re-emission of energy in the form of waves of shorter wave-length. This phenomenon has received the name of calorescence.

Tyndall's experiment consisted in the formation of a dark heat focus by means of a large condensing lens, the visible radiations being filtered out by means of a solution of iodine in bisulphide of carbon, contained in a glass cell. A piece of blackened platinum foil held in the focus was speedily raised to a red-heat. In this case the emission of light is a pure temperature effect. Certain substances, however, possess the peculiar property of emitting light when illuminated, without any appreciable rise in temperature. The emitted light is usually of a different color from that of the exciting radiation, and the emission may continue for some time after the illuminating light is cut off.

If the emission ceases as soon as the exciting radiations cease to fall upon the substance, the phenomenon is called Fluorescence: if it persists for an appreciable time, the term phosphorescence is applied to it. In general, fluorescence is only exhibited by gases and liquids, phosphorescence by solids, though we sometimes find the terms confused, the term fluorescence being applied to uranium glass and certain crystals. It is, perhaps, best not to attempt to draw a sharp line between the classes of phenomena, for it has been shown recently that we may have a gradual transition of fluorescence into phosphorescence.

**Fluorescence.**—The name fluorescence is derived from fluor spar, a native fluoride of calcium, a substance which was first observed to exhibit this peculiar emission of light. The subject was first investigated by J. Herschel (*Phil. Trans.*, 1845, p. 143) and Sir David Brewster (*Trans. of Edin.*, 1846, part ii., p. 3), who examined solutions of sulphate of quinine, which emit a brilliant blue light in all directions when illuminated with a beam of sunlight.

Herschel found that light which had traversed the solution was incapable of exciting any further emission, and that the blue luminosity was confined to the surface. He termed the phenomenon epipolic dispersion, believing that he was dealing with a new type of polarization. The light on entering the solution became "epipolarized," a lateral emission or dispersion resulting from the process, and this epipolarized light was incapable of exciting further fluorescence. Brewster found, however, that by employing an intense beam of light, the blue emission marked the entire path of the beam, and he accordingly changed the name to "internal dispersion."

It occurred to Stokes that the blue light "dispersed" by the quinine solution might not be the blue light of the illuminating beam, but a new creation due to the absorption of more refrangible radiations. This would explain the inability of the light to excite further fluorescence after it had already traversed a sufficient thickness of the solution, the rays effective in provoking the emission being removed by absorption. Experiments verified this surmise, establishing the general law that the fluorescent radiations are always of longer wave-lengths than those of the light which excites them. This change in the wave-length can be very easily observed by interposing colored glass in the path of the illuminating beam. A beam of sunlight, from which the orange-yellow and green has been removed by means of a sheet of dense cobalt glass, is concentrated with a lens upon a few crystals of uranium nitrate. Although the illuminating beam is of a deep blue violet color the crystals shine with a brilliant green light. A piece of the ordinary canary glass, which is colored with oxide of uranium, can be substituted for the nitrate crystals. Small vases of this glass, which can now be found in almost any glass-store, are suitable for the experiment. It can be easily recognized by the greenish color which it assumes when held in sunlight, which is in marked contrast to the yellow color of the transmitted light. One of the best substances for the exhibition of fluorescence is an aqueous solution of uranin, an alkaline salt of fluoresceine. The solution should be extremely dilute. It is best to begin with pure water contained in a rectangular glass tank, the light from an arc-lamp or the sun being brought to a focus at

the center of the tank. On adding a drop or two of a fairly strong solution of the dye the path of the beam becomes luminous, shining with a brilliant green light. On adding more of the dye the fluorescence retreats towards the region where the light enters the solution, owing to the increased absorption of the rays which are capable of exciting the fluorescence.

Other solutions can be easily prepared which fluoresce with different colors. An aqueous solution of aesculin, which can be prepared by pouring hot water over some scraps of horse-chestnut bark, shines with a beautiful blue light, while an alcoholic solution of chlorophyll, which can be prepared by soaking green leaves in strong alcohol, exhibits a red fluorescence. Among other substances which exhibit the phenomenon may be mentioned, solutions of sulphate of quinine (acidified with a few drops of sulphuric acid) and paraffin oil, both of which fluoresce with a blue light. (See page 443 for list.)

**Methods of investigating Fluorescence.**—An exceedingly simple and ingenious method was devised by Stokes for detecting fluorescence and phosphorescence, which is applicable to cases where the emitted light is so feeble as to be overpowered by the irregularly reflected light. This method depends on the change of wave-length which accompanies both phenomena. Two screens are prepared, one of which transmits the violet and blue, absorbing the green, yellow and red, while the other absorbs the violet and blue, transmitting the rest of the spectrum. Dense cobalt glass combined with a thin sheet of signal green glass, or a solution of cuprammonium, makes a suitable blue violet screen; while yellow glass or a solution of bi-chromate of potash will answer for the other. The two together should be practically opaque even to a fairly strong light.

If, now, a powerful beam of light is admitted to a dark room or box through the blue screen, objects illuminated by it will be invisible through the yellow screen unless they fluoresce or phosphoresce, that is, give out less refrangible radiations than those which fall upon them. Stokes succeeded in showing that ordinary paper, cotton, bones, ivory, leather, cork, horn and many other substances exhibit the phenomenon.

This method, while admirably adapted to the detection of fluorescence, is not suited to the study of the relation between the wave-lengths of the fluorescent and incident light.

The method adopted by Stokes was analogous to Newton's method of crossed prisms; a very narrow and intense solar spectrum was thrown upon the surface of the liquid under investigation, the fluorescence resulting in this case from monochromatic light of varying wave-length.

This spectrum was then viewed through a prism held in such a position as to deviate the spectrum in a direction perpendicular to its length, as shown in Fig. 284, in which *AB* is the undeviated and

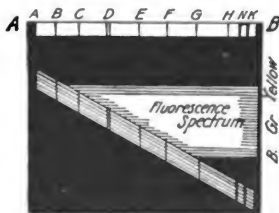


FIG. 284.

$A'B'$  the deviated spectrum, as it would appear if the surface of the liquid merely reflected or scattered light without fluorescence. This spectrum will always be visible to a greater or less degree, owing to the fact that even the surface of a transparent liquid scatters a certain amount of light without change of wave-length.

In the case of fluorescent or phosphorescent substances, we have in addition the complete fluorescent spectrum produced by the monochromatic illumination at each point of the spectrum. The fluorescent spectra together form the broad band shown in the figure, from which it is apparent that the wave-length of the fluorescent light is never less than that of the light which causes it. Any exception to Stokes's law would give rise to an extension of the band on the other side of the deviated spectrum, as indicated by the dotted line.

Stokes's law, that the waves of the fluorescent light are never shorter than those of the exciting rays, was questioned by Lommel, who believed that he could detect the complete fluorescent spectrum of Magdala red, which contains red, yellow and green rays, when the fluorescence was excited by sodium light alone.

Hagenbach investigated this same substance and came to a different conclusion. The eye-piece of a spectroscope was replaced with a screen perforated with a vertical slit, upon which the spectrum was focused. An image of this slit, illuminated in monochromatic light, was thrown by means of a lens partly upon the surface of the fluorescent solution, and partly upon a small piece of white porcelain placed close to the surface. The porcelain reflected only the monochromatic light, while the liquid emitted the fluorescent light. On viewing the image

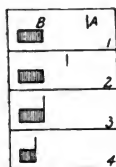


FIG. 285.

image of the slit, reflected from the porcelain  $A$ , appeared as shown in Fig. 285 (1). The two images were separated by a wide gap, the fluorescent light being much less refrangible than the exciting rays. By turning the prism of the spectroscope, the wave-length of the light from the slit was increased, which caused the image  $A$  to approach  $B$ , the gap between becoming narrower as shown in (2). No change in the fluorescent spectrum was observed. On still further increasing the wave-length,  $A$  came into coincidence with  $B$ . Up to this point the fluorescent spectrum remained unchanged, but from now on a further increase in the wave-length resulted in a contraction of the spectrum  $B$ , as shown in (4), no radiations appearing in it of shorter wave-length than those in the image  $A$ . An exception to Stokes's law would have made itself manifest as a faint illumination in the region indicated by the dotted line in (4).

Lommel then repeated his experiments, using both sodium light and monochromatic light from a spectroscope, and found as before that not only Magdala red but a number of other substances showed evidences of emitting fluorescent waves shorter than the exciting ones. He distinguished three distinct classes of fluorescence. Bodies belonging to the first class were capable of giving out their *complete* fluorescent spectrum when excited by any radiations exciting fluorescence. Under the

second class were grouped substances of which the fluorescent spectrum contained no shorter radiations than the exciting ones. A third type, which he called composite fluorescence, embraced substances having a fluorescent spectrum consisting of two parts, similar respectively to the spectra shown by substances of the first two classes. These substances behaved as would a mixture of a substance belonging to the first class with one belonging to the second.

A very careful study of the subject has been made recently by Nichols and Merritt (*Phys. Rev.*, June and July 1904), who measured with a spectrophotometer the distribution of intensity in the fluorescence spectrum, when the wave-length of the exciting light was varied. They found that in the case of all the substances which they examined there were marked exceptions to Stokes's law, the position of the maxima of the fluorescence spectrum being independent of the wave-length of the exciting light. They were able to produce powerful fluorescence when the exciting light was of much greater wave-length than that at the center of the fluorescent band.

An aqueous solution of fluorescein is admirably adapted for the purposes of illustration. In Fig. 286 we have the intensity curves of the fluorescent spectra *A*, *B* and *C* when excited by approximately monochromatic light cut out of a spectrum in the region *A'*, *B'* and *C'*. The fluorescence is seen to be most intense when the exciting wave-lengths lie on the edge of the fluorescent spectrum which is towards the violet, *C'* but still of considerable intensity when the illuminating light is made up of a band on the red side of the point of most intense fluorescence.

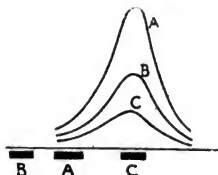


FIG. 286.

**Polarized Fluorescence.**—The fluorescent light emitted in an oblique direction from the surfaces of isotropic media was found by Millikan (*Phys. Rev.*, Sept. and Nov. 1895) to be polarized in a manner similar to that of light obliquely emitted by white hot surfaces. This polarization results from refraction of the light as it passes out of the medium into the air.

Certain crystals possess the remarkable property of emitting a polarized fluorescence. The most interesting case is that of magnesium platino-cyanide, which can be prepared by the addition of a solution of magnesium sulphate to a solution of barium platino-cyanide, until no further precipitation of barium sulphate takes place. The colorless solution is filtered, evaporated and crystallized. The crystals, which are of a deep red color, have most remarkable optical properties, showing a brilliant green surface color on the sides of the prisms, while the ends selectively reflect a deep violet light, which is polarized even at normal incidence. The fluorescence of the crystals can be best observed by exciting them in a concentrated beam of sunlight which has been passed through a piece of dense cobalt glass. We will suppose the crystal to be standing upright, upon one of its bases, and the incident beam horizontal. If the fluorescent light, which emerges from the sides of the prism, be observed through a

Nicol prism, it will be found that the color is orange yellow when the polarizing plane of the Nicol is perpendicular to the axis of the prism, and scarlet when the plane is parallel to the axis. If the exciting light be polarized horizontally, the color of the fluorescent light is yellow, changing to red as the plane of polarization is rotated through  $90^\circ$ .

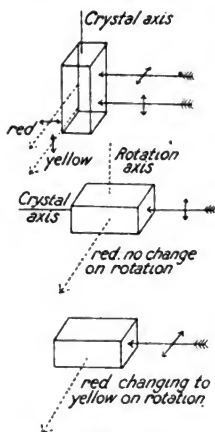


FIG. 287.

Examination with a second Nicol shows that the orange-yellow fluorescence is polarized perpendicular, the scarlet parallel to the axis of the crystals. If, now, the crystal be placed in a horizontal position, and the beam of unpolarized light directed against an end surface, the fluorescent light will be unpolarized and of a scarlet color. If the incident light be polarized in a vertical plane, and the crystal be turned on a vertical axis so as to vary the angle of incidence, the red color changes to yellow. If, on the other hand, the plane of polarization be horizontal, no change is observed on turning the crystal. This shows that the change from red to yellow takes place as the angle which the direction of vibration makes with the crystal's axis changes from  $90^\circ$  to  $0^\circ$ , the direction of

vibration being perpendicular to the plane of polarization.

The results obtained in the two positions of the crystal are in perfect agreement, as will be seen by reference to Fig. 287, in which the exciting rays are indicated by solid arrows, the fluorescent rays by dotted arrows, the direction of vibration in each case being indicated by double-headed arrows.

**Variation of the Intensity with the Angle of Emission.**—The intensity of the light emitted by selfluminous solid and liquid substances varies as the cosine of the angle of emission (Lambert's law). On this account the intrinsic intensity is not increased by foreshortening of the source, *i.e.* by viewing it in an oblique direction. This is not, however, the case with fluorescent light, as is indicated by some as yet unpublished experiments by the author. If the light from a powerful spark discharge between cadmium electrodes is allowed to fall upon the face *AB* of a prism of crown glass, the surface shines with a beautiful blue light, the intrinsic intensity being much greater when it is viewed from *D* than when viewed from *C*. Apparently the total light sent off from the surface is independent of the direction, which is also the case with X rays. This condition only holds, however, *within* the fluorescent medium, as is easily seen by viewing the other side of the luminous surface in different directions.

The effect may also be beautifully shown by dissolving a little uranin (fluorescein) in a beaker of water, arranging a cadmium spark close to

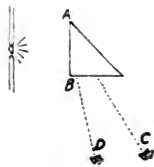


FIG. 288.

the surface and viewing the surface from below. The fluorescence is of dazzling intensity, when the surface is foreshortened almost into a line. The spark light is necessary, for we require a *surface* effect, and the ultra-violet radiations scarcely penetrate the surface. With other lights the fluorescence is distributed through the mass of the liquid.

**Change of Absorption produced by Fluorescence.**<sup>1</sup>—The very remarkable fact was discovered by Burke that a substance, while fluorescing exercised a different absorbing power on a beam of light passing through it at right angles to the beam which excited the fluorescence. This seems to be strong evidence in favor of the hypothesis which we made, that the action of the light caused a chemical change in the substance, which was, however, unstable, the original condition being established very soon after the light is cut off. Burke experimented with cubes of uranium glass, measuring photometrically the intensity of the absorption as influenced by the passage of a beam of light through the cube in a different direction. Various corrections were necessary owing to the superposition of the fluorescent light on the light which was being observed.

**Phosphorescence.**—The term fluorescence, strictly speaking, should be limited to the cases of the gases and liquids, since, in the case of solids, the emission of light continues for an appreciable time after the exciting radiations are cut off. In some cases the emission of light may continue for several hours, in others it lasts for but a very small fraction of a second after the exciting rays are removed. Most remarkable is the prolonged emission of Balmain's luminous paint, a sulphide of calcium, which glows in the dark for many hours after exposure to a strong light. It appears probable that something analogous to a chemical change is produced by the action of the light, the condition being unstable and the process reversing as soon as it is screened from the action of the light. This reversing of the process may be prolonged or rapid, and is accompanied by the emission of light. The energy of the absorbed light is stored in the substance in the form of potential energy of the atoms. At very low temperatures the condition brought about by the action of light may be more or less stable, as has recently been shown by Dewar. A fragment of ammonium-platino-cyanide was cooled by means of liquid hydrogen and exposed to a strong light. On removing it to a dark room no trace of phosphorescence was perceived, but on removing the crystal from the chilled tube and allowing it to warm up, it presently bursts into a brilliant green phosphorescence.

A partial stability can be shown at ordinary temperatures with Balmain's luminous paint. If this be kept in absolute darkness for twenty-four hours it becomes non-luminous. A further emission of light may, however, be produced by concentrating invisible infra-red radiations upon it. This at first sight appears like an exception to Stokes's law, but if the experiment be continued for a few minutes the luminosity ceases. The infra-red radiations have merely carried on the reverse process originally produced by violet light, further than it is able to go spontaneously. Not until the powder has again been exposed to light and kept over night in the dark can the experiment be repeated. The

<sup>1</sup> *Phil. Trans.*, 191, p. 87, 1898.

color of the phosphorescent light obtained in this way differs from that of the light by the spontaneous breaking down of the molecular condition produced by the light, being distinctly greenish instead of deep blue.

If the Balmain paint be exposed to infra-red radiations while it is phosphorescing the luminous energy is liberated much more rapidly. This property has been utilized for obtaining records of the infra-red spectrum. A sheet of glass is painted over with luminous paint and exposed to sunlight.

If a spectrum is then thrown upon it it will be found that the effect of the red and infra-red region is to render the portions of the surface upon which they fall, at first more luminous than their surroundings. The emission soon ceases, and on examination we find a dark region where the infra-red radiations have destroyed the phosphorescence. Very good records have been obtained in this way of the infra-red solar spectrum by Draper, Becquerel and Lommel. The phosphorescent plate, after exposure to the spectrum, was placed in contact with a photographic plate, by means of which the record was made permanent.

**Duration of the Phosphorescence. Phosphoroscope.**—While some phosphorescent substances remain luminous for a considerable time after their exposure to light, the majority cease to give out visible radiations in a few seconds after the exciting radiations cease to fall upon them. An instrument was devised by Becquerel for examining substances in complete darkness, a small fraction of a second after their exposure to a brilliant light. This instrument, which is known as the phosphoroscope, consists of two metal discs mounted side by side on the same axle. The discs are perforated with an equal number of apertures which are arranged out of step, and are driven at high speed by a train of cog-wheels. The substance to be examined is placed between the discs, and a strong beam of light directed upon it through the apertures of one of them. If the eye be brought close to the other disc the object will be seen only at the moments when the light beam is cut off, and it will be visible therefore only by phosphorescent light. The discs are mounted in a cylindrical metal box, to screen the substance from all light except that reaching it through the perforations.

With this instrument it is possible to observe an object one one-thousandth of a second or less after its illumination. Becquerel found that phosphorescence was much more common than had been supposed. The salts of the alkali metals, compounds of aluminium and nearly all organic compounds were found to be phosphorescent. Compounds of the heavy metals for the most part showed no trace of luminosity, the salts of uranium and platinum being marked exceptions however.

All solid fluorescent substances were found to be phosphorescent; fluorescent liquids, on the other hand, showed no trace of the phenomenon. The author has found the same to be true for sodium vapor. A simple phosphoroscope with a single revolving disc can be set up in a very few minutes, and gives excellent results with uranium glass, uranium nitrate and other phosphorescent substances. The disc, which is 30 or 40 cms. in diameter, can be made of cardboard with holes about half a centimeter in diameter punched at regular intervals around its circumference. The distance between the holes should be about 2 cms.,

not less. The disc can be mounted on the shaft of a small electric motor, or on a whirling table, or it may even be mounted on a lead pencil held in vertical wooden supports and set in rotation with a top string. A beam of sunlight, reflected from a mirror, is focused on one of the holes; the diverging cone is received by a second lens on the other side of the disc and again brought to a focus, this time upon the object under examination, *e.g.* a lump of nitrate of uranium. On setting the disc in motion, and viewing the object through the small holes, taking care not to get in the way of the light, the phosphorescence can be easily observed. Obviously the eye must be moved about until the position is found in which the incident light is cut off from the object when it is exposed to view. By laying the crystals upon white paper the effect is more striking, for it is then apparent that we are seeing the crystals by their own light alone. The room should be made as dark as possible, of course, though the phosphorescence is apparent even in a room brilliantly illuminated with sunlight.

**Influence of Temperature**—It has been shown by Dewar (*Chem. News*, 70, 252, 1894; *Proc. Chem. Soc.*, 10, 171) that many substances at low temperatures exposed to light do not fluoresce until they are warmed, the change produced appearing to be stable at low temperatures. It would be interesting to examine the absorption spectrum of a fluorescent substance at low temperatures before and after its exposure to light. In this way direct evidence of a molecular change might be obtained. It is even possible that prolonged exposure to a brilliant light might produce a change that could be recognized in other ways. It is still apparently an open question as to how much luminous energy can be stored up in a fluorescent body at a low temperature.

Balmain's paint, cooled in solid carbonic acid and ether, and exposed to a powerful light while at a low temperature, emits no light until it is removed from the freezing mixture. The influence of temperature can be well shown by painting a sheet of metal with the paint, exposing it to sunlight and then heating it suddenly with a Bunsen burner in a dark room. The sudden rise of temperature is accompanied by a brilliant emission of light. Dewar found other substances, however, which phosphoresced only at low temperatures. Gelatine, celluloid, paraffin, ivory, and horn, which at ordinary temperatures only exhibited the feeblest traces of phosphorescence, became very luminous at  $-180^{\circ}$ . The following substances were found especially brilliant at  $-180^{\circ}$ : acetophenon, benzophenon, asparagin, hippuric and uric acids, disphenyl, salicylic acid, and egg shells. Ammonium platino-cyanide showed no luminosity at  $-180^{\circ}$  until the liquid air was poured off, when it immediately lighted up like a lamp.

**Theories of Fluorescence.**—The fact that fluorescence is always accompanied by absorption shows that the phenomenon is in some way connected with the vibration of the electrons.

At first sight it may appear as if all that is necessary is to assume that the ions set in vibration by the ether waves become themselves sources of radiation. If this is the case all absorbing media should fluoresce, which is not the case. Moreover, we have seen that we can have a gradual transition from fluorescence to phosphorescence, and

it is unthinkable that phosphorescence, of even a small fraction of a second's duration, can be simply the radiation of an electron which continues in vibration, after once having been set in motion. Even if the phosphorescence lasted but  $1/1000$  of a second the electron would be obliged to perform of itself 600 billion vibrations. Though the damping of the vibration of an electron by radiation can be shown to be small, it is scarcely possible that a vibration, once started, can continue for such a length of time without receiving energy from some source. Another difficulty is the circumstance that with monochromatic light as the exciting radiation, the fluorescent radiation is distributed over a wide range of wave-lengths chiefly of lower refrangibility.

A satisfactory theory of fluorescence must first of all distinguish between absorbing media which fluoresce and those which do not; furthermore, it must explain the change of wave-length and the increase in the duration of the emission when the substance is in the solid form.

As a matter of fact no satisfactory theory exists but it is instructive to examine briefly into the attempts which have been made to establish one. The fundamental assumption which has been made in every case is that the fluorescent light is emitted by atoms or electrons which are set in vibration by the light-waves. This explanation was given by Stokes, and it may be in part correct, though it fails to show why all absorption is not accompanied by fluorescence. It can hardly be extended to phosphorescence, and as the two phenomena are so closely related it seems probable that some at least of the facts of fluorescence are not to be accounted for on the simple assumption of a forced vibration, but must be referred to some complicated chemical process.

A theory was developed by Lommel, who built up an equation of motion of an atom vibrating under the impact of light-waves. This equation was similar in form to the one which we have already discussed in the treatment of the Helmholtz dispersion theory.

The molecular condition of the substance has apparently as much to do with its power of fluorescing as its chemical constitution. For example, many substances which in the solid state show no trace of fluorescence, when dissolved in various liquids become powerfully fluorescent. Other substances possess the property both in the solid and dissolved states, some to a greater degree in the former, others in the latter. Still others, such as Barium platino-cyanide, which in the crystalline form are powerfully fluorescent, show no trace of the phenomenon in solution. For this behavior we have at the present time no satisfactory explanation, no theory of fluorescence having up to the present time been formulated which is capable of explaining even the simplest facts.

**Effect of the Solvent on the Intensity of Fluorescence.**—The solvent exercises a very marked influence upon the intensity of the fluorescence. This subject was carefully investigated by Knoblauch.<sup>1</sup> He found that the intensity of the fluorescent light was strictly proportional to the intensity of the exciting light, but that this intensity varied greatly when the substance was dissolved in different liquids, as is shown in the following table:

<sup>1</sup> *Annalen der Physik*, 54, page 193, 1895.

	Water.	Glycerine.	Methyl alcohol.	Ethyl alcohol.	Acetone.	Isobutyl alcohol.	Amyl alcohol.	Ethyl ether.	Gelatine.	Xylol.	Toluol.	Benzol.
Magdala red			4	4			3				1	1
Eosine (sodium)	1	2	6	5			4		3			
Phenosafranine	1	6	7	9	11	9	10			4	3	2
Fluorescein lithium	2	3	5	4			1					
Chrysolin	2	3	3	3		3	1					
Chrysaniline			1	2			3					
Curcumin			1	2		3	4					
Aesculine	3	3	3	3	1	3	2					
B. Phenylnaphtylamin			5	5	3	5	4	2		1	1	1
Phenanthrene				1								2
Anthracene				4	3	5	4	4		5	5	5
Petroleum				5	4			3		6	6	6
Fluoresceine			1	2			3					

The figures indicate the order of intensity, 10 indicating very strong fluorescence, 1 very feeble.

This table is of use in the preparation of fluorescent solutions, and shows us that some solvents are better adapted to certain substances, other solvents, however, to others.

Knoblauch explains the effects observed as due partly to the influence of the dielectric constant of the medium upon the intensity of the emitted radiation, and partly to the difference in the degrees of dissociation of the dissolved molecules. It is, however, difficult to bring any satisfactory theory to bear upon observations of this nature, for too little is known about the actual nature of fluorescence.

**Fluorescence of Vapors.**—The molecular condition being much simpler in the case of gases and vapors than in that of liquids and solids, we should naturally expect that the most valuable data would come from the investigation of their behavior with respect to fluorescence. Unfortunately only a few vapors have been found up to the present time which exhibit the phenomenon. Of these sodium is by far the most interesting.

The fluorescence of vapors has been studied by Wiedemann and Schmidt (*Wied. Ann.*). Anthracene and a number of other organic compounds were vaporized in exhausted glass bulbs and illuminated with a concentrated beam of light. The experiment is not difficult to repeat. A pinch of anthracene is put in a bulb of glass 5 cms. in diameter, exhausted and sealed off from the pump. Light from an arc-lamp or the sun is concentrated by means of a large lens of short focus, on a point at the center of the bulb, which is then warmed as uniformly as possible by waving the flame of a Bunsen burner about it. A cone of deep violet fluorescent light will presently appear within the bulb. A similar phenomenon has been observed by the author in a bulb containing a small crystal of iodine. The fluorescence in this case can be observed at room temperature; warming the bulb slightly increases the

intensity of the fluorescent light, which is never very bright however. The fluorescence disappears as soon as the vapor becomes dense enough to exhibit much color. Whether this is due to re-absorption of the emitted light, or to an absence of fluorescence at the higher pressure, has not been determined. The admission of air or any other chemically inert gas destroys the fluorescence, which only manifests itself in a high vacuum. It is worthy of remark that the magneto-rotatory power of absorbing vapors disappears almost completely in air at atmospheric pressure; and in the case of sodium vapor, at least, the same electrons seem to be responsible for the magnetic rotation and the fluorescence. We will now consider one of the most remarkable cases of fluorescence known, that of the vapor of metallic sodium, the study of which is throwing a great deal of light upon the mechanics of molecular radiation.

**The Fluorescence of Sodium Vapor.**—Wiedemann and Schmidt were the first to observe the fluorescence of this vapor. The metal was heated in exhausted glass bulbs and the spectrum of the fluorescent light examined with a spectroscope of low dispersion. The spectrum was found to consist of broad bands or flutings in the green, a continuous region in the red, and a hazy yellow band coinciding with the *D* lines. This spectrum was first photographed by Wood and Moore (*Astrophysical J.* and *Phil. Mag.*, 1903) with a concave grating of short focus, and appeared to be the complement of the absorption spectrum taken under similar conditions. Glass bulbs could not be used owing to the rapid blackening of the glass due to the reduction of silica by the metal. Steel tubes were used, the ends of which were closed with glass plates cemented on with sealing-wax. On illuminating the vapor with monochromatic light furnished by a spectroscope it was found that when the vapor was excited by blue-violet light the yellow end only of the fluorescent spectrum manifested itself, together with an emission of light of the same color as the exciting light. As the wavelength of the exciting light was increased, *i.e.* changed gradually to green, the fluorescent region moved down the spectrum, so to speak: certain changes appeared to take place in the position of the bands, which indicated that the subject was worthy of a most careful investigation. The work was carried on by Wood during the following year, the apparatus being gradually improved and the dispersion of the spectroscope increased. Very remarkable relations between the nature of the exciting light and the fluorescent light were found, which we will now consider.

The apparatus in its final form consisted of a seamless tube of thin steel 3 inches in diameter and 30 inches long, with a steel retort at its center in which a large amount of sodium could be stored. The retort was made by fitting two circular discs of steel to a short piece of tubing, just large enough to slip snugly into the larger tube. The circular ends of the retort were provided with oval apertures as shown in Fig. 289 (1). The retort was half filled with sodium, the molten metal being poured in through one of the apertures. It was then introduced into the tube and pushed down to the center, after which the plate glass ends were cemented on as shown in the figure. This arrangement prevented the rapid diffusion of the vapor, and enabled a large supply

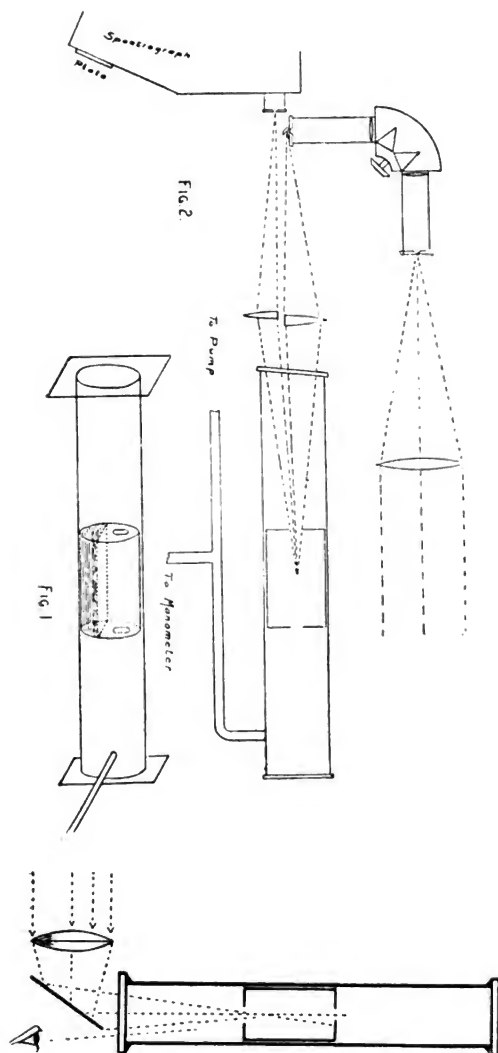


FIG. 289.

of metal to be kept at the center of the tube. The tubes used in the earlier work required re-charging after two hours' continuous operation, while the retort tube could be operated for several hundred hours on a single charge.

If a steel tube cannot be procured, a large brass tube will answer the purpose. The retort is not necessary except for long continued photographic study, the lump of sodium being placed at the center of the tube. The conical beam of light can be thrown in in an oblique direction, and the fluorescence observed through the opposite end of the tube.

After exhausting the tube to a pressure of a millimeter or two a Bunsen burner is placed under the retort, and a cone of sun or arc light focused on the oval aperture by means of a lens of long focus placed to one side of the tube, and a small piece of mirror-glass as shown in the lower diagram of Fig. 289. The other end of the tube should be covered with a black cloth.

As soon as the sodium vapor begins to form a brilliant spot of green fluorescent light will be seen at the aperture of the retort. The spectrum of the light is made up of a large number of hazy lines, which in the yellow and yellow-green region are arranged in groups or bands (Plate V., Fig. 1), which lie close together in the vicinity of the  $D$  lines, widening, however, as the blue region is approached. This spectrum is shown in the colored frontispiece (Fig. 4). Coincident with the  $D$  lines there appears a hazy band (the surrounding region being nearly devoid of light), which, if the vapor is not too dense, can be resolved into a double line, the components of which coincide with  $D_1$  and  $D_2$ . These lines only appear when the vapor is stimulated with light of the wave-length of the sodium lines, *i.e.* the fluorescent light can be regarded as an emission of light by the electrons, in virtue of the vibrations excited by the incident light. This was proven in two ways: first, by illuminating the vapor with the light of a very intense sodium flame, which provoked a bright yellow fluorescence; secondly, by illuminating it with light from a spectroscope, and varying the wave-length continuously. The yellow band only appeared when the spectroscope furnished light of the wave-lengths of the  $D$  lines. Further work, it is hoped, will show whether the  $D_1$  vibration is independent of that giving rise to  $D_2$ . To solve this interesting question it will be necessary to illuminate the vapor with the light of  $D_2$  only, and ascertain whether both lines are present or not in the fluorescent spectrum, a difficult observation, but one which can doubtless be made.

The rest of the fluorescent spectrum behaves in a very different manner. Stimulating the vapor with light of a deep violet color from the spectroscope produces no effect; as the wave-length is gradually increased a yellowish fluorescence appears, which spectroscopic examination shows to be made up of two parts, a re-emission of the same wave-lengths as those absorbed (blue) and the extreme yellow end of the fluorescent spectrum, comprised between wave-length 571 and 505. As the wave-length of the exciting light is further increased, the point of maximum fluorescence moves down the spectrum, the first bands or groups of lines disappearing. In other words, as the exciting light

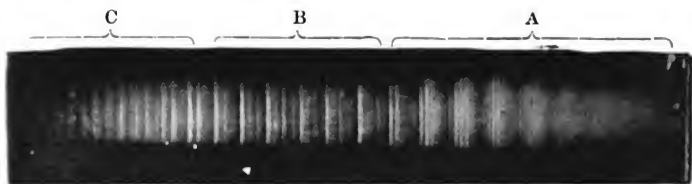


Fig. 1.

D lines.

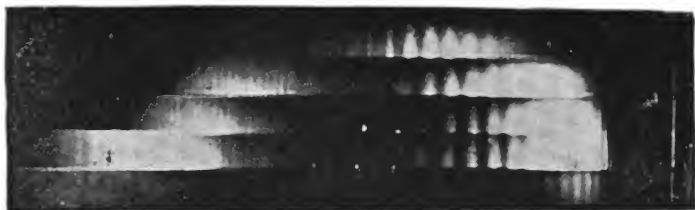


Fig. 2.

a.

b.

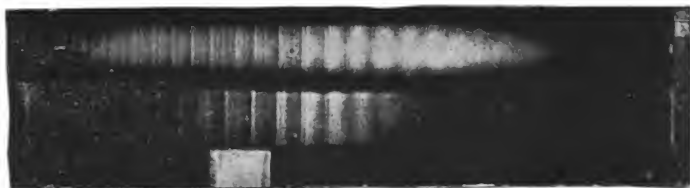


Fig. 3.

E.

D.

F.

A.

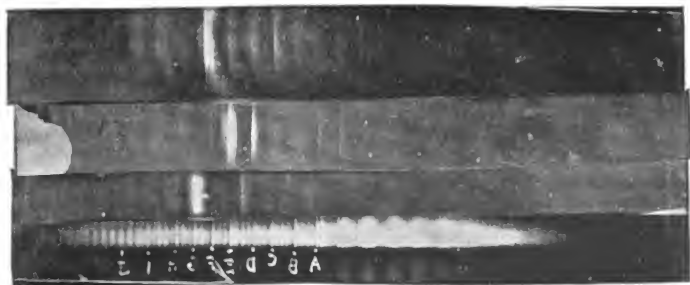


Fig. 4.

D lines.

PLATE V.

To face p. 446.

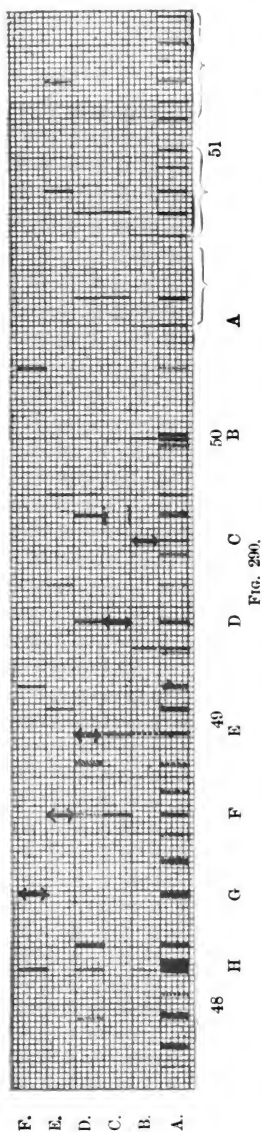
moves up the spectrum, the fluorescent light moves down the spectrum. There is, however, in all cases an emission of light of the same wave-length as the exciting light. These relations will be better understood by referring to Plate V., Fig. 2, where a number of photographed spectra are shown one above the other, and to the colored frontispiece (Fig. 5). The region of the spectrum excited is represented by the luminous band on the left hand side of the spectra. Still more remarkable is the fact that the positions of the groups of lines change as the wave-length of the exciting light is changed. This can be easily seen in Plate V., Fig. 2, the bands in the fluorescent spectra being out of step at one point. The cause of this singular phenomenon was subsequently ascertained, and will be considered presently.

Stokes's law is violated in a most flagrant manner, especially when the exciting light is near the middle of the fluorescent spectrum. This is well shown in the lower spectrum (*b*) of Plate V., Fig. 3, in which the range of the spectrum covered by the illuminating light is indicated by the white band below the spectrum. The spectrum (*a*) was obtained by illuminating the vapor with white light.

The most remarkable phenomenon of all appears, however, when the slits of the monochromatic illuminator are narrowed to the width of a hair. The stimulating light now is limited in range to the width of one of the sodium absorption lines. In other words, we take hold of, and shake, but one of the many electrons which make up the molecule. The fluorescent spectrum is now very weak, and the eye must be carefully rested if the remarkable changes which accompany a change of wave-length of the exciting light are to be made out. It is at once apparent that the character of the spectrum is much altered, and as the wave-length of the exciting light is slowly altered, the lines of the fluorescent spectrum appear to move about in the liveliest manner. The whole spectrum appears in motion, the luminous bands moving in a rippling manner, like moonlight on water. The motion is soon seen to be an illusion due to the continual disappearance and re-appearance of the bright lines, the phenomenon reminding one forcibly of the scintillations produced on a zinc sulphide screen by the radium bombardment.

Exposures of five or six hours were necessary for the recording of the spectrum on the photographic plate, and at the present time only half-a-dozen satisfactory plates have been secured. These show clearly, however, that certain lines of the fluorescent spectrum are associated with one another. A number of the photographs are reproduced in Plate V., Fig. 4, together with a portion of the complete spectrum. The photographs suggest that the electrons are arranged into groups, and that the stimulation of any member of a group sets the entire group in vibration, without, however, disturbing any of the other groups. In the records thus far secured, the brightest lines are in the immediate vicinity of the line of wave-length equal to that of the exciting light, but the earlier photographs obtained with less homogeneous excitation show that in some cases the region of strongest fluorescence is far removed from the excited region.

The spectra obtained with highly homogeneous stimulation furnish a clue to the cause of the shifting of the groups of lines previously



referred to. These groups extend from  $\lambda = 505$  to  $\lambda = 571$  and the photographs indicate that *one* line from each group together form a series, the lines being simultaneously excited. This has only been observed in the three groups at 505, 509, and 514, owing probably to insufficient exposure. The lines are too faint even in these groups to reproduce, and a drawing has been made of the spectra obtained with five different stimulations (Fig. 290). A portion of the complete spectrum obtained with white light is shown in the lower diagram *A*. The line at which the stimulation has been applied is indicated in each of the other diagrams by a double arrow. In diagram *B* the stimulation is at  $\lambda = 4962$ , and the first line in each of the three groups (which are bracketed) appears. The first group is incomplete in all cases, the first two lines only appearing. In diagrams *C* and *D*, with the stimulations at  $\lambda = 4935$  and  $4895$ , we obtain the second line of each group, the electrons stimulated in these two cases belonging to the same series.

In the *E* diagram we have an excitation which gives us the third line of each group. Longer exposures will doubtless enable these relations to be traced throughout a wider range of wave-lengths, but from what is already known it is clear that the shift in the apparent position of a group of lines results from a change in the distribution of the intensity among the individual lines constituting the group.

These experiments show in a striking manner the complexity of the piece of machinery which we call the sodium molecule. Professor Rowland once said that a molecule is much more complicated than a piano. In most cases all that we have been able to do is to strike the entire keyboard at once, but in the case of sodium it seems possible to strike one key at a time. A study of the fluorescent spectra of other vapors will doubtless do much to clear up the mystery of the mechanism of molecular radiation.

**Transformation of Light into Chemical Energy.**—In the cases of substances which

emit other than reflected light when illuminated it has been shown that the emission is in all probability due to the formation of unstable groupings of the molecules or atoms, the breaking down of which is accompanied by the liberation of radiant energy.

We have now to consider cases in which stable re-arrangements of the molecules or atoms are produced by the action of light, *i.e.* a transformation of light into molecular or atomic energy. Two classes of phenomena come under this head. We may have an element transformed into some allotropic modification, or we may have a chemical compound decomposed into its constituents, or a chemical compound formed from its elements.

**Molecular Changes.**—We will first consider a few cases in which light produces a specific action upon elementary bodies. Ordinary white phosphorus is transformed into the red modification by the action of light, while sulphur is changed into the insoluble variety.

Ultra-violet light, if sufficiently intense, changes oxygen into ozone. The new mercury vapor lamps, in quartz tubes, emit the short waves in such profusion that the odor of ozone is as noticeable as during the operation of a large static machine.

Amorphous selenium, which is fairly transparent, is a non-conductor of electricity, and under the action of light passes over into what is usually termed the metallic modification, which is opaque and conducts electricity. Still more remarkable is the fact, discovered in 1872 by May, that the metallic form conducts better when illuminated than when in the dark.

**Chemical Changes.**—Examples of chemical decomposition are very numerous. Peroxide of hydrogen is rapidly decomposed, by exposure to light, into water and oxygen, the re-combination of which, or the "burning" of the water if it could be accomplished, would liberate in the form of heat the chemical energy into which the light has been transformed. Chloride of nitrogen decomposes explosively when illuminated.

Many salts of silver, gold, iron, platinum uranium, etc., are decomposed by the action of light, these actions being at the bottom of all photographic processes. In certain cases the decomposition may be only started, a molecular instability being imparted by the action of light, the continuation of the process being effected by reducing agents (developers). Another interesting example of photo-chemical decomposition has been pointed out by Tyndall. The vapor of amyl nitrite in a glass flask is colorless and transparent, until illuminated by a powerful beam of sun or arc-light, when a dense white cloud at once forms, the products of decomposition condensing to liquid drops which scatter the light. Light which has previously been filtered through the vapor of amyl nitrite is unable to bring about this decomposition, showing that the effective waves are absorbed.

A remarkable transformation of luminous into chemical energy on a vast scale is the breaking up of the carbonic acid of the atmosphere, which takes place in the leaves of plants. Some of the oxygen is liberated in the free state, and some enters into the organic compounds which the plant forms, by the subsequent combustion of which we may recover the original energy of the light in the forms of both heat and light.

The decompositions which we have enumerated above are for the most part effected by the more refrangible violet and ultra-violet rays.

If these rays are greatly in excess other very remarkable transformations are produced. Oxygen is changed into ozone, as we have seen. Permanganate of potash solutions are almost instantly bleached, and many other chemical substances break down, which under ordinary conditions are stable. The author has observed the liberation of a gas from glycerine (probably dissolved air) and also from distilled water contained in quartz vessels, under the action of the ultra-violet rays from the cadmium spark.

**Chemical Combination.**—Examples of chemical combination resulting from the action of light are not as common. The best known case is the union of chlorine and hydrogen to form hydrochloric acid. The mixture of the two gases is best obtained by the decomposition of strong hydrochloric acid by electricity, the operation being conducted by feeble lamplight. The gas which comes off first, contains an excess of hydrogen, owing to a solution of a part of the chlorine in the acid. Subsequent portions may be collected in small glass bulbs, blown in strings and separated by fine thin walled tubes, which are broken and closed with warm sealing wax, care being taken not to have the wax on fire. The bulbs should be kept in a dark box until wanted. On exposing one to sunlight or the light of burning magnesium, or the electric arc, a violent explosion is said to occur. Precautions should of course be taken to avoid injury from the flying fragments of glass.

This experiment is not an easy one to repeat. The author has succeeded, only after repeated trials, in producing an explosion, and apparently others have met with the same difficulty.

It is probable that, in this case, we are really dealing with a decomposition, the action of the light being to separate the diatomic molecules into free hydrogen and oxygen atoms, which then unite.

The oxidation of metals, the action of chlorine on organic compounds, and the oxidation of organic compounds are accelerated by the action of light, though it is quite probable that here, as in the case just cited, the real action of the light is of the nature of a decomposition which is preliminary to the formation of the new compound.

**Transformation of Luminous into Electrical Energy.**—Cases of this sort are worthy of mention, though in none of them is the transformation direct. In the thermopile, when illuminated by light, we have a liberation of electrical energy, which is, however, due only to the heating action of the radiation. In the photo-electric cell, which consists of two silver plates coated with silver chloride and immersed in dilute acid, a feeble current flows through a wire joining them, when one of them is illuminated by light. In this case the action of the light is primarily chemical, the current being the result of the chemical decomposition. The case is somewhat analogous to phosphorescence, the difference being that here the reversion of the process set up by the light liberates electricity instead of light.

**Luminescence and the Radiation of Electrons.**—The general term luminescence has been proposed by Wiedemann for all cases in which we have an emission of light from all causes other than high temperature. Fluorescence and phosphorescence are termed by him photo-

luminescence, the emission being due primarily to the action of light. Tribo-luminescence covers cases where light is produced by friction, as when two lumps of sugar are rubbed together in the dark or crystals of uranium nitrate are crushed. If the radiation is the direct result of the passage of an electric current, as appears to be the case in vacuum tube discharges, the phenomenon is termed electro-luminescence. Chemi-luminescence is applied to cases where chemical action is the exciting cause, as the well-known case of the emission of light by phosphorus and other substances undergoing slow oxidation. One other type, thermo-luminescence, in which an emission of light is produced by warming the substance, is not to be confounded with the true temperature emission, since the necessary temperature is far below that commonly designated as a red-heat. Fluor-spar and diamonds come under this class, a specimen of the former in the possession of the author emitting light when warmed by the hand only. It has been proven, however, that in these cases a preliminary exposure to light is necessary, the emission ceasing after a short time, only to be renewed by exposing the crystals to the action of light, the case being analogous to that of Balmain's luminous paint already referred to.

Kirchhoff's law governing the relation between the emission and absorption of light of a given wave-length cannot be applied to cases of luminescence.

In many cases it is not at once apparent whether the emission is due to high temperature or to some one of the causes above enumerated, an example being the much discussed case of the very high emission of the Welsbach incandescent mantle, which is obviously in part a true temperature emission, but which some investigators have attributed in part to phosphorescence.

In the modern theories of absorption we find frequent use made of the conception of a re-emission of radiant energy by an electron which is vibrating in unison with the incident light waves. The emitted energy should, however, be of the same wave-length as that of the exciting waves, and while we have plenty of examples of photo-luminescence, it seems probable that in these cases the phenomenon is extremely complicated, for the emitted radiation consists of a heterogeneous mass of waves, usually of slower period than that of the incident light. Repeated efforts have been made by various investigators to detect a lateral emission of yellow light by sodium vapour when in the act of absorbing sodium light. The phenomenon has at last been observed: a dense mass of non-luminous sodium vapour radiating a brilliant yellow light when illuminated by the light from a very intense sodium flame. The same phenomenon has been observed in the region of the channelled absorption; in this case, however, radiations of other wave-lengths are emitted, as well as those of the same period as that of the exciting light. This seems to be the first case found of the phenomenon, which it may perhaps be well to style *resonance radiation*, to distinguish it from fluorescence. The intimate connexion between fluorescence and phosphorescence, and the almost indisputable evidence that the latter phenomenon is associated with chemical changes produced in the substance, makes it appear probable that fluorescence and resonance radiation are two entirely different phenomena, though the former is doubtless caused in some roundabout way by resonance.

## CHAPTER XIX.

### LAWS OF RADIATION.

IN the present chapter we shall discuss the laws which govern the emission of light by bodies in virtue of their temperature. Of the physical processes which are at work we know but little. In the heated body we believe that the molecules are in a rapid vibratory motion, which increases in violence as the temperature is raised, but the precise mechanism by which this energy is transformed into radiant energy is but imperfectly understood. The simplest assumption appears to be that the molecular collisions in some way throw the electrons into vibration, or increase the amplitude of their vibrations, and that these radiate energy into space. The same thing does not, however, occur in the case of a gas, for the same molecules can be heated to a much higher temperature—several thousand degrees even—without emitting a particle of light. Mercury may be heated “white-hot” in a sealed quartz tube, but mercury vapor can be heated to the highest temperatures at our command without emitting any visible light. It would be extremely interesting to study the emission of light by some fluid which had a critical temperature of about  $700^{\circ}$ , observing the luminosity as the liquid passed over into the gaseous state. The phenomenon might be studied in a quartz tube (which emits but little light, for reasons which will appear presently) if a suitable substance could be found. Strutt has experimented with mercury with a view of measuring the electrical conductivity of the vapor at the critical temperature, but even thick walled capillaries of fused quartz exploded, or yielded to the pressure, before any evidence of the approach of the critical state appeared.

The electrons of the gas molecules can be made to emit light by the stimulus of electrical discharges, or chemical changes, and in one or two cases (iodine and sodium) by virtue of temperature alone.

All substances in the *solid or liquid* state emit light as soon as their temperature is raised above  $500^{\circ}$  C. The intensity of the light varies, however, with the nature of the substance, being greatest for substances which absorb light strongly. A *perfectly* transparent solid or liquid would not emit light even at the highest temperature. No such substance is known, however, though a bead of microcosmic salt heated in a loop of platinum wire by means of a blast lamp, comes pretty near to fulfilling the required conditions. We shall presently investigate

the laws which govern the emission of light by various substances, and in particular the emission by a substance which is perfectly black, *i.e.* perfectly absorbing. No substance has this property, though by an experimental artifice we can produce a radiator which will give out radiation identical in every respect with the radiation which would be given out by a perfectly black body at the same temperature.

**The Relation between Emission and Absorption.**—We will now investigate the very intimate relation which exists between the emission of heat or light waves by a substance, and its power of absorbing the same waves.

That some relation existed between the emissive and absorbing power in the case of radiant heat, was indicated by the experiments of Leslie, Melloni, Provostaye and Desains, and others. Their measurements were, however, made for the most part with apparatus of insufficient sensitiveness, the spectrum regions being only roughly determined by means of absorbing screens. It was determined, however, without question that bodies which possessed a strong emissivity, acted also as powerful absorbers of the radiant heat which they emitted, and the approximate equality of the emitting and absorbing powers was recognized.

**Ritchie's Experiment.**—This relation was shown by a very simple and ingenious experiment devised by Ritchie (*Pogg. Ann.*, 28, p. 378, 1833). Two air-tight metal chambers were connected by a glass tube containing a drop of fluid, the whole forming an air thermometer.

Between them a third metal chamber of the same size was mounted, which could be heated by filling it with boiling water. One surface of this heat radiator was covered with lamp-black, the other with the substance under investigation, for example powdered cinnabar. (See Fig. 291.) The surface of the air thermometer which faced the radiating lamp-black surface was coated with cinnabar, while the surface which faced the cinnabar radiator was coated with lamp-black. With

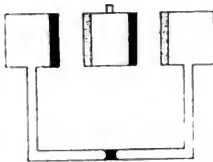


FIG. 291.

the apparatus arranged in this way no movement of the fluid drop occurred when boiling water was poured into the radiator, which established the fact that the emitting and absorbing powers were equal. Let us assume that the lamp-black radiates powerfully, the cinnabar feebly. The powerful radiations coming from the former are but slightly absorbed by the latter, while the feeble radiations from the latter are strongly absorbed by the former, the heating of the two chambers of the thermometer being equal. Calling  $E$  the amount of heat emitted by the cinnabar and  $A$  its absorbing power,  $e$  and 1 corresponding expressions for the lamp-black, the lamp-black surface emits an amount of radiant heat  $e$ , of which the cinnabar surface absorbs the amount  $eA$ . The cinnabar surface emits an amount  $E$ , which the lamp-black completely absorbs (since its absorbing power = 1). The equality of temperature indicated by the thermometer shows us that

$$eA = E \text{ or } \frac{E}{e} = A. \text{ Now } \frac{E}{e} \text{ is the ratio of the amount of heat emitted,}$$

by cinnabar to the amount emitted by lamp-black at the same temperature. This ratio we will call the emissivity of the cinnabar, and our equation shows us that it is equal to the absorbing power.

**Kirchhoff's Law.**—This relation was reduced to a more definite form in 1859 by Kirchhoff, and independently by Balfour Stewart, who showed that it must be true for each wave-length in the emitted spectrum, and formulated the law which has since gone by his name. "*At a given temperature the ratio between the emissive and absorptive power for a given wave-length is the same for all bodies.*" The theoretical considerations from which Kirchhoff's law is deduced will be discussed later on in the chapter. It may be remarked that we frequently meet with the statement that the absorption of light by flames which contain the vapors of metals is a necessary consequence of this law. This is by no means the case, for Kirchhoff's law is only to be applied to radiation which results from temperature. In the case of the emission by flames, the phenomena are probably connected with chemical changes which are taking place.

Cotton has called attention to the fact that there are two distinct relations which are almost invariably confused: a qualitative rule, which connects the absorption and emission for a given substance, and a quantitative rule, which establishes relations between different bodies. From the former we can only draw the conclusion that if a body emits certain radiations it absorbs them when they come from without. It may, however, absorb other radiations, as is illustrated by the selective absorption of colored substances. For these radiations  $\frac{E}{A} = 0$ , since at the temperature in question there is no emission of visible radiations.

**Absorption by Flames.**—The absorption by flames and the reversal of spectral lines are special cases of the qualitative rule. These cases have been studied by Gouy, who sought to determine by experiment whether flames were transparent to the radiations which they emitted. His method consisted in comparing photometrically the light of different thicknesses of radiating gas. If no absorption occurred, doubling the thickness should double the intensity of the illumination. He found, in the case of every line examined, that after the line had attained a certain brightness absorption manifested itself. If  $E$  and  $A$  be the emissive and absorptive powers of unit thickness of the flame and we increase the thickness, we shall find that the emissive power (*i.e.* the intensity sent out by the thick layer) approaches  $\frac{E}{A}$  as a limit.

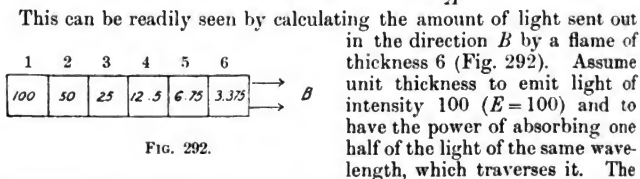


FIG. 292.

intensity of the light from the element 1 is reduced to 50 by the absorption of element 2, and on emergence from element 6 has an intensity of only 3.37. The total amount of light emerging from

element 6 is obviously the sum of the numbers in the different elements, which in this case is 197.62. If the number of elements were increased indefinitely this sum would be 200, the limiting intensity which is equal to  $\frac{E}{A}$ , as defined above.

It should be possible therefore to reverse all lines by a sufficient increase in the thickness of the flame, for the line will appear darker than the background of the continuous spectrum when the intensity of the light of the given wave-length emitted by the flame is less than the intensity of the continuous spectrum at the point in question. By increasing the thickness we progressively increase the absorption, while the intensity of the flame remains practically constant after a certain thickness is reached.

It by no means follows, however, that all vapors which emit radiations of definite wave-length will show the corresponding wave-lengths reversed, even when a very thick layer is used. The above argument is based on the fact that there is a certain absorbing power. If  $A=0$  reversal can never occur. Now it is a noteworthy fact that reversals are only obtained in the case of the bright lines in metallic spectra. The lines in the spectra of the metalloids have never been reversed. This shows that Kirchhoff's law does not even apply qualitatively in these cases; in other words, there is powerful selective emission without any corresponding selective absorption. In such cases  $\frac{A}{E}$  is infinite.

Summing up, we find that for colored substances, absorbing without a corresponding emission,  $\frac{E}{A}=0$ , for flames which show reversal of spectrum lines  $\frac{E}{A}$  is finite, for luminous gases which show no absorption  $\frac{E}{A}=\infty$ .

**Temperature Radiation of Gases.**—Kirchhoff's law states that in the case of radiation which results solely from temperature,  $\frac{E}{A}=\text{const.}$  for all bodies at the same temperature. The value of the constant is a function both of the temperature and the wave-length, and is equal to the emissive power of a perfectly absorbing body, that is a body which at the temperature in question completely absorbs, without reflection, all radiation falling up, no matter what its wave-length. This amounts to saying that at a given temperature no substance can emit more light of a given wave-length than a perfectly black body. Paschen has compared the intensity of the light of the two  $D$  lines in the sodium flame with the total intensity of a region, completely enclosing the  $D$  lines, in the continuous spectrum of a black substance heated in the same flame. The total intensity of the  $D$  radiation was more than twice as great as that of the region of the continuous spectrum which enclosed them, from which the inference can be drawn that something other than temperature is concerned with the emission of light by the sodium flame. The same thing was found by Kayser and Paschen in the case of the ultra-violet bands of the arc, which were much brighter than a

corresponding region of the spectrum of the positive crater, notwithstanding the fact that the temperature of the latter is higher than that of the arc proper.

There are, however, certain cases in which we have a true temperature emission of a gas. Carbon dioxide, when heated, emits an infra-red radiation, the spectrum showing a very sharp band at  $\lambda = 5.12\mu$ . Paschen (*Wied. Ann.*, 51, page 1, 1894) found that a layer of the gas 7 cms. thick emitted and absorbed as strongly as a layer 33 cms. thick. This indicated that the radiation from a 7 cm. layer could be regarded as the equivalent of the radiation from a layer of infinite thickness, or in other words, the radiation of a black substance at the same temperature. By heating the gas in a tube, and measuring the intensity of the emitted radiation with a spectro-bolometer, Paschen found that for all temperatures between  $150^\circ$  and  $500^\circ$  the intensity of the radiation of wave-length  $5.12\mu$  was only a little below that of a black body at the same temperature. As a black body he used a smoked strip of platinum heated by an electric current. This investigation is practically the only one which has been made to test Kirchhoff's law in the case of substances which give a discontinuous temperature-emission spectrum. The proof of the law follows from the fact that for a layer from which the radiation is the equivalent of that from an infinitely thick one, we have the relation  $\frac{E}{A} = e$ , the emissivity of a black body.

as we can at once see by comparing the equation  $eA = E$ , given at the beginning of the chapter with the relation deduced on page 454, namely, that as the thickness of an emitting absorbing layer increases, the intensity of the radiation leaving it approaches the value  $\frac{E}{A}$

as a limit. An interesting conclusion has been drawn by Kayser from Paschen's experiment. The thickness of the layer necessary in order that the radiation may equal that of a black body will vary for the different lines in the spectrum, the greatest thickness being necessary

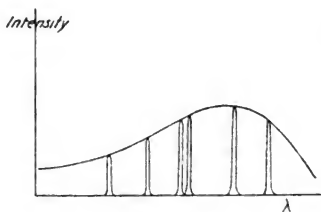


FIG. 293.

for the wave-lengths for which the absorption is a minimum. If we employed a layer so thick that the conditions were realized for all of the lines, and plotted the intensities of the lines as ordinates, and the wave-lengths as abscissae, the curve joining the points should be the emission curve of a black body at the same temperature as shown in Fig. 293.

We may perhaps apply the same reasoning to the case of the bright lines of flames. The sodium flame, for example, exhibits in addition to the *D* lines a very faint line in the green. By increasing the thickness of the flame the ratio of the intensity of the *D* lines to that of the green line should become less. Multiplying the flame by repeated reflections between two parallel mirrors amounts to the same thing as

increasing its thickness, and Wanner (*Wied. Ann.*, 68, page 143) found that a sodium flame placed between two concave silvered mirrors showed the green line with great distinctness. The *D* lines were not increased in brilliancy in anything like the same proportion, and appeared much broadened, a faint continuous spectrum appearing in addition. Attempts made by the author to repeat this experiment both by the use of mirrors and a sodium flame, over a meter in length, gave negative results.

Very few cases are known in which visible radiations can be obtained by merely heating a gas or vapor. An immense amount of work has been done by Pringsheim and others in endeavors to obtain a luminous emission from gases as a result of high temperature alone. Efforts in this direction have been almost, without exception, in vain, and Pringsheim came to the conclusion that, at least for temperatures which could be commanded in the laboratory, gases remained dark.

There are, however, one or two exceptions to this rule, which were studied by Salet and Evershed. Iodine vapor when heated to a temperature of only six or seven hundred degrees gives off a reddish orange light. The experiment is best performed by arranging a small spiral of platinum wire, which can be heated by a current, in a test-tube in which a little iodine is vaporized by means of a Bunsen burner. An orange-colored flame is seen to rise from the hot wire. A similar phenomenon has been observed in the case of sodium vapor, but is not as easily reproduced as the iodine emission. The best arrangement is a long steel tube containing metallic sodium, and highly exhausted, heated by a row of burners, or better in one of the electrical ovens made by Heraeus of Hanau. Evershed was of the opinion that the spectrum was continuous, but by employing a vapor of small density Koenen (*Wied. Ann.*, 65, page 256) succeeded in resolving it into bands, which corresponded to the bands seen in the absorption spectrum. It is not difficult to see how a continuous spectrum might easily result when a thicker layer or a denser vapor was used, for we have only to apply the principles involved in the case suggested by Kayser to a spectrum consisting of bands, the intensity having a finite, though different value, for each wave-length.

**Temperature Radiation of Solids and Liquids.**—The radiation of solids and liquids is especially adapted to the proof of Kirchhoff's law, since in these cases we can be sure that it is the result of temperature alone.

That the emission of light by heated substances is proportional to the absorption can be easily shown by heating a fragment of a piece of decorated china in a blast-lamp. The design emits much more light than the white background, owing to its stronger absorbing power. It follows at once from Kirchhoff's law that a substance, perfectly transparent for a given region of the spectrum at all temperatures, will emit no radiations in this region, no matter how hot it be heated. Sodium phosphate possesses this property to a high degree in the region of the visible spectrum, and if a bead of it be heated to the temperature of a white heat in a loop of platinum wire by means of a blast lamp, it emits scarcely any light, though the wire glows vividly.

Small specks of impurities in the fluid mass glow like stars on the dark background. It is frequently stated that ruby glass when heated emits an excess of green light, but the phenomenon is not very striking, if it exists at all. Cobalt glass was investigated by Rizzo (*Atti. acc. Torino*, 29, 424, 1894), who was unable to establish any relation between its emitting and absorbing power. His apparatus was not very sensitive however, and it is difficult to draw very definite conclusions from his observations. (See Appendix D.)

The emission and absorption of rock-salt has been studied by Abramczyk (*Wied. Ann.*, 64, p. 625, 1898). Unfortunately he made use of absorbing screens, instead of spectroscopic dispersion, and his results cannot on this account be regarded as wholly trustworthy. He found, however, that the heat emission consisted of two parts, one of which was stopped by a salt plate, while the other was freely transmitted. He found that 40% of the radiation from the salt was reflected by a polished plate of the same material, from which he inferred that there was a selective emission at a region in the spectrum corresponding to the position of the band of metallic reflection observed by Rubens and Nichols. Rubens and Aschkinass have pointed out, however, that a strong emission is not necessarily to be expected at a band of metallic reflection. Though this region is one of relatively strong absorption, the absorbed portion may be only a small fraction of the amount reflected, and a high value of the emissivity is not to be expected, as it is the absorption proper, and not reflection, that is related to the emission. In fact high reflecting power is usually associated with low emissivity, as is shown by the small ratio of the emissivity of a polished metal surface to a surface of the same metal brought into a spongy condition, i.e. into the state of platinum black. A hint has been given in the chapter on Absorption as to the physical explanation of the increased absorption in this case, and it is instructive to reverse the reasoning and apply it to emission. As we shall see later the case is not unlike that of a hollow body, the emission from the interior of which we shall show is equal in intensity to that of the radiation of a perfectly black body at the same temperature.

In the case of substances which are not transparent, as we have seen, a portion of the incident energy is reflected and a portion absorbed. If the intensity of the light is 1, the absorbed fractional part  $A$ , and the reflected part  $R$ , we have  $1 = R + A$  or  $A = 1 - R$ . Substituting this value in our formula  $\frac{A}{E} = e$  gives us an expression for Kirchhoff's law, in which the relation between emissivity and reflecting power is established:

$$\frac{E}{1 - R} = e, \text{ the emissivity of a black substance.}$$

This formula has been verified by Rosenthal (*Wied. Ann.*, 68, p. 783), who investigated the emission and reflection of quartz, mica, and glass, with a spectrometer and thermo-element, and compared the results with the values calculated from the above formula. The low emissivity at regions of the spectrum corresponding to those of the maxima of the

reflection curve is clearly shown in Fig. 294, and the close agreement between the observed values and those calculated from the above formula may be regarded as a most excellent proof of the law. As will be seen the law has been proven quantitatively for but few substances. That there is a relation between absorption and emission in a large number of cases, where Kirchhoff's law cannot be expected to hold, is evident. Some of these cases we have already considered.

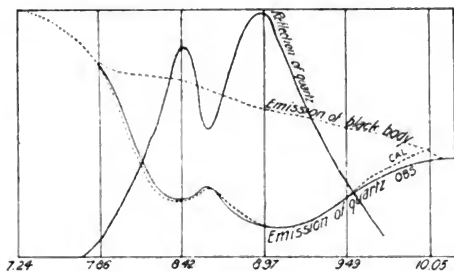


FIG. 294.

A noteworthy example often quoted as a proof of the law is the remarkable relation between the emission of heated oxides of erbium and didymium, and the absorption spectra of the same oxides and solutions of the salts of the metals. The oxides, unlike all other solid substances, when heated to incandescence, show in addition to a continuous spectrum a number of bright bands, which correspond in position to the absorption bands, at least approximately. This can be easily shown by dipping a platinum wire in a concentrated solution of erbium chloride and heating it in a Bunsen burner, the continuous spectrum of the white hot oxide being crossed by a number of bright bands.

We have in this case the law holding qualitatively for very different temperatures of the emitting and absorbing substance. Even in the case of fluorescence we have an indication of the law, for, as we have seen in the chapter on Fluorescence, a substance *while fluorescing* absorbs more strongly the waves of the same wave-length as those which it is emitting.

**Emission of Polarized Light.**—Certain crystals, tourmaline for example, have an absorbing power which differs according to the plane in which the vibrations are taking place. Suppose the crystal to be so oriented that its absorptive power is greatest for horizontal vibrations. We might expect, on heating the crystal, to find a preponderance of horizontal vibrations in the emitted light. This was found to be the case by Kirchhoff, who heated a crystal in a Bunsen flame and found that, on viewing it through a double-image polarizing prism, one of the images was distinctly brighter than the other.

A quantitative proof of Kirchhoff's law in the case of glowing tourmaline has been made by Pflüger (*Annalen der Physik*, 7, p. 806, 1902), who measured with a spectrophotometer the absorption and

emission of the crystal at the same temperature and for the same wave-length. If  $J$  is the intensity of the incident light,  $D$  that of the transmitted light, and  $R$  that of the reflected, the intensity of the absorbed light  $A = J(1 - R - D)$ .  $R$  was calculated from the reflection formula  $R = \left(\frac{n-1}{n+1}\right)^2$ , while  $D$  and  $E$  were observed with the spectrophotometer. Designating by  $E_o$  and  $E_e$  the emissivity for vibrations parallel respectively to the vibrations of the ordinary and extraordinary rays, and by  $A_o$  and  $A_e$  the corresponding absorptive powers, we have, if Kirchhoff's law can be extended so as to include the direction of the vibration,

$$\frac{E_e}{A_e} = \frac{E_o}{A_o} \quad \text{or} \quad \frac{E_e}{E_o} = \frac{A_e}{A_o}.$$

After eliminating all sources of error, Pflüger obtained as final values for the two ratios,

$$\frac{A_e}{A_o} = \cdot 650, \quad \frac{E_e}{E_o} = \cdot 641,$$

a very beautiful verification of the law as applied to anisotropic media.

**Deduction of Kirchhoff's Law.**—The law of Kirchhoff can be deduced from purely theoretical considerations. Kirchhoff's method is free from serious objections, but assumptions are made which cannot be regarded as truths without further treatment. Moreover, his method involves the consideration of bodies which really have no existence, such as perfect reflectors and perfectly transparent substances.

The most logical and concise treatment is due to Pringsheim (*Verh. d. deutsch physik. Ges.*, 3, pp. 81-84, 1901). Consider a ball  $\kappa$  composed of any material enclosed in a hollow vessel, opaque to radiation of all wave-lengths, and uniformly heated to any given temperature. The ball emits in unit time the total radiation  $E$ , while there falls upon it from the walls in the same time the amount  $e$ , of which the fraction  $Ae$  is absorbed. Since by Carnot's principle the temperature cannot change, the amount of radiation emitted by the ball must equal the amount absorbed, as  $E = Ae$ . If the ball is made of a conglomerate of different substances, some parts of its surface may absorb more strongly than others. Suppose we rotate the ball: the amount of energy  $e$  falling upon it will only be changed by an infinitely small amount, since only the part of the radiation which came originally from the body, and is reflected back from the walls, can be responsible for the change: the amount of this which falls across the body is of course very small. If, however, the radiation from the walls is not uniform, *i.e.* if it has especial states of polarization, or is more intense in certain directions than in others, the amount of heat absorbed by the body would be changed by its rotation. If, for example, the rotation brought a strongly absorbing surface element into the path of an especially intense ray coming from the wall, the absorption of heat would be increased.

The equation  $E = Ae$  shows, however, that this is impossible; therefore the radiation in the hollow vessel has similar properties in every direction.

The total radiation  $e$  is made up of waves of all possible lengths

between 0 and  $\infty$ . Call  $e_\lambda$  the radiant energy comprised between the limits  $\lambda$  and  $\lambda + d\lambda$ , then

$$e = \int_0^\infty e_\lambda d\lambda.$$

Further, let  $A_\lambda$  be the absorption coefficient of  $\kappa$  for waves of length  $\lambda$ : the total absorbed energy is then

$$Ae = \int_0^\infty A_\lambda e_\lambda d\lambda,$$

or

$$E = \int_0^\infty A_\lambda e_\lambda d\lambda.$$

Now let  $\kappa$  be brought into another hollow vessel of different material but at the same temperature. The emission of  $\kappa$  remains the same, also its absorption coefficient  $A_\lambda$ , for waves of the designated length.

If in the present case the radiation  $e_{1\lambda}$  which falls upon  $\kappa$  is different from  $e_\lambda$  (in the previous case), we should have

$$\int_0^\infty A_\lambda e_\lambda d\lambda = \int_0^\infty A_\lambda e_{1\lambda} d\lambda.$$

Since, however,  $A_\lambda$  is quite independent of  $e_\lambda$  the above equation can only hold if

$$e_\lambda = e_{1\lambda}.$$

By comparing this with  $E = Ae$  we see at once that the radiation  $e$  is equal to that which  $\kappa$  would emit if it were perfectly absorbing, i.e. if  $A = 1$ . This shows us that the radiation within a hollow vessel heated to a uniform temperature is independent of the material and shape of the vessel, and is identical in every respect with the radiation emitted by a perfectly absorbing body at the same temperature.

Consider now the radiant energy which a surface element  $ds_1$  of the body  $\kappa$  sends to the distant surface element  $ds_2$  of the vessel. We will define the emission coefficient  $E_\lambda$  of  $\kappa$  as the single radiation  $E_\lambda d\lambda$ , which in unit time reaches  $ds_2$  from  $ds_1$ . This radiation has a wave-length  $\lambda$  and any state of polarization. In a hollow vessel of uniform temperature,  $ds_1$  gives out a radiation similar to that of a black body at the same temperature. The total energy of wave-length  $\lambda$  and of a given state of polarization which reaches  $ds_2$  from  $ds_1$  is therefore  $e_\lambda d\lambda$ , if  $e_\lambda$  is the emission coefficient of a black body under similar conditions.

We thus have  $e_\lambda = E_\lambda + G_\lambda$  if we define  $G_\lambda d\lambda$  as the energy of wave-length  $\lambda$ , and of a definite state of polarization, which, coming originally from the walls of the vessel, and transmitted by, or reflected from, the body  $\kappa$ , reaches  $ds_2$  from  $ds_1$ .

We must now determine the value of  $G_\lambda$  and substitute it in the above equation. The radiation from the inner surface of the vessel which reaches  $ds_2$  by reflection or refraction from  $ds_1$  is equal to the amount which, leaving  $ds_2$ , reaches the inner surface by way of  $ds_1$ . Of all the waves of length  $\lambda$  which leave  $ds_2$  the amount reaching  $ds_1$  in unit time is  $e_\lambda d\lambda$ , of which an amount equal to  $A_\lambda e_\lambda d\lambda$  is absorbed, while the remainder  $(1 - A_\lambda)e_\lambda d\lambda$  are in part reflected and in part

transmitted, and pass off to the inner wall again. This remainder is equal to  $G_\lambda d\lambda$ , and substituting the value  $G_\lambda = (1 - A_\lambda)e_\lambda$  in the equation

$$e_\lambda = E_\lambda + G_\lambda$$

gives us

$$e_\lambda = E_\lambda + (1 - A_\lambda)e_\lambda$$

or

$$E_\lambda = A_\lambda e_\lambda,$$

an equation which expresses Kirchhoff's law, showing that the emission coefficient of any substance for any given wave-length, divided by its absorption coefficient for the same value of  $\lambda$ , is equal to the emission coefficient of a perfectly black body, for, as we have seen above,  $e_\lambda$  represents the latter quantity.

That the intensity of the radiation from the inner surface of a hollow vessel is independent of the nature of the material can be shown by placing a fragment of decorated china in a porcelain crucible heated over a Bunsen burner. If the cover of the crucible is put on, a small opening being left through which the interior can be viewed, and the flame of a second burner be directed upon it, so as to bring the whole to a nearly uniform temperature, the decorations on the china will be quite invisible, the radiations from them being equal to the radiation from the rest of the surface. As we have seen, if the china is heated in the open air the dark portions radiate more strongly, the design appearing brighter than the background. The cause of the equality in the case of an enclosed radiator can be very simply stated. The radiation is made up of two parts, the emitted and the reflected, the latter coming from the heated walls. Dark portions of the material emit more powerfully than white portions, since their power of absorption is greater; on the other hand, they reflect scarcely any of the radiation from the walls. The white portions, which emit feebly, reflect powerfully, and, owing to the proportionality between emission and absorption, a perfect balance is secured.

This principle is now made use of in experiments pertaining to radiation. In studying the nature of the radiation of perfectly absorbing bodies as a function of temperature, it was formerly the custom to make use of an electrically heated strip of platinum with a smoked surface. Such a radiator cannot, however, be brought to a high temperature, owing to the oxidation of the carbon. For high temperature work it was customary to coat the strip with platinum-black, or copper oxide.

Such radiators cannot be regarded as perfect, and at the present time the heated hollow chamber is almost exclusively used, the radiation to be examined escaping through a small hole.

**The Perfect Black Radiator.**—While the principle that the radiation within a closed space at a uniform temperature is identical with the radiation of a perfectly black body had been recognized for many years, Wien and Lummer (*Wied. Ann.*, 56, page 451, 1895) were the first to actually prepare radiators acting on this principle, and make use of them in experimental work.

For studying the intensity of the radiation at low temperatures and the distribution of energy in the spectrum of the radiation, a hollow cylinder of brass, blackened on the inside, can be used. The cylinder is provided with a small aperture, and is surrounded by a steam

jacket, or embedded in a mixture of sodium and potassium nitrate, and the whole packed in felt. The smaller the size of the hole in comparison to the internal capacity of the cylinder, the more nearly does the emerging radiation compare with that of an ideal black body. For high temperature work a cylinder of platinum or porcelain, electrically heated, can be employed, or even a hollow iron ball heated in a gas furnace. Kayser has proposed a very simple device, which, though superior to an electrically heated strip of blackened platinum, is not as good as a hollow vessel. Two strips of platinum, one provided with a narrow slit, are mounted opposite to one another and heated to the same temperature by a current. The principle is of course the same as that of the device just considered.

Paschen (*Wied. Ann.*, 60, page 719, 1897) has proposed still another device. A glowing carbon filament is mounted at the center of a hollowed silvered sphere. Assuming the silver to reflect all of the energy, it can be regarded as a hollow vessel having the same temperature as the carbon filament. The radiation escapes as before through a small hole. This same device has been applied to the bolometer, the absorbing strip being mounted at the center of a hollow spherical chamber silvered on the inside. All radiation not absorbed at once by the bolometer is returned to it by the reflecting surface. In this way it is possible to prepare a perfectly black bolometer.

**Equilibrium between Radiation and Material Bodies.**—In the deduction of the remaining laws of radiation we shall employ largely a conception due to Bartoli, which, though it cannot be carried out experimentally, leads to important laws which can be verified in other ways. The idea in brief is to apply the principles of thermo-dynamics to radiation, performing a cyclical process similar to Carnot's cycle, employing vibrating ether instead of a gas as the working substance.

The radiation within a hollow vessel can only be in equilibrium with the walls or with bodies in the interior, when it is of the same nature as the radiation emitted by the walls or the bodies contained within the vessel.

To get an idea of exactly what we mean by equilibrium between radiation and a material body we will consider the following case:

Suppose we have a hollow vessel the walls of which are perfect reflectors, which contains only ether. If we fill this cavity with monochromatic radiation, say that of the sodium flame, by opening a door in the wall and allowing the light to enter, which, of course, can be done perfectly well in theory, the radiation will, if we close the door, be reflected back and forth within the vessel for ever. It will neither change in intensity nor alter its wave-length; in other words, it is in equilibrium with the reflecting walls. We shall now prove that a perfect reflector is the only body with which this radiation can be in equilibrium, with the exception of the flame which originally emitted the light. Suppose we introduce a small fragment of absorbing matter within the cavity of the reflecting vessel. It will immediately absorb the monochromatic sodium radiation as fast as this radiation falls upon it, and in a very short space of time the monochromatic waves will have vanished completely. The temperature of the absorbing body will be slightly elevated, and it will emit long heat-waves, the energy

being distributed over a wide range of wave-lengths, the range and distribution depending on the temperature of the body. This radiation will now fill the cavity in place of the sodium radiation, and it will be in equilibrium with the absorbing body, *i.e.* a permanent state is speedily reached, after which there is no further change.

**Pressure of Radiation.**—The radiation within the vessel exerts a pressure upon the walls and upon the surface of the absorbing body. As we are to make use of this pressure in the derivation of laws it will be well to investigate it somewhat in detail.

Maxwell, in his electro-magnetic theory, showed that radiation must exert a pressure when it falls upon a reflecting or absorbing surface. As this pressure is the foundation upon which the laws of radiation have been built, we will briefly consider the phenomenon.

Maxwell showed that when plane electro-magnetic waves fall in a normal direction upon a perfectly absorbing surface, the pressure exerted on unit area is equal to the energy contained in unit volume of the vibrating medium.

That a pressure is exerted by heat (or light) waves may be proven by making use of the idea of Bartolli. Consider a cylinder, composed of some material which reflects perfectly, closed at the ends by black plates at temperatures  $T_1 > T_2$  (Fig. 295). Introduce a screen  $S$ , made also of a reflecting material, which divides the cylinder into two compartments. The body at temperature  $T_1$  will fill the upper compartment with radiation of energy corresponding to its temperature.  $T_2$  (at a lower temperature), will fill the lower compartment with radiant energy of less density. Let  $B$  represent a movable reflecting diaphragm, provided with a sliding door, which, when open, allows the energy from  $T_2$  to fill the middle compartment. Now close the door



FIG. 295.

and raise the diaphragm or piston. The volume of the middle compartment is decreased, and the density of the radiant energy "trapped" within it is increased. On removing the screen  $S$  laterally, which we can do without performing work, the diaphragm will drive the radiant energy above it into the body  $T_1$ . We have thus taken energy (or heat) from a body at low temperature and carried it to one of high temperature, which by the second principle of thermo-dynamics is impossible, unless mechanical work is done in the operation. This work can only have resulted from the overcoming of a pressure exerted upon the diaphragm, the vibrating medium resisting compression in the same way that a gas does. This pressure becomes greater as the volume is diminished owing to the increase in the energy density. In the case of the compression of a gas, the molecules rebound from the moving piston with increased velocity, consequently the force of each blow, and the number of blows per second, are increased.

In the case of compressed radiation the mechanism is not so easy to follow; as we shall see presently, reflection from a moving diaphragm decreases the wave-length by an amount proportional to the distance through which the diaphragm moves (provided the rest of the vessel is reflecting). This means that the number of waves which strike it per

second will be increased. The amplitude, as we shall see presently, remains the same, and the increase of energy density is due solely to the fact that more waves are present in unit length of the train after the compression than existed before the motion of the diaphragm. Such a process as that described above cannot even be approximately realized experimentally. It is no less valuable however, as our inability to carry it out is due solely to mechanical difficulties and our inability to obtain a substance which reflects perfectly.

An admirable treatment of the mechanical pressure of radiation has been given by Larmor (*Encycl. Brit.*, vol. 32, "Radiation"). Consider a wave train travelling along the  $x$  axis incident upon a perfect reflector, which is travelling in the opposite direction with a velocity  $v$ . The displacement in the incident wave train is

$$\xi = a \cos m(x + ct)$$

and in the reflected train

$$\xi' = a' \cos m'(x - ct).$$

The position of the reflector at time  $t$  is given by

$$x = vt.$$

The disturbance does not travel into the reflector, and must therefore be annulled at its surface. Thus when  $x = vt$  we must have  $\xi + \xi' = 0$ . This gives us  $a = -a'$  and  $m'(c - v) = m(c + v)$ .

The amplitude of the reflected disturbance is therefore equal to that of the incident one, while the wave-length is altered in the ratio  $\frac{c-v}{c+v}$  or  $1 - \frac{2v}{c}$  approx., when  $\frac{v}{c}$  is small.

The energy of the wave-train is half potential and half kinetic, and is given by the integration of  $\rho \left( \frac{\partial \xi}{\partial t} \right)^2$  along the train, in which  $\rho$  = density.

In the reflected train it is therefore augmented, when equal lengths are compared, in the ratio  $\left( \frac{c+v}{c-v} \right)^2$ , but the length of the train is

diminished by the reflection in the ratio  $\frac{c+v}{c-v}$ . This increase in energy per unit time can arise only from work done by the advancing reflector against pressure due to the radiation. The pressure per unit surface must therefore be equal to the fraction  $\frac{2}{c-v}$  of the energy in the length

$c + v$  of the incident wave train; thus it is the fraction  $\frac{c^2 - v^2}{c^2 + v^2}$  of the total density of energy in front of the reflector belonging to both the incident and reflected trains.

When  $v$  is small compared with  $c$  this makes the pressure equal to the density of the vibrational energy, in accordance with Maxwell's electro-dynamic formula.

The pressure due to light was, for a long time, sought for in vain. The disturbing effects of "radiometric action," or the reaction pressure of gas molecules rebounding from the surface heated by the radiation,

completely masked the very small effect which was looked for. As early as 1754 an attempt was made by De Mairan and Du Fay to detect the pressure of light. This was of course in the days of the corpuscular theory, and the looked-for pressure was that due to the arrest of the flying corpuscles. Fresnel, Zöllner, Bartolli, and Crookes also searched in vain for evidences of the pressure, the experiments of the latter, however, resulting in the discovery of the radiometer. The pressure was first observed by Lebedew in 1900 (*Rapp. près au Congrès de Phys.*, 2, 133, Paris, 1900) and by Nichols and Hull independently at about the same time. Though the latter investigators were anticipated by Lebedew by some months, their investigation was conducted with greater care, and the errors due to gas action were more carefully eliminated.

**Experiments of Nichols and Hull.**<sup>1</sup>—The inability of previous observers to measure the pressure due to radiation was due to the apparent impossibility of separating the effect from the so-called “radiometer” action. Thin vanes were employed to detect the pressure, and the radiation warmed the side on which it fell. When such a condition exists the gas exerts a greater pressure on the warm than on the cold side, and in general this pressure is vastly greater than the true radiation pressure.

Nichols and Hull finally succeeded in eliminating the gas action by employing a suspended vane made of two circular discs of thin glass silvered on one side. By employing a reflecting surface the pressure is double that exerted upon a black surface, and the heating is reduced to a minimum. By measuring the deflections when the glass and silver sides were illuminated in succession the gas action could be calculated, for the silver surface is the one heated in both instances. This is due to the fact that the radiation, before its impact upon the vane, has passed through a number of lenses and plates of glass, and is consequently robbed by absorption of all rays capable of heating a glass surface. It is at once apparent that when the radiation falls upon the glass surface the gas pressure and the light pressure are opposed, while when the silver surface is illuminated they act together, *i.e.* in the same direction. Larger deflections are of course observed in the latter case than in the former. To still further eliminate gas action, the ballistic method was adopted; it had been observed that some seconds' or even minutes' exposure to the radiation were required before the gas pressure reached its maximum, while the radiation pressure is of course instantaneous. Very short exposure, were consequently given, and the ballistic deflection of the vane was observed by means of a mirror and scale.

By an elaborate series of experiments the investigators determined the most suitable pressure for the air in the chamber in which the vane was suspended, the pressure, in other words, at which the gas action was at a minimum.

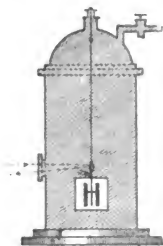


FIG. 296.

<sup>1</sup> *Proc. Am. Acad.*, xxxviii., April 1903; *Phys. Rev.*, xiii. 293, [1901].

This pressure proved to be about 16 mms. of mercury. After measuring the value of the radiation pressure, the energy of the radiation was determined, by allowing it to fall upon a blackened silver disc. The rise of temperature of the disc was determined by means of iron-constantan thermo-junctions imbedded in the disc. From these energy measurements the pressure to be expected was calculated.

The radiation employed was that of an arc-lamp, either with or without absorbing screens. After correcting for all possible sources of error, the following values were obtained:

Radiation.	Pressure in $10^{-5}$ dynes observed.	Pressure calculated from energy measurements.
Through air only	$7.01 \pm .02$	$7.05 \pm .03$
Through red glass	$6.94 \pm .02$	$6.86 \pm .03$
Through water cell	$6.52 \pm .03$	$6.48 \pm .04$

These experiments can be regarded as establishing in a quantitative manner the existence of the Maxwell-Bartoli pressure, which measured in dynes per sq. cm. is equal to the energy contained in unit volume of the radiation. A reflecting surface doubles the energy density in the medium in front of it by superposing the reflected beam upon the incident.

The gas action was subsequently eliminated by Hull (*Phys. Rev.*, May, 1905) by enclosing the reflecting and absorbing surfaces in thin glass cells, as proposed in the earlier paper. The silvered side of a thin cover glass was placed in contact with the blackened side of a similar glass, and the whole enclosed by means of two other thin glasses as shown in Fig. 297. Two cells of this description were mounted upon opposite ends of a torsion arm suspended in a receiver from which the air could be removed. When the light falls upon the blackened surface and is absorbed, the temperatures of the two outer glass surfaces of the cell are the same, since they are separated from the heated surface by equal thicknesses of glass and air. The gas action should therefore be equal on the two surfaces. Any gas action occurring within the cell will produce no effect, owing to the equality of action and reaction. Hull found that the ratio of the deflections obtained when the silvered and blackened surfaces were illuminated in succession, agreed with the calculated ratio to within 2 %, showing that the "radiometer" action had been practically eliminated.



FIG. 297.

**Tangential Component of Radiation Pressure.**—An interesting experiment was described by Poynting at the Cambridge Meeting of the British Association in 1904, in which gas action is completely eliminated.

When radiation is incident upon an absorbing surface in an oblique direction, the pressure has a component parallel to the surface. In the case of a reflecting surface this tangential force cannot be detected, since the incident and reflected beam give rise to equal and opposite forces parallel to the surface. The magnitude of the force, when  $E$  is the

energy density,  $\mu$  the fraction reflected, and  $\alpha$  the angle of incidence, is given by

$$F = \frac{E}{2}(1 - \mu) \sin 2\alpha.$$

The existence and magnitude of the force was observed with the apparatus shown in Fig. 298. Two thin glass discs were mounted on the ends of a fine glass rod, the system being suspended by a quartz fibre in a brass box provided with glass windows. One of the discs was silvered, the other blackened, and the pressure within the case was reduced to 1 cm. Sunlight, or the beam from an arc lamp, was directed against the black disc at an angle of  $45^\circ$ . Gas action due to heating will give rise to a pressure normal to the surface, but there will be no tendency to rotate the suspended

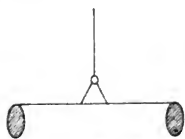


FIG. 298.

system. The tangential component of the radiation pressure on the other hand will produce a deflection, the magnitude of which can be read with a mirror and scale.  $E$  was calculated from the observed deflection, and was found to be  $5.8 \cdot 10^{-6}$  dynes, while a direct measurement of  $E$ , by the heating of a silver plate, gave the value  $6.5 \cdot 10^{-6}$ .

**The Stefan-Boltzmann Law.**—An empirical law was deduced by Stefan from observations made by other observers on the intensity of the total radiation from bodies at different temperatures. The law states that the complete emission  $S$  of a black body is proportional to the fourth power of the absolute temperature  $T$ , or

$$S = aT^4,$$

in which  $a$  is a constant.

This same law was subsequently deduced from theoretical considerations by Boltzmann (*Wied. Ann.*, 22, p. 291, 1884), who availed himself of the ingenious conception by which Bartoli proved that radiation must exert a pressure. Consider a hollow cylinder of unit cross section, the walls of which are black, and of infinitely small heat capacity. The ends of the cylinder are also black, but of infinitely large heat capacity. Within the cylinder is a frictionless black piston, in contact with the left-hand end plate of the cylinder, which has an absolute temperature  $T_0$ . The opposite end plate has a lower temperature  $T$  (Fig. 299). The radiant energy exerts a pressure on the piston, which in the case of plane-waves parallel to the surface is equal to the radiant energy in unit volume of the ether. Let  $\Psi(T)$  be the energy in unit volume. Since the energy is travelling in all possible directions, the pressure on unit surface will not be  $\Psi(T)$  but  $\frac{1}{3}\Psi(T)$ . (Compare with the calculation of the pressure due to molecules moving in all directions, in the Kinetic Theory of Gases.) We have then  $\frac{1}{3}\Psi(T) = f(T)$ , the pressure at temperature  $T$ .

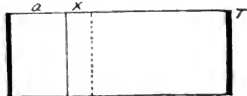


FIG. 299.

Now let the piston move forward a distance  $a$ , under the influence of the pressure of the radiant energy coming from the high temperature

plate  $T_0$ . The heat energy leaving  $T_0$  is partly spent in doing the work  $af(T_0)$  on the piston, and partly in filling up the volume " $a$ " with radiant energy. This process obviously corresponds to the isothermic expansion of the Carnot cycle, the filling of the space  $a$  with energy corresponding to the heating of the gas. The amount of heat which leaves the plate  $T_0$  is  $a[\Psi T_0 + f(T_0)]$ .

We will now introduce a screen impervious to heat immediately in front of  $T_0$ , which prevents further radiation into the space to the left of the piston. This corresponds to placing the cylinder in Carnot's cycle upon an insulating stand. The pressure to the left of the piston is greater than that to the right, owing to the higher temperature of the plate which filled this portion of the cylinder with radiation. The piston will therefore move forward until the energy per unit volume is the same on both sides. Allowing this adiabatic expansion to take place we have  $d[(a+x)\Psi T] = -f(T)dx$ , the characteristic equation for an adiabatic process (see any Thermodynamics). In this expression  $T$  is of course variable. During this process the volume to the right of the piston has been still further diminished, and an amount of heat energy represented by  $(a+x)[\Psi(T) + f(T)]$  due to diminishing the volume and work done enters the plate at temperature  $T$ .

Since the process is reversible we have by the second law of thermodynamics,

$$\frac{(a+x)[\Psi(T) + f(T)]}{T} = \frac{a[\Psi(T_0) + f(T_0)]}{T_0} = C,$$

in which  $x$  and  $T$  are variables.

Writing  $(a+x)[\Psi(T) + f(T)] = CT$ ,

or, for simplicity,  $(a+x)(\Psi + f) = CT$ ,

$$(\Psi + f)d(a+x) + (a+x)d\Psi + f = \frac{(a+x)(\Psi + f)}{T}dT,$$

and subtracting,  $\Psi d(a+x) + (a+x)d\Psi = -f dx$  (adiabatic equation)

gives us  $(a+x)df = \frac{(a+x)(\Psi + f)}{T}dT$ ,

$$(\Psi + f)dT = T df,$$

or, inserting the  $(T)$  which we omitted above,

$$\Psi(T)dT + f(T)dT = T df(T).$$

Substituting for  $f(T)$  its equivalent value  $\frac{1}{3}\Psi(T)$

gives  $\frac{4}{3}\Psi(T)dT = \frac{1}{3}Td\Psi(T)$ ,

$$\frac{d\Psi(T)}{\Psi T} = 4 \frac{dT}{T},$$

$$\Psi = aT^4.$$

**Proof of Stefan's Law.**—The law was first deduced empirically from observations made on the rate of cooling of a blackened thermometer bulb.

Lummen and Pringsheim (*Wied. Ann.*, 63, page 395, 1897) proved the law over a range of temperatures included between 100° and

1300° C. by measuring the intensity of the radiation from a hollow chamber (black body) by means of the bolometer.

The constant  $a$  has been determined in absolute measure by Kurlbaum (*Wied. Ann.*, 65, p. 746, 1898), who heated the bolometer strip (screened from the radiation) by means of an electric current of known strength to the same temperature to which it was raised by the radiation. The radiation was thus determined in absolute units by calculating the Joule heat developed by the current. The value found was

$$a = 1.71 \cdot 10^{-5} \frac{\text{erg}}{\text{sec.}} = 0.408 \cdot 10^{-12} \frac{\text{gr. cal.}}{\text{sec.}}$$

**Optical Pyrometers.**—Various types of pyrometers have been designed for measuring high temperatures by optical methods. Fery's instrument is based upon the law of total radiation. It consists of a telescope with a fluorite objective, in the focus of which is mounted a sensitive thermo-couple as shown in Fig. 300. To use the instrument one has only to point it at the object, the temperature of which is to be measured, *e.g.* the interior of a blast-furnace, and focus the image upon

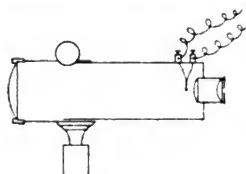


FIG. 300.

the thermo-junction by means of the eye-piece, which is moved with the latter by means of the rack and pinion wheel at *B*. The temperatures are read with a galvanometer.

Other optical photometers have been devised, based upon the laws which we are about to study.

**Temperature of the Sun.**—The sun's temperature has been computed by measuring the total radiation. Assuming the solar disc to be a black body, and taking for the value of the solar constant 3 gr. cal. per minute the computed temperature comes out a trifle over 6000°.

**Change in the Spectrum of a Black Body with the Temperature.**  
**Wien's Laws.**—Making use of a conception similar to the one by means of which Boltzmann deduced Stefan's law, but extending it by the introduction of the consideration of the change in wave-length which occurs when radiation is reflected from a moving mirror, Wien (*Wied. Ann.*, 46, p. 633; 52, p. 132) arrived at a formula which expressed the change in the spectrum of a heated black body with its absolute temperature. As is well known, when a solid or liquid is heated the longer heat-waves appear first, then red light, and finally at still higher temperatures the violet and ultra-violet. If we measure the energy at different points in the spectrum with the bolometer and plot these values as ordinates, with the wave-lengths as abscissae, we obtain the energy curve for the emission at the temperature in question. The maximum of this curve moves towards the region of the shorter waves as the temperature is increased, but there is an increase in the height of every ordinate; in other words, the curve does not move bodily down the spectrum. It seems very remarkable that the form and position of this curve can be determined by considering merely the motion of reflecting pistons moving in a closed cylinder, the ends of

which radiate at different temperatures. This, however, is precisely what was done by Wien, whose treatment we will now consider. Consider a cylinder of unit cross-section, the walls of which reflect diffusively all of the incident energy, while the ends are composed of black material, of infinite heat capacity, at temperatures  $T_2$  and  $T_1$  ( $T_2 > T_1$ ). The cylinder is divided into three compartments by means of movable pistons composed of perfectly reflecting material and furnished with openings which can be closed by means of trap-doors. At the beginning we have things arranged as shown in Fig. 301 and the perfect radiator  $T_1$  fills up compartments 1 and 2 with radiant energy, of density  $\Psi(T_1)$ . The density in compartment 3 is greater, namely  $\Psi(T_2)$ .

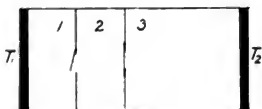


FIG. 301.

The trap-door is now closed and the partition moved towards 3, a distance  $\delta x$ , such that the radiation in 2 has the same density as that in 3. The distribution of energy in the spectrum is now the same in compartments 2 and 3, since, if this were not the case, there would of necessity be rays of a certain wave-length  $\lambda$  in compartment 3, which have a greater energy density than the waves of corresponding  $\lambda$  in compartment 2. This being the case, we could cover the opening in the moving plate with a screen composed of some material transparent to waves of this particular length, but reflecting all others. On opening the trap-door more energy would pass from 3 to 2 than passed back in the opposite direction, and the density in 2 would become greater. On closing the door and removing the screen, the plate would be moved to the right until the pressure became equal on both sides, furnishing an amount of work  $A$ . A small amount of heat leaves the black body at temperature  $T_2$  in the form of radiation of wave-length  $\lambda$ , to restore the original condition. This heat is the equivalent of the work  $A$ . The trap-door is now opened and the plate brought back to its original position, no work being required.

The door is now closed, and the plate which separates 1 and 2 is driven back to its original position (distance  $\delta x$ ), by which the work is gained which was originally spent in moving the plate through the distance  $\delta x$ . If we now open the door in the plate, we have the original state of things, the body at temperature  $T$  neither having given up nor received heat, while the other body at temperature  $T_2$  has given up heat corresponding to the amount which passed through the selectively transparent screen, and furnished the work  $A$ . By the second law of thermo-dynamics work cannot be derived by a cyclic process in the case in which a single reservoir gives up heat in such a manner that all of it is transformed into work.

We conclude therefore that when the energy density is the same in compartments 2 and 3, the distribution of energy in the spectrum is also the same.

Going back now to the main part of the problem. The motion of the piston which condensed the energy in 2 until it had the same density as that in 3, is accompanied by a shortening of the wave-lengths of the

reflected energy, for a moving mirror will, by Doppler's principle, alter the lengths of all waves incident upon it. Though the shortening depends on the velocity of the mirror, in the present case it depends only upon the total distance through which the mirror moves. This is due to the fact that the rays are repeatedly reflected from the mirror, and if the mirror moves with slow velocity more reflections will occur from it during its movement, the increased number of reflections compensating for the slower velocity. For normal incidence, if the velocity of the mirror is  $v$ , the wave-length after one reflection is

$$\lambda' = \frac{c - 2v}{c} \lambda.$$

As we have seen, we can regard  $\frac{1}{3}$  of the total radiation as incident normally, and we will assume that a single reflection shortens the waves by an amount  $h$ .

Plotting the original density distribution for wave-lengths  $\lambda$  and  $\lambda + h$ , we will determine the effect of a single reflection upon this portion of the energy curve. In Fig. 302 the curve  $AB$  represents

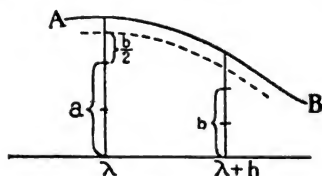


FIG. 302.

the energy distribution before reflection, the ordinates representing total densities of energy for different values of  $\lambda$ . Since we can regard  $\frac{1}{3}$  of the energy as incident normally, if we disregard the rest we shall have  $\frac{2}{3}$  of the energy at  $\lambda$  unaffected, represented by the ordinate  $a$ . Of the energy at wave-length  $\lambda + h$ ,  $\frac{2}{3}$  is unaffected (ordinate  $b$ ), while  $\frac{1}{3}$  is shortened in wave-length to the value  $\lambda$ ; consequently this portion must be added above  $a$  as indicated. If we do this for all values of  $\lambda$  we shall obtain the dotted curve, which represents the energy distribution after one reflection. If  $f_1(\lambda)$  represents this distribution, and  $\Phi(\lambda)$  the original distribution, we can express the above change as follows:

$$f_1(\lambda) = \frac{2}{3} \Phi(\lambda) + \frac{1}{3} \Phi(\lambda + h) = \Phi\left(\lambda + \frac{h}{3}\right).$$

If the radiation is reflected  $n$  times, we have

$$f_n(\lambda) = \Phi\left(\lambda + \frac{nh}{3}\right).$$

The change in the distribution of energy can thus be represented by considering  $\frac{1}{3}$  of the rays as shortened by an amount  $nh$ .

If now  $(a - x)$  is the distance between the pistons, we have for  $n$ , while the piston moves a distance  $dx$ ,  $n = \frac{dx}{2(a - x)} \frac{c}{v}$ , in which  $c$  is the velocity of the radiation and  $v$  the piston's velocity. After  $n$ -fold reflection, we have

$$\lambda_n = \left(\frac{c - 2v}{c}\right)^n \lambda = \left(\frac{c - 2v}{c}\right)^{\frac{dx}{2(a - x)} \frac{c}{v}} \lambda = \left[\left(1 - \frac{2v}{c}\right)^{\frac{dx}{2(a - x)} \frac{c}{v}}\right] \lambda.$$

For the limit  $c = \infty$ ,

$$\lambda_n = e^{-\frac{dx}{a-x}} \lambda, \quad \text{since limit of } \left(1 + \frac{\theta}{n}\right)^n = e^\theta, \text{ when } n = \infty,$$

or writing  $\lambda_n = \lambda + nh$ , in which  $nh$  is infinitely small, of the order  $dx$ ,

$$nh = -\frac{dx}{a-x} \lambda.$$

Now  $f(\lambda)$ , the distribution of energy after  $n$ -fold reflection, is given by

$$f(\lambda) = \Phi\left(\lambda + \frac{nh}{3}\right) = \Phi(\lambda + d\lambda);$$

$$\therefore d\lambda = -\frac{dx}{3(a-x)} \lambda,$$

which gives us the change of wave-length due to the motion of the piston through a distance  $dx$ .

Integrating the above,  $\lambda = \sqrt[3]{\frac{a-x}{a}} \lambda_0$ , in which  $\lambda_0$  is the value for  $x=0$ , i.e. before the motion of the piston commenced.

Let  $E$  be the total energy in compartment 2, when  $x=0$ : its density is then

$$\Psi = \frac{E}{a-x}.$$

If  $x$  increases by  $dx$ , the energy intensity is increased by diminution of volume, and work done against the radiation pressure, by an amount

$$\frac{d\Psi}{dx} dx = \left\{ \frac{dE}{dx} \frac{1}{a-x} + \frac{E}{(a-x)^2} \right\} dx = \left\{ \frac{dE}{dx} + \Psi \right\} \frac{dx}{a-x}.$$

The pressure on the piston is  $\frac{1}{3}\Psi$ , therefore the work is

$$\frac{dE}{dx} dx = \frac{1}{3}\Psi dx, \text{ and } d\Psi = \frac{4}{3} \frac{\Psi}{a-x} dx,$$

$$\Psi = \left( \frac{a}{a-x} \right)^{\frac{4}{3}} \Psi_0,$$

in which  $\Psi_0$  is written in place of the original  $\Psi = \frac{E}{a-x}$ , for  $x=0$ ,

$$\frac{\Psi}{\Psi_0} = \left( \frac{a}{a-x} \right)^{\frac{4}{3}}, \text{ and as we have seen above, } \frac{\lambda}{\lambda_0} = \left( \frac{a-x}{a} \right)^{\frac{1}{3}};$$

$$\therefore \frac{\Psi}{\Psi_0} = \frac{\lambda_0^4}{\lambda^4}.$$

We can now write the expression  $\frac{\Psi}{\Psi_0}$  in terms of the absolute tem-

peratures, by applying the Stefan-Boltzmann Law. If  $\Psi_0$  is the energy at  $T_0$  and  $\Psi$  the energy at  $T$ ,

$$\frac{\Psi}{\Psi_0} = \frac{T^4}{T_0^4}$$

and we get at once  $T\lambda = T_0\lambda_0$ .

This expression may be interpreted in the following way: On raising the temperature of a black body from  $T_1$  to  $T_2$ , the ordinates of our energy curve move towards the short wave-lengths by an amount such that the product of the corresponding abscissa and the temperature remains constant for each ordinate. The maximum ordinate at say wave-length  $\lambda_m$  for temperature  $T_1$  will therefore at temperature  $T_2$  occupy the position  $\lambda'_m$ , such that  $\lambda_m T_1 = \lambda'_m T_2$ : in other words, as the temperature rises, the summit of the energy curve drifts towards the region of shorter wave-lengths.

We can find, by the aid of this displacement law the distribution of energy for any temperature, if the distribution for some given temperature is known.

Plot as before a  $\Phi(\lambda)$  and  $\lambda$  energy curve; the area of the curve is equal to the total energy  $\Psi$ .

To pass to a curve for another temperature, we take a vertical strip at  $\lambda_0$  of width  $d\lambda_0$ , the area of which is  $\Phi d\lambda_0$ ; this strip is displaced by the temperature change to, say,  $\lambda$ .

The quantity of energy in the strip  $\Phi_0 d\lambda$  must remain constant.

$$\therefore \Phi d\lambda = \Phi_0 d\lambda_0,$$

$$\Phi = \Phi_0 \frac{d\lambda_0}{d\lambda} = \Phi_0 \frac{T}{T_0}.$$

It will be observed that thus far we have neglected the circumstance that the total energy increases with the temperature, as represented by Stefan's law.

Taking this into account by itself, we have

$$\Phi = \Phi_0 \frac{T^4}{T_0^4}, \text{ and by combining this with } \Phi = \Phi_0 \frac{T}{T_0},$$

we obtain as the complete expression

$$\Phi = \Phi_0 \frac{T^5}{T_0^5},$$

and our new ordinate at  $\lambda$  must be equal to ordinate at  $\lambda_0$  multiplied by the ratio  $\frac{T^5}{T_0^5}$ .

Wien's two laws as applied to the wave-length at which we have the maximum energy may be written as follows:

$$\lambda_m T = A \text{ (const.)},$$

$$E_m T^{-5} = B \text{ (const.)},$$

in which  $E_m$  is the energy at the maximum.

Lummer and Pringsheim have tested these two laws by measuring the energy curves of a heated black body over a range 621-1650 Absolute. Their results are given in the following table :

$T$ .	$\lambda_m$ .	$E_m$ .	$A = \lambda_m T$ .	$B = E_m T^{-5}$ .
1650	1.78	270	2928	2246 . $10^{-17}$
1260	2.35	69	2959	2176
1094	2.71	34	2956	2166
908	3.28	13.6	2980	2208
723	4.08	4.3	2950	2166
621	4.53	2.03	2814	2190

Neither of the two equations, however, give us any information regarding the actual distribution of energy in the spectrum of a black body.

To express this we require an expression which represents  $E$  as a function of  $\lambda$  and  $T$ .

A number of formulae have been developed which we will briefly discuss.

**Complete Radiation Formulae.**—Wien (*Wied. Ann.*, 58, p. 662), by the consideration of a peculiar type of radiator, deduced a formula connecting  $E$  with  $\lambda$  and  $T$ . The radiator is considered as a hollow vessel filled with a gas mixture capable of emitting waves of all lengths. He assumes that every molecule emits only a single wave-length, which depends on its velocity, the intensity of which wave is a function of this velocity. Further, the intensity  $\Phi(\lambda)$  of the radiation between the limits  $\lambda$  and  $\lambda + d\lambda$  is proportional to the number of molecules vibrating with periods corresponding to wave-lengths within this range. From these assumptions he derived the formula

$$E = \frac{c}{\lambda^5} e^{-\frac{\beta}{\lambda T}}.$$

This formula represents the energy distribution very well if it is not applied to too long waves. Lord Rayleigh has pointed out that the energy at a definite wave-length, as represented by the formula, does not increase indefinitely with the temperature, but approaches a limit. For visible waves this limit would only be reached at temperatures far beyond our command, but for  $\lambda = 60\mu$  (Rest-strahlen from Sylvite) the limiting value of the intensity would occur at about  $1000^\circ$ .

Planck has deduced a radiation formula of different form from electromagnetic considerations :

$$E = \frac{c\lambda^{-5}}{e^{\lambda T} - 1}.$$

This formula has been confirmed in a remarkable manner by the work of Rubens and Kurlbaum (*Ann. der Physik*, 4, p. 649, 1901), who measured the intensity of the radiation of  $\lambda = 51\mu$  emitted by a black body over a temperature range comprised between 85 and 1773

Absolute. Their results are given in the following table, together with values calculated from the formulæ of Wien and Planck.

$T$ .	$E$ obs.	$E$ cal. (Wien).	$E$ cal. (Planck).
85	-20.6	-107	-21.9
193	-11.8	-48	-12
293	0	0	0
523	+31	+63	+30.4
773	64.5	96	63.8
1023	98.1	118	97.2
1273	132	132	132
1523	164	141	166
1773	196	147	200
$\infty$	—	194	$\infty$

This table shows not only the close agreement between the observed values and those calculated from Planck's formula, but also that at a temperature of 1773, a value of  $E$  was obtained, larger than the limiting value 194 for infinite temperature, calculated from Wien's formula.

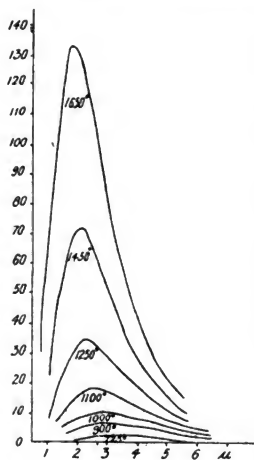


FIG. 303.

For short waves Wien's formula is perfectly satisfactory. It is questionable, however, whether it is anything more than an empirical formula, for many objections have been raised against the methods employed in its deduction. The energy curves for a black body are represented in Fig. 303 for temperatures between 750 and 1650. The shift of the maximum towards the region of shorter wave-lengths with increasing temperature is clearly brought out. These curves were made from observations by Lummer and Pringsheim.

In the case of the sun's spectrum the point of maximum energy is shifted much nearer to the visible region. The solar intensity curve as actually measured, is of course greatly modified by atmospheric absorption. Fig. 304 is

from a record obtained by Langley, and shows the relative energy distribution in the solar spectrum, and in the spectrum of a black body at various temperatures. The spectra were obtained by means of a prism, which by crowding the energy, so to speak, in the infra-red portion, on account of the small dispersion in this region, does not give us a true value for the position of the maximum. The deep valleys in the solar curve represent atmospheric absorption.

**Energy Distribution in the Spark Spectrum.**—As we have seen, there is a very rapid drop in the energy as we pass from the red to the

violet of the spectrum of a white hot body. Pflüger,<sup>1</sup> in a series of remarkably interesting experiments, has found that in general the reverse is true in the case of the bright-line spark spectra. He investigated the energy distribution in the spark spectra of a large number of metals with a bolometer, and found that the largest deflections were obtained in the remote ultra-violet: in other words, the

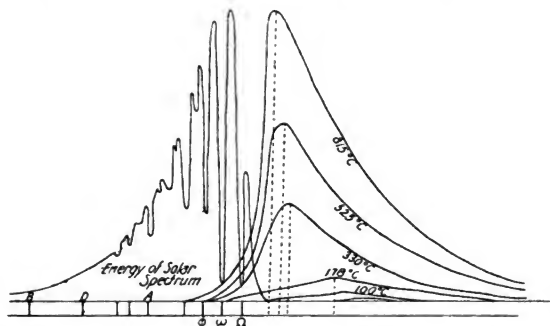


FIG. 304.

ultra-violet lines were "hotter" than the red or infra-red. We have as yet no law governing the energy distribution in discontinuous spectra, but Pflüger's experiments probably may be taken as marking the beginning of a subject to which a chapter may be devoted in text books twenty years hence. His results for the zinc spark are given in the following table, from which an energy curve can be constructed. It is instructive to compare the curve thus obtained with curves on page 476.

Wave-lengths.	Gal. defl.	Wave-lengths.	Gal. defl.	Wave-lengths.	Gal. defl.
199	50	244	35	515	80
200	195	252	85	570	45
203	225	261	20	650	50
204	205	275	25	800	80
206	280	335	60	900	85
206 +	160	360	15	1. $\mu$	80
208	220	395	45	1.2	75
211	60	465	40	1.4	60
Deflections under 10.				Deflections under 10.	
				From here on, decreasing.	

<sup>1</sup> *Annalen der Physik*, 13, page 890, 1904.

## CHAPTER XX.

### SCATTERING OF LIGHT BY SMALL OBSTACLES. OPTICAL RESONANCE.

**Scattering of Light by Small Particles.**—If a beam of light is passed through a transparent medium containing in suspension small particles, the refractive index of which differs from that of the surrounding medium, light will be given off by the particles in all directions. If the particles are very small, the color of the scattered light is blue, and it is more or less completely plane polarized, the direction of vibration being perpendicular to the direction of the incident beam. If the incident beam is plane polarized to start with, no light is scattered by the cloud of particles in directions parallel to that of the incident vibration. In the case of particles of the order of magnitude of the light waves, the amount of light scattered increases as the wave-length is decreased, which explains the preponderance of blue always observed in these cases. The subject was investigated experimentally by Tyndall with clouds precipitated by the chemical action of light upon the vapor of iodide of allyl. Whenever the particles obtained were sufficiently small, the laterally emitted light was blue in color and polarized in a direction perpendicular to the incident beam. Tyndall was unable to explain the polarization, and imagined that it contradicted Brewster's law, there being no angle of maximum polarization, as in the case of reflection from flat surfaces of isotropic media. That there is really no contradiction here is at once apparent when we consider that the nature of the phenomenon is radically different in the two cases. In ordinary reflection we are dealing with surfaces large in comparison with the wave-length, and the amount of reflected light is independent, or nearly so, of the wave-length. While, in the case which we are considering, regular reflection in the ordinary sense does not occur, and the intensity of the light is a function of the wave-length. By making the particles sufficiently small we may obtain a violet of great intensity and purity. The fog formed by the condensation of sodium vapor has been observed by the author to give a deeper color than any of the other media heretofore employed. The experiment, however, is rather difficult to perform, and a description of it would be out of place here. The blue color is easily seen in tobacco smoke rising from the end of a lighted cigar.

On standing, the smoke particles appear to collect into larger aggregates and the blue color disappears. This is usually the case with smoke exhaled from the mouth. The blue color of the sky has its origin in a similar action exerted either by small dust particles or even by the molecules of air themselves. The subject has been exhaustively studied by Lord Rayleigh, who explains the phenomenon in the following way: "Conceive a beam of plane-polarized light to move among a number of particles all small compared with any of the wavelengths. The foreign matter may be supposed to load the ether so as to increase its inertia without altering its resistance to distortion. If the particles were away the waves would pass on unbroken, and no light would be emitted laterally. Even with the particles retarding the motion of the ether the same will be true if, to counterbalance the increased inertia, suitable forces are caused to act on the ether at all points where the inertia is altered. These forces have the same period and direction as the undisturbed luminous vibrations themselves. The light actually emitted laterally is thus the same as would be caused by forces exactly the opposite of those acting on the medium otherwise free from disturbance, and it only remains to see what the effect of such forces would be. In the first place there is necessarily a complete symmetry around the direction of the force; the disturbance consisting of transverse vibrations is propagated outwards in all directions from the center; and in consequence of the symmetry the direction of vibration in any ray lies in the plane containing the ray and the axis of symmetry; that is to say, the direction of vibration in the scattered or refracted ray makes with the direction in the incident or primary ray the least possible angle. The symmetry also requires that the intensity of the scattered light should vanish for the ray which would be propagated along the axis. For there is nothing to distinguish one direction transverse to the ray from another. Suppose for distinctness of statement that the primary ray is vertical, and that the plane of vibration is that of the meridian. The intensity of the light scattered by a small particle is constant, and a maximum for rays which lie in the vertical plane running east and west, while there is no scattered ray along the north and south line. If the primary ray is unpolarized, the light scattered north and south is entirely due to that component which vibrates east and west, and is therefore perfectly polarized, the direction of its vibration being also east and west. Similarly any other ray scattered horizontally is perfectly polarized, and the vibration is performed in the horizontal plane. In other directions the polarization becomes less and less complete as we approach the vertical."

What actually occurs as a physical process, that is the exact manner in which the particles load the ether, is not definitely stated. We shall have no difficulty in remembering that the direction of vibration of the light scattered in a direction perpendicular to the incident beam, is the direction in which no light is scattered when the incident beam is plane polarized, if we imagine the obstacle actually set in vibration. Transverse waves would then not be given off in the direction in which the vibration takes place, i.e. in the direction parallel to the vibrations of the incident

light. But it is inconceivable that particles as small even as the molecules are actually thrown into vibration as rapid as those of light. We may imagine if we like that the contained electrons are set into vibration as a whole, i.e. that the center of gravity of the system is periodically displaced by the electric forces in the light waves. This would be equivalent to "loading the ether," and though it may not be a correct conception, serves perhaps as something tangible to fix the mind on. This conception neglects everything analogous to dispersion, the free periods of the electrons not being considered, and forces of restitution being neglected. We are merely concerned with the inertia of the system as a whole, which we conceive as having no free period of vibration.

The intensity of the scattered light as a function of the wave-length, for obstacles of fixed size, small in comparison to  $\lambda$ , was calculated by Lord Rayleigh (*Phil. Mag.*, xli., pages 107-120, 274-279 (1871)).

Let  $i$  be the ratio of the amplitude of the incident to that of the scattered light, and  $T$  the volume of the disturbing particle. If  $r$  is the distance from the particle of a given point, the value of  $i$  at this point is shown to be proportional to  $\frac{T}{\lambda^2 r}$ , i.e. the amplitude varies inversely as the square, and the intensity as the fourth power of the wave-length.

Observations were made of the distribution of energy in the spectrum of the light of the blue sky by comparing its spectrum with the spectrum of direct sun-light, diffused by white paper. These values were compared with values calculated on the assumption that the intensity of the scattered light (i.e. blue-sky light) varied as  $\frac{1}{\lambda^4}$ . The

two sets of values are given in the following table for four of the Fraunhofer lines:

$C$	$D$	$b_3$	$F$	
25	40	63	80	calculated.
25	41	71	91	observed.

Lord Rayleigh's formula for the intensity of the scattered light in a direction making an angle  $\beta$  with the incident ray is, if the incident light is unpolarized,

$$A^2 \frac{(D' - D)^2}{D'^2} (1 + \cos^2 \beta) \frac{m\pi T^2}{\lambda^4 r^2},$$

in which  $A^2$  is the intensity of the incident light,  $D'$  and  $D$  the densities of the particles and the medium in which they are immersed,  $m$  the number of particles, and  $\lambda$  the wave-length.

The formula shows that the intensity is twice as great in the direction from which the light comes originally as in a direction perpendicular to it.

**The Residual Blue.**—Tyndall found that as the particles in his precipitated clouds increased in size, the blue color disappeared, the scattered light appearing white. If, however, it was received through a Nicol prism held in the position in which it would ordinarily extinguish the scattered light, the blue color appeared again in increased splendor. This blue color he named the "Residual blue."

Lord Rayleigh considers this phenomenon in a subsequent paper (*Phil. Mag.*, xii., page 81), and shows that if the incident light is polarized with its vibrations parallel to the  $z$  axis, the intensity of the light scattered along the  $z$  axis varies as the inverse 8th power of the wave-length, so that the residual blue is purer than the blue seen under ordinary conditions. With smaller particles no light at all would be seen in this direction under the conditions specified. It remains to be seen whether in this case there is any direction in which the scattered light vanishes. Lord Rayleigh derived an equation which showed that zero illumination was to be expected in a direction inclined backwards, *i.e.* towards the source of light, and this was found to be the case. The experiments were made with a precipitate of sulphur, obtained by adding a small quantity of dilute sulphuric acid to a weak solution of hyposulphite of soda. The more dilute the solution, the slower is the process of precipitation, and the slower the change in the size of the particles. Solutions of such strength that no precipitate appears for four or five minutes will be found to give the best results, and the process can be arrested at any stage by the addition of a few drops of ammonia. The experiment should be performed in a dark room, a beam of sunlight rendered convergent by means of a long focus lens being passed through a glass tank containing the solution. The scattered light should be examined by a Nicol prism. Quoting from the paper above referred to, "In the early stages of the precipitation polarization is complete in a perpendicular direction and incomplete in other directions. After an interval the polarization begins to be incomplete in a perpendicular direction, the light which reaches the eye when the Nicol is in the position of minimum transmission being of a beautiful blue, much richer than anything that can be seen in the earlier stages. This is the moment to examine whether there is a more complete polarization in a direction somewhat more oblique, and it is found that with  $\theta$  positive (*i.e.* towards the source) there is in fact an oblique direction of more complete polarization, while with  $\theta$  negative the polarization is less perfect than in the perpendicular direction itself."

The mathematical treatment of the subject is too long to be given in detail, and as an abbreviated treatment is unsatisfactory it is omitted entirely. Reference should be made to the original papers. In one of his more recent papers Lord Rayleigh has shown that the blue light of the sky can be regarded as caused by the scattering power of the air molecules themselves, in the absence of any suspended particles.

**Scattering of Light by Metal Spheres.**—The case where light is scattered by metal spheres, small in comparison to the wave-length, has been treated theoretically by J. J. Thomson (*Recent Researches*, p. 437). He finds that the scattered light, produced by the incidence of plane-polarized light, vanishes in the plane through the center, at right angles to the magnetic induction in the incident wave along a line making an angle of  $120^\circ$  with the radius to the point at which the wave first strikes the sphere, and it does not vanish in any direction other than this. The direction in which the scattered light is plane polarized will thus be inclined at an angle of  $120^\circ$  to the direction of the incident light. It is thus seen that the law is quite different from

that which holds in the case of non-conducting particles, when the scattered light vanishes at all points in a plane normal to the magnetic induction, where the radius vector makes an angle of  $90^\circ$  with the direction of the incident light. When the light is scattered by a conducting sphere the points of complete polarization are on the surface of the cone, whose axis coincides with the direction of propagation of the incident light, and whose semi-vertical angle is  $120^\circ$ .

**Optical Resonance. Electrical Resonance.**—In the chapter on Dispersion we have seen that the velocity of waves in dispersing media is modified by the presence of charged electrons, which have definite periods of vibrations, and which are thrown into oscillation by the periodic electric forces of the light waves.

We have an analogy in the case of electro magnetic waves, which differ from light waves only in the comparative slowness of the period. Oscillatory electrical disturbances are set up in strips of metal of suitable dimensions by the passage of Hertzian waves, and we shall see presently that the velocity of the waves will be changed if the metal strips or resonators are sufficiently numerous. The maximum resonance effect will occur when the natural period of the resonator agrees with that of the wave. Garbasso (*Atti. Acc. di Torino*, 28 (1893)) found that a system of tin-foil strips was much more opaque to the radiation of a Hertz oscillator, when the period of the latter agreed with the natural period of the electrical vibration in the tin-foil strips. The greater part of the incident energy was reflected, the phenomenon being the electrical analogue of surface color.

Garbasso and Aschkinass (*Wied. Ann.*, 53, page 534 (1894)) subsequently found that if the tin-foil strips were distributed in space in the form of a prism, the phenomena of dispersion were exhibited, the electro-magnetic waves being deviated in different degrees, according to their lengths.

If the resonators were immersed in fluid dielectrics, such as benzol, ether, or acetone, the free periods were altered and the position of maximum absorption in the electrical spectrum was shifted towards the region of longer waves (Aschkinass and Schaefer, *Ann. de Physik*, 5, page 489 (1901)). This is analogous to the shift in the position of the absorption band, which occurs when a dye is dissolved in different solvents.

In the above case we are clearly dealing with electrical vibrations which occur in the metal strips taken as a whole, and not with vibrations going on within the molecule, as is the condition assumed in the electron theory of dispersion.

The question naturally arises as to whether it is possible to reduce the dimensions of the metal strips to such a point that their free periods are of the order of magnitude of the periods of light or heat waves.

**Resonance for Heat Waves.**—The first attempts in this direction were made by Rubens and Nichols (*Wied. Ann.*, 6', page 456 (1897)), who ruled a thin silver film, deposited on glass, into narrow strips, and then cut the strips into suitable lengths by cross ruling. The reflecting power of the plates, for the very long heat waves (*Rest-strahlen*)

obtained by multiple reflections from fluorite surfaces, was studied, with the direction of the strips first parallel and then perpendicular to the direction of the electric vector in the heat waves. The width of the strips averaged about  $5\mu$ , and the lengths were made equal to 6, 12, 18, and  $24\mu$ , the cross ruling being omitted in one case, which gave a plate with resonators infinitely long in comparison with the waves.

In all cases the plates showed a higher reflecting power when the direction of vibration in the heat waves (polarized by reflection) coincided with the long axis of the resonators. The highest reflecting power was exhibited by the plate on which the cross ruling had been omitted, which suggested the polarizing power of wire gratings observed by Rubens and Du Bois (*Wied. Ann.*, 49, page 593, 1893). The authors were of the opinion that unquestionable evidence of electrical resonance was shown, which seems highly probable. For many reasons more conclusive data could be obtained with resonators of smaller dimensions, which would respond to waves corresponding to regions of the heat spectrum where there was a greater abundance of energy, and accordingly a series of experiments were commenced by Nichols and Hull in collaboration with the author, which are still in progress. By depositing films of gold on glass by means of the cathode discharge and ruling under oil (which prevented the tearing of the film by the diamond), it was found possible to rule resonators measuring only  $1.6$  by  $3\mu$ . These plates were ruled with one of Rowland's machines, and were far superior in appearance to the earlier plates used by Rubens and Nichols, which were ruled by hand with a small dividing engine. Under the microscope the plates appeared almost flawless, the rectangles being sharply cut even at the corners. Unfortunately, however, no trace of resonance could be found with these plates, which appears to be due to the excessive thinness of the films, which were not wholly opaque to light. Rubens has, moreover, pointed out that, in order to secure sharp resonance, it is necessary that the clear spaces between the strips should be much wider than the strips, the reverse being true in the present case. The experiments are still in progress, and efforts are being made to obtain finely ruled plates of the required specifications.

**Polarizing Action of Gratings.**—Closely related to the phenomena which we have just discussed is the polarizing action which gratings exert on light. Fizeau (*Ann. de Chim. et Phys.* (3), 63, 385, 1861) noticed as early as 1861 that light is partially polarized by passage through a narrow slit. The same phenomenon was found by H. Dubois (*Wied. Ann.*, 46, 548, 1892) in the case of gratings made of fine silver wire. In all of these cases the direction of polarization (*i.e.* electric vector) of the transmitted light was parallel to the slit or wires. Fizeau, however, recorded that light which had passed through an extremely narrow slit less than  $0.1\mu$  in width was polarized perpendicular to the direction of the slit. On widening the slit to the dimension of the light wave  $0.5\mu$ , the direction of polarization of the transmitted light turned through a right angle.

Dubois and Rubens have (*Ber. der Deut. Phys. Gesell.*, 2, 2, 1904, p. 77) investigated the polarizing action of wire gratings for the long heat waves reflected from fluorite and rock-salt. They found that as the wave-length was increased the polarization of the transmitted

light, which was parallel to the wires, increased to a maximum, then diminished, becoming zero for a certain wave-length, after which further increase in the wave length was accompanied by a polarization of the transmitted light perpendicular to the wires, which increased in amount with increasing wave-length. This is in agreement with the investigation of Hertz, who found that a wire grating was more transparent for electric waves when the direction of the wires was perpendicular to the direction of the electric vector than when it was parallel to it. The investigations were confined to the energy directly transmitted by the grating, *i.e.* observations were made of the central image, and *not* in the diffraction spectra. The results found with a platinum grating are given in the following table :

Wave-length $\lambda$ .	$Q_m$ .	$Q_p$ .	$P_r$ .	$u^2 = \frac{Q_p}{Q_r}$ .
5	0.223	0.198	0.248	0.80
25.5	0.297	0.230	0.265	0.63
51.2	0.535	0.332	0.738	0.45

In this table  $Q_m$  represents the percentage transmitted when the incident light is unpolarized,  $Q_i$  and  $Q_p$  the transmission of light, the vibrations of which are perpendicular and parallel respectively to the direction of the wires.

We see in the first place that the transparency for all three types of radiation increases with increasing wave-length, but that the increase is greater for the perpendicularly-polarized light. The ratio  $u^2 = \frac{Q_p}{Q_i}$ , therefore decreases with increasing wave-length. The increase in the transparency for unpolarized radiations with increasing wave-length is due to the fact that the diffraction spectra disappear, with the exception of those of the first order, with the result that the intensity in the central image is increased. For the longest waves,  $51\mu$ , reflected from rock-salt, which are larger than the grating constant, diffraction in the ordinary sense no longer occurs, and the grating transmits practically one-half of the incident light.

It will be observed that in these experiments the state of polarization of the transmitted light is similar to that found by Fizeau with a slit smaller than the wave-length of light, *i.e.* the polarization direction of the transmitted heat-waves is at right angles to polarization direction of transmitted light.

As we shall see presently, the condition which prevails in the present case can be observed with visible light, if the elements of the grating are made small enough.

**The Color of Light Diffracted by Screens, showing Selective Absorption.**—Some remarkable effects were observed by Gouy (*Comptes Rendus*, xcvi., page 1573, 1884), and more carefully studied by Wien (*Inaug. Diss.*, Berlin, 1886), of the colored light diffracted into the region of the shadow by thin sheets of copper, gold, silver, etc.

Wien focused sun-light upon the highly polished edges of thin plates of various metals, and observed that light was diffracted far

into the region of the shadow, the edge of the plate appearing luminous. The color of the light varied in a remarkable manner with the nature of the metal, appearing red with copper and gold screens, orange with silver, yellow and yellow-green with platinum and tinfoil. The color only appeared when the edge was clean and quite free from dust; it was complementary to the color most strongly absorbed by the metal, and polarized with the vibration perpendicular to the diffracting edge. If the incident light was polarized to start with, the color was only seen when the vibration was perpendicular to the edge. The phenomenon is evidently related in some way to resonance, vibrations being set up in the metal along the edge, which emit energy into the region behind the screen. In addition to the colored light, Wien found that white light was also present, and that it could also be cut off by a Nicol prism, though its plane of polarization appeared to depend upon the azimuth in which the incident light was polarized, and also upon the angle of diffraction.

Still more recent observations have been made by F. Braun (*Ann. de Phys.*, 16, p. 1, 1905), who worked with extremely fine metal gratings obtained by passing the discharge of a large battery of Leyden jars through platinum or silver wires held in contact with a glass plate. The metal vaporized by the discharge deposited itself in the form of fine metal striae perpendicular to the wire. An examination of these plates between crossed Nicols showed that certain spots were transparent for light polarized with the electric vector perpendicular to the direction of the metal strips.

The best results were obtained with platinum wires .044 of a millimeter in diameter, through which the discharge of a battery of twenty jars (capacity 40,000 cm.) was passed. This phenomenon had previously been observed by Kundt in metal films obtained by the cathode discharge from a fine wire, which stood perpendicular to a glass plate. When observed between crossed Nicols the film appeared as a brightly illuminated disc, with a dark cross, the arms of which were parallel to the planes of polarization, the center of the cross being always at the point of the conical metal film, *i.e.* the point above which the cathode stood. Kundt regarded the phenomenon as a consequence of the orientation of the metal particles, and designated it as double refraction, although he clearly recognized the difficulty of the conception of double refraction in an isotropic metal. Although his films appeared homogeneous under the microscope, it appears probable, in view of the more recent investigations, that they were in reality metal gratings, the elements of which were arranged radially around a central point.

Another method of obtaining metal gratings of extreme fineness was devised by Braum, based upon an observation made by Ambrown, that thin sections of pine wood stained with chloride of gold exhibited dichroism. The color of the transmitted light varied with the direction of the polarization of the incident light. Experiments were made with sections of various kinds of wood, stained by immersion in a 2 per cent. solution of chloride of gold, followed by exposure to light, which reduced the metal in the microscopical cells of the material. The results of the experiments indicated that the metal was deposited

in the form of microscopic strips, the whole constituting a metal grating of extremely fine structure. It was also shown by Braun that the absorption of the light by the grating was greater when the direction of polarization was parallel to the direction of the metal strips than when perpendicular to them. This was accomplished by mounting the grating in the bulb of a small air thermometer, and observing the motion of the fluid index when the direction of polarization was changed.

**Optical Resonance.**—The first experiments on optical resonance were made by Wood (*Phil. Mag.*, April 1902, Oct. 1902, Aug. 1903). It was observed that films of the alkaline metals, deposited on the inner walls of exhausted glass bulbs, exhibited brilliant colors which could not be explained by any of the known laws of interference or diffraction. Examination with the highest powers of the microscope revealed the fact that the deposits were granular, the size of the metal grains being of the order of magnitude of the light waves. It seemed highly probable that the phenomenon was the optical analogue of the experiment performed by Garbasso with the tin-foil strips, the metal granules having free periods of electrical vibration of the order of the periods occurring in the case of visible radiation. The films are very easily prepared in the following manner:

A number of small bulbs are blown of the form shown in the same figure, and a piece of sodium or potassium is cut up under ligroin into blocks about 3 mm. on each edge. These are introduced into the bulbs as quickly as possible after wiping off the fluid, and the stems of the bulbs drawn down to a small bore for subsequent sealing.

They are quickly fastened to the branched tube and exhausted.

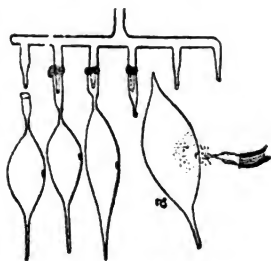


FIG. 305.

It is a good plan to heat the metal until it fuses, while the bulb is still on the pump; the whole bulb may be warmed by a Bunsen flame to drive off adsorbed air. If the exhaustion is carried down to the point where the mercury begins to hammer, it is generally sufficient. The bulbs are now sealed off from the pump, and may be put away for future use, or experimented with at once. A burner should be made by drawing out a glass tube, which will give a pointed flame about half a centimeter high, and the tip of the flame allowed to play against the

spot on the bulb where the metal lies (Fig. 305, *a*). Sometimes the whole bulb will suddenly flash a deep violet or blue, and sometimes the film will develop more slowly. A chain of three or four bulbs may be made, the sodium heated in one, and the clean molten metal shaken into the others, drops of various sizes sticking to the bulbs. Colored films can then be formed by heating these clean drops in the bulbs. This shows that the oxide present in the first lump has nothing to do with the production of the color.

If the metal is heated at one end of a rather long tapering bulb, the color is most intense near the metal and gradually fades away to

nothing at the other end of the bulb. If the bulb is placed in strong sunlight with a black background, it is seen that in some places where the deposit is too slight to show much color by transmission, the light is scattered or diffused, and this diffused light is colored. The claret-colored or purple film, where the deposit is slight, scatters a green light, the surface appearing as if fluorescent. Now the spectrum of the transmitted light in these purple films has a heavy absorption-band in the yellow-green, consequently the scattered light is the complementary color of the transmitted. If the film is greenish blue, the scattered light is reddish. The microscope shows that in these deposits, which have the power of scattering light, the individual particles are rather widely separated, that is, the distance between them is large in comparison to their diameters. The appearance of a bulb in strong light is very much as if certain portions of its interior surface had been painted over with a solution of fluorescein. No trace of regular reflexion is shown by these films, except of course the reflexion due to the glass. The particles are so far apart that they apparently act as independent sources, the interference necessary for rectilinear propagation not being present. If the incident light is polarized, the fluorescent light (as we may call it for convenience) is also polarized, which is not the case for ordinary diffuse reflexion. On the resonance theory, we may regard this fluorescent light as the energy radiated from the resonators, as a result of their forced vibrations. Passing now to a part of the film where the color of the transmitted light is deeper, we find that there is no longer any trace of this fluorescent light. The color absent in the transmitted light is now regularly reflected, the particles being so close together that interference, as imagined by Huygens, takes place.

It appears as if the case was very similar to the hypothetical one considered by Planck in his paper on absorption, which has been already alluded to. It will be remembered that Planck considers the energy stopped by his resonators, as re-emitted by them, either as diffused light or regularly reflected light, the diffusion and reflection being of course selective. This seems to be precisely what occurs in the present instance, the particles diffusing or reflecting regularly according to their proximity.

On portions of the bulb close to the heated spot, the metal is deposited in granules too large to show resonance colors, a silky lustre being exhibited by reflected light. This is obviously the ordinary diffusion or diffraction produced by small particles.

**Effect of Changing the Surrounding Medium.**—It has been shown by Aschkinass and Schaefer that the length of electro-magnetic waves to which a system of resonators respond is increased by immersing the resonator system in a medium of high dielectric constant. The same phenomenon occurs in the case of the sodium and potassium films. The bulbs usually contain traces of hydrocarbon vapor, which can be condensed upon the inner wall by touching a spot on the outside with a piece of ice. It is a good plan to moisten the bit of metal with a little ligroin before its introduction into the bulb. The color changes are most remarkable. Pink and purple films become blue, while pale apple-green films change to a deep blue violet, as deep as dense

cobalt glass. Blue films often became perfectly transparent, the absorption band, originally in the red, moving out of the visible spectrum entirely. Spectroscopic examination showed that the immersion of the resonators in the liquid dielectric caused the absorption band to move towards the region of longer wave-lengths, as it should do according to theory.

**Changes produced by Oxidation.**—If the tip of one of the bulbs is cut off, the entrance of the air causes the colored film to vanish like a flash. In some instances a momentary change of color was noticed before the film disappeared. To lengthen the process the expedient was adopted of drawing the end of the bulb out into a long fine capillary, with a bore less than one one-hundredth of a millimeter. In this bulb a film of a deep pink color was formed, and on cutting off the tip of the capillary the color changed to blue, and the film vanished almost immediately. The small amount of air necessary to efface the films is indicated by the fact that at the end of an hour there was still a fairly good vacuum in the bulb, notwithstanding the fact that the end of the capillary had been open all the while. If the process of oxidation be made very slow, by employing a very long capillary, and the process watched under the microscope, the black particles slowly become dim, and finally fade away. The microscope merely shows us the diffraction-disc due to the opaque particle. This becomes dimmer as the size of the particle is reduced, without any apparent change of size. Examination with the spectroscope shows us that, during the process of oxidation, the absorption-band sometimes moves out of the spectrum through the red end, and sometimes merely fades away without any motion.

**Colors of Granular Films of Gold and Silver.**—To ascertain whether the color effects were common to all metals in a state of fine subdivision, experiments were made with gold and silver, the former obtained by electrical discharges in high vacua from a gold cathode, the latter by employing Carey Lea's solution of allotropic silver.

The color of the gold deposit varies with the conditions under which the deposition takes place. Gold cathodes of two forms were employed, a flat plate about 3 cms. square, and a thick wire, screening off the radiation from all but the tip with a mica screen. The most interesting deposits were obtained from the small source. In one instance the film showed a brilliant green surface color, resembling fuchsine, the transmitted light having a purple tint. Owing to the transparency of the film a good deal of white light is mixed with the selectively reflected light; this can be cut off with a Nicol if the reflection takes place at the polarizing angle for glass, and the colored light from the film, which is unpolarized, then appears in great purity. One plate showed patches of brilliant carmine red, deep blue, and green, of a surprising intensity and saturation. The color of the selectively reflected light depended somewhat on the angle of incidence, a phenomenon observed also in the case of the sodium and potassium films. Increasing the angle of incidence changed the color from green to blue; the period of vibration of the resonator system appears therefore to be less when the angle of incidence is large.

If the glass plate is placed near the tip of the gold wire, the green

deposit, similar to gold leaf in its optical properties, is deposited at the center. The color of the green film is probably due to the same causes which operate in the case of gold leaf, *i.e.* to molecular resonance. These films are not granular, the metal vapor not condensing into drops before reaching the glass. Surrounding this is a film appearing light yellow by transmitted light, and bluish by reflected light. This seems to be what we should expect, for the smallest particles, which will resonate for blue light, will be deposited when the distance from the cathode is a little greater from that at which the molecular deposit occurs. Increasing the distance, we get larger particles, and the point of maximum resonance moves up into the green, giving us a purple film with green surface color. At a still greater distance we get particles large enough to resonate for red, and the film appears deep blue by transmitted light. All of these variously colored films can be changed into the green structureless film by heating. We may regard the change as due to the fusing together of the resonators. Silver films showing brilliant colors can be prepared by employing a solution of so-called allotropic silver described by Lea. Three solutions are prepared: a 30-per-cent. one of ferrous sulphate, a 40-per-cent. one of sodium citrate, and a 10-per-cent. one of silver nitrate. Fourteen c.cms. of the citrate solution are mixed with 10 c.cms. of the ferrous sulphate solution, to which is then added 10 c.cms. of the silver nitrate solution. A dense black precipitate immediately forms, and the whole is at once poured into a filter. As soon as the liquid has entirely run through, the precipitate is washed with 10 cms. (*not more*) of distilled water. This is to remove the salts which make the precipitate insoluble. After the water has entirely passed through the filter, about 25 c.cms. of distilled water are poured into the filter, and the blood-red solution which runs through collected. As it does not keep very well, it is best to prepare it on the day on which it is to be used.

A sheet of glass is washed clean, rinsed with fresh water, and the wet surface rubbed over with some shreds of gelatine. It is then drained for a few seconds and dried on a hot plate. A little of the silver solution is flowed over it, the surplus being drained off. If too much gelatine has been used, precipitation is apt to take place, the deposit taking the form of floating shreds of a reddish membrane. If no considerable precipitation occurs, the plate, which should have been quite warm when flowed, is placed once more on the hot plate until dry. The films formed in this way are usually deep red in color, though sometimes patches of deep violet form, with sharply defined edges. Violet patches may be easily formed in the following way: When the plate is about half dry, with a steaming film and a few small pools of the hot solution, it is removed from the hot plate, held at an angle, and treated with a few drops of alcohol, which are allowed to run down across the still damp portion of the plate. These portions speedily dry into a most gorgeous mosaic of red, purple, and violet patches, the experiment being especially striking in the lantern, as it occupies but a few seconds, and the color-display spreads over the plate like the blaze of a sunset.

Any desired depth of color can be obtained with these films by merely allowing more or less of the solution to remain on the plate.

Some are of such a deep red that they are almost opaque, a gas-flame being barely visible through them. The light which does get through is regularly transmitted, that is, the films are not turbid media. The spectroscope shows that the absorption band is wider and less sharply defined than is the case with some of the purple potassium films, which have a rather narrow and very black band in the yellowish green. This can be explained by assuming that there is not a great regularity in the size of the particles, and consequently less sharp selective resonance. These films are transparent to the entire ultra-violet region, even down to the last cadmium lines, of wave-length 22 or thereabouts.

**Anomalous Dispersion of the Films.**—Prisms were made by the evaporation of the silver solution between a plate of glass and a piece of glass tubing, the method being similar to the one employed by Pflüger in the preparation of cyanine and fuchsine prisms. If the colors are due to resonance, anomalous dispersion should be observed in crossing the absorption band, which in the case of the silver films occurs in the green.

This was found to be the case, the prisms deviating red light in a measurable degree, though transmitting blue light without sensible deviation. It is highly probable that a similar phenomenon would be shown by the sodium and potassium films, though these have not been investigated up to the present time owing to the experimental difficulties involved.

**Colors of Lippmann Photographic Plates.**—An exhaustive study has been made by Kirchner (*Ann. der Physik*, 13, p. 239, 1904) of the colors exhibited in transmitted light by Lippmann plates. These plates, after development, appear reddish-brown, green, or blue, according to the developer used. The color phenomena have nothing to do with the presence of silver laminae, which are chiefly responsible for the colors seen in reflected light in Lippmann's color photographs, for they are equally pronounced in the case of plates immersed in benzol during their exposure to light. Standing waves and the formation of the laminae are of course out of the question in such a case.

Kirchner came to the conclusion that the color was due to the resonance of the minute silver aggregates reduced in the film by the action of the developer. In ordinary photographic plates these are too large to serve as resonators for visible radiations, and the plates appear black in consequence, the light being stopped by the opaque masses.

The position of the absorption band was found by Kirchner to shift its position when the dry films were moistened, the band moving towards the blue. This is in accordance with theory, for the refractive index of wet gelatine is much less than that of dry. The dispersion of the colored films was also investigated and found to be anomalous. The refractive indices for the various wave-lengths are given in the following table:

DISPERSION OF REDDISH-BROWN SILVER FILM.

$\lambda$	500	525	550	560	570	575	589	600
$n$	1.542	1.557	1.567	1.585	1.595	1.601	1.555	1.535

The absorption band begins in the red and extends to the blue, increasing in intensity. There is a decrease in the values of  $n$ , as the observations are pushed into the band, which is in accordance with theory, and the maximum value of  $n$  is found on the red sides of the band, though perhaps further within the band than is usually the case with absorbing media.

These results are interesting, as confirming those obtained by Wood with granular deposits of the alkali metals and allotropic silver. Kirchner's work was practically completed when the paper above referred to was published, his results having been obtained independently.

The subject of optical resonance has also been studied extensively by Kossonogoff, whose papers appeared in the *Physikalische Zeitschrift* for 1903. Resonating films were prepared by blowing a spray from an atomizer charged with a solution of the metal salt, upon a strongly-heated glass plate. He also obtained evidences of resonance in granular deposits of non-metallic substances, and made a careful study of the colors of the wings of butterflies, which he regards as due to similar causes. Bock, in the same journal, has published results which are claimed to show that minute water drops also exhibit the phenomenon.

**Colors in Metal Glasses and Metallic Films.**—An extremely interesting treatment of the colors exhibited by glasses which are stained with metallic oxides, and by the films of the alkali metals deposited in vacuo, has been given by Garnett (*Phil. Trans of Roy. Soc. Lond.*, Series A, vol. 203, page 385). Colored glasses are supposed to owe their color to the presence of minute globules of the metal. Colloidal solutions of the metals act in a similar manner, and recently the presence of the particles has been detected in both cases by Siedentopf and Szigmondy, by means of their so-called ultra-microscopic method, which is merely oblique illumination pushed to the limit. A powerful beam of light issuing from a horizontal slit is brought to a focus by means of a microscope objective, within the glass or liquid under investigation. The small particles scatter some of this light and appear as minute diffraction discs of light, when a microscope is focused on the illuminated plane (Fig. 306). The phenomenon is analogous probably to the scattering of light by the air molecules or small particles suspended in the atmosphere, the action of which has been exhaustively treated by Lord Rayleigh. A simpler device, due to Cotton, is shown in the lower part of the figure, in which the illuminating beam is prevented from entering the microscope by total reflection.

Garnett shows that the colors of the glasses can be accounted for by the presence of the small metal spheres, and explains a number of very curious effects observed by Siedentopf and Szigmondy, some of which we will now consider.

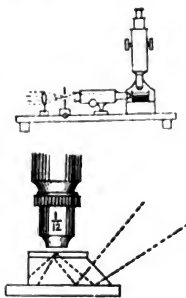


FIG. 306.

**Polarization Effects of Ultra-Microscopic Particles.**—Some very curious and interesting effects were observed by Siedentopf and Szigmondy in the case of gold particles when the illuminating beam was plane polarized. If the plane of the vibration was perpendicular to the plane containing the illuminating ray and the microscope, the little diffraction discs appeared everywhere in the field, and were of uniform illumination. The scattered light was polarized in the same plane as the incident. This condition is shown at *a* (Fig. 307). The

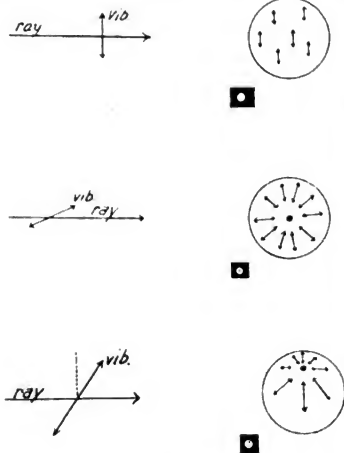


FIG. 307.

field of the microscope is represented by the large circle, and the direction of vibration of the scattered light by the arrows. The appearance of the diffraction discs is shown below. If, however, the incident light vibrated parallel to the above defined plane, the particles scattered no light in the vertical direction, and the diffraction discs were formed by the oblique rays gathered in by the objective. In this case each diffraction disc appeared with a black spot at its center as shown. Moreover, since the light scattered in any given direction by the particles comes to a focus at a given point in the second focal plane of the microscope, a black spot will be found at the center of this plane. This condition is shown at *b*. If the incident vibration is inclined at  $45^\circ$ , the spot appears on the sides of the diffraction discs, and there is a dark region on one side of the second focal plane as in *c*. The direction of the vibration at the second focal plane is indicated in each case by the double-headed arrows in the large circles.

The explanation of these appearances was not given by Siedentopf and Szigmondy, but Garnett discusses them very fully in the paper referred to. Another very remarkable effect observed was the splitting of each diffraction disc into a pair, when the gold particles were not smaller than  $1\mu$ . One of the discs was reddish, the other yellow-green, and they were all oriented in the same way, showing that the effect was one of diffraction, and not due to any actual dual structure of the minute particles. No explanation of this curious phenomenon has been given, but it is probably related in some way to the diffraction of colored light (red) by thin plates of gold observed by Wien.

The effects described indicate that the gold particles must be spherical, for oblong particles would not completely polarize the scattered light (as Lord Rayleigh has shown in his papers dealing with

the polarization of the light of the sky), and we should not observe the dark spot in the diffraction discs, or the dark region in the second focal plane of the microscope.

We will now consider the effects of these small gold spheres upon the color of the transmitted light. The subject has been very fully discussed by Garnett, who has investigated the subject from the standpoint of the electro-magnetic theory. His treatment is much too long to give in full, but we can examine to advantage the general method of attack and some of the conclusions.

Let light of wave-length  $\lambda$  fall on a metal sphere of radius  $a$ , refractive index  $n$ , and absorption coefficient  $\kappa$ . Further, let

$$N \equiv n(1 - i\kappa) = \sqrt{\epsilon'},$$

$\epsilon'$  being the complex dielectric constant.

This case has been considered by Lord Rayleigh (*Phil. Mag.*, xliv., page 28, 1897), who showed that the sphere excited by a periodic electric force  $E_0$ , emits the waves which would be emitted by a Hertzian doublet, which at time  $t$  was of moment equal to

$$\frac{N^2 - 1}{N^2 + 2} a^3 E_0.$$

If there are a large number of spheres in close proximity, the electric force exciting each one will be  $E'$ , i.e. the force  $E_0$ , together with forces due to the neighboring doublets. This force  $E'$  causes the polarization

$$f(t) = a^3 \frac{N^2 - 1}{N^2 + 2} E'.$$

If the average moment of a doublet be  $f(t)$ , and there are  $n$  doublets per unit volume, the polarization of the medium will be  $f'(t) = nf(t)$ .

By means of analyses by Lorentz and by Larmor it can be proved that

$$E' = E_0 + \frac{4\pi f'}{3} = E_0 + \frac{4\pi}{3} na^3 \frac{N^2 - 1}{N^2 + 2} E',$$

provided the doublets are distributed through a space large in comparison to the wave-length.

This gives us

$$E' = \frac{E_0}{1 - \frac{4\pi}{3} na^3 \frac{N^2 - 1}{N^2 + 2}}, \text{ so that } f = \frac{E_0 \frac{N^2 - 1}{N^2 + 2} a^3}{1 - \frac{4\pi}{3} na^3 \frac{N^2 - 1}{N^2 + 2}}.$$

By substitution of these units in Maxwell's equation, the complex dielectric constant of the medium containing the spheres is found to be

$$\epsilon' = 1 + \frac{3D \frac{N^2 - 1}{N^2 + 2}}{1 - D \frac{N^2 - 1}{N^2 + 2}},$$

in which  $D$  is written for  $\frac{4\pi}{3} na^3$ , denoting the volume of the metal per

unit volume of the medium. This is for spheres in vacuo: in glass of refractive index  $\mu$  the equation becomes

$$n'(1 - i\kappa') = \epsilon' = \mu^2 + \frac{3\mu^2 D \frac{N^2 - \mu^2}{N^2 + 2\mu^2}}{1 - D \frac{N^2 - \mu^2}{N^2 + 2\mu^2}}.$$

The optical constants of the medium  $n'$  and  $\kappa'$  thus depend only on  $D$ , the relative volume of the metal, and not on the size of the spheres, restricting them, however, to sizes small in comparison to  $\lambda$ . By reducing the above equation and substituting in it the values for  $N$  and  $\mu$ , the absorption coefficient can be found for any given value of  $D$ .

Now  $D$  varies with the nature of the glass. The gold glass as first prepared is colorless, becoming red on re-heating, the process causing the metal spheres to form within the body of the glass. "Excretion of the metal" Garnett calls it. Colorless gold glass turned red on exposure to the emanation of radium, and it is probable that the blue color of X-ray tubes, and tubes which have contained radium, is due to the excretion of metallic potassium or sodium by the radiation. Sir William Ramsay exposed glass containing silver to radium rays and found that it turned yellow. Quartz glass is not colored, as no metal is present.

Elster and Geitel (*Wied. Ann.*, 59, page 487, 1896) found that salts of the alkaline metals, colored by the action of cathode rays, exhibited photo-electric properties, which suggested the presence of free metal; this supports the view held regarding the coloration of glasses by X-rays and radium rays.

Garnett took the values of  $N$  calculated from Drude's tables of the optical constants of the metals, and the values of  $D$  calculated from the total gold content of the glass, and the observations of Siedentopf and Szigmondy, and showed that the medium should be much more transparent for red than for yellow light.

Values of  $n$  and  $\kappa$  for green and blue light not being available, the best that could be done was to infer that, since yellow is less freely transmitted than red, the medium is still more opaque to green and blue.

Garnett next develops an expression for the intensity of the scattered light, and finds that the amplitude at any point of the light emitted from a sphere is proportional to

$$\left| \frac{N^2 - \mu^2}{N^2 + 2\mu^2} \right| \frac{a^3}{\lambda^2}.$$

The  $\lambda^2$  in the denominator indicates that the scattered intensity increases with the inverse 4th power of the wave-length, but that it is also dependent on  $N$ , i.e. on the optical constants of the metal.

Calculations showed that yellow light would be scattered more powerfully than red, from which it was inferred that green would be still more powerfully scattered, which is in agreement with the observations of Siedentopf and Szigmondy.

Certain types of gold glass scattered a muddy red light however. In this case the particles are probably so large that they reflect light in

the ordinary sense, and, as we know, gold reflects red light in greater excess than any of the other colors.

In an appendix to the paper, the transmission of gold and silver glass has been calculated for red, yellow, green, and blue light, from values of  $n$  and  $\kappa$  given by Rubens. The colors, in the order of the degree in which they were transmitted, were found to be: for gold glass,—red, yellow, blue, green; for silver glass,—yellow, red, green, blue. Certain gold glasses appear blue by transmitted light, and it appears probable that large particles (diameter  $> .0001$ ), by reflecting out the red and orange, give the glass a blue color.

**Colors of Granular Metallic Films.**—In the second part of the paper above referred to Garnett examines the conditions which hold in cases where the metallic granules are deposited in thin films. The equations already given were developed on the assumption of a uniform polarization in the medium, which is only the case when the spheres are distributed in three dimensions. For a two-dimensional distribution, in the  $xy$  plane, it is shown that the complex dielectric constant in the direction of the  $xy$  axes is the same as for the medium in bulk, whereas the constant in the direction of the  $z$  axis may be quite different. If this were the case the film would behave like a doubly refracting substance, the "optic axis" being perpendicular to the film.

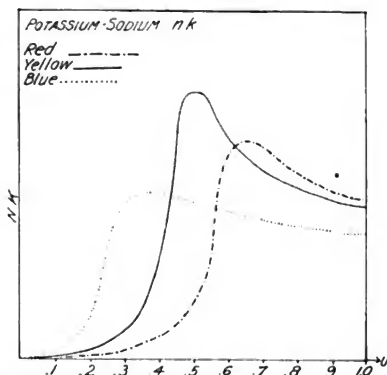


FIG. 308.

It is found that for films of thickness greater than  $\frac{2}{3}$  of  $\lambda$  the absorption is governed by  $n\kappa$ , while in the case of films less than  $\frac{1}{2.5}\lambda$  it is governed by  $n^2\kappa$ . Curves are given showing how the absorption depends on  $D$ , the volume of metal per unit volume of the medium. The values of  $n\kappa$ , or  $n^2\kappa$ , are plotted as ordinates, and the values of  $D$  as abscissae. In the case of a non-granular film of solid metal it is evident that  $D=1$ .

Garnett was able to explain all of the effects observed by Wood in

the case of the sodium and potassium films deposited in exhausted bulbs, at least all of the effects which fell within the scope of his equations. The curves for a potassium sodium amalgam are given in Figs. 308, 309, and show how the absorption depends on the value of

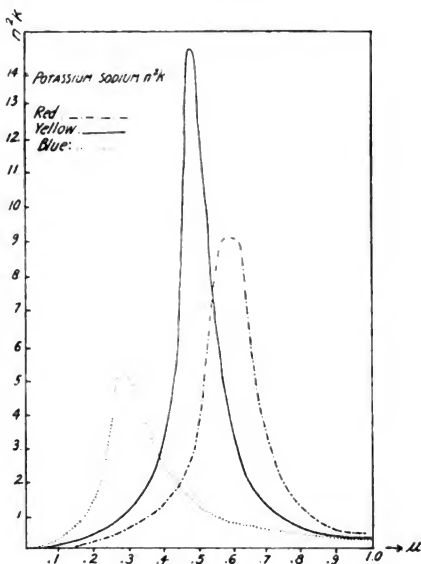


FIG 309.

*D*. The upper curve represents the condition for a thick film. For  $D=1$ , i.e. solid metal, the absorption is strongest for red and weakest for blue. For  $D=.5$ , that is for a film with equal volumes of metal and empty spaces, the absorption is strongest for yellow, while for  $D=.3$  the blue is practically the only color absorbed. For thin films we find that for  $D=.5$  the yellow is very powerfully absorbed, which agrees with Wood's observation that when the conditions were such as to cause an absorption band in the yellow, the band was much blacker and narrower than when it occurred in some other part of the spectrum.

The graphs for gold indicated that for  $D=1$  the color in the case of very thin films of gold leaf should be blue. This was the color observed by Mr. Beilby in the case of the thinnest leaf which could be procured. For thick films the graphs showed that the color of the transmitted light should be green. Many very curious and interesting changes, observed by Mr. Beilby (*Proc. Roy. Soc.*, 72, page 226), in the colors of thin gold films, produced by heating and annealing, are discussed and explained by Garnett, whose paper is by far the best on the optical

properties of metallic films which has appeared up to the present time.

It appears that the theory of optical resonance can be regarded as in a measure confirmed by these investigations, though the phenomenon is very much more complicated than in the case of large resonators and electro-magnetic waves. The optical constants of the metal enter as a factor, and for very small particles at least the absorption depends not on their size, but on the total bulk of metal in unit volume.

## CHAPTER XXI.

### THE NATURE OF WHITE LIGHT.

PREVIOUS to Newton's classic experiments on the decomposition of white light by a prism, it had been supposed that every refraction of light actually produced color, that is, the color was produced within the dispersing piece. Newton's experiments apparently showed, however, that the colors were actually present in the original light, the function of the prism being merely to separate them. At the present time, however, the view most generally held resembles in some respects the idea which prevailed previous to the time of Newton. Colored light implies a greater or less amount of regularity in the sequence of the waves. If a grating or prism yields us light which the eye is unable to distinguish from the light of the sodium flame, it is pretty certain that the luminous impulses are following one another at approximately the same intervals, as in the case of the radiation from the sodium molecule. The question now is, "Were these regular wave-trains present originally in the white light, or have they been manufactured by the grating?" If the former is true, how are we to regard a vibration which is made up of an infinite number of regular trains of waves, each, however, of different wave-length? Obviously the motion of the ether must be regarded as the resultant of all the component trains, and the regularity of sequence, as we usually understand it, would disappear wholly in wave-motion of this nature.

If, however, the regular wave-trains, or colored rays, are manufactured by the prism, we very naturally ask whether Newton's discovery was really a discovery after all.

The problem can be attacked both from the experimental and theoretical standpoint. Let us see first what evidence experiment furnishes.

Interference fringes were obtained by Fizeau and Foucault with white light and with a path-difference of some 50,000 wave-lengths. This experiment has been held, by many eminent authorities, to indicate that the colored components of the white light consist of regular wave-trains, the vibrations of which take place without sensible change of phase for at least 50,000 complete periods.

The first to question this conclusion was Gouy (*Journ. de Phys.*, 5, page 354, 1886), who showed that the experiment of Fizeau and

Foucault showed nothing whatever about the regularity of the vibrations in white light, the number of fringes observable, or the allowable path distance, depending solely upon the resolving power of the spectroscope used for their detection.

Gouy made use of Fourier's theorem, by which any function  $f(x)$ , between limits  $x = -c$  and  $x = +c$ , may be represented by the sum of a series involving the sines and cosines of  $\frac{\pi x}{c}$ .

$$f(x) = a_0 + a_1 \cos \frac{\pi x}{c} + a_2 \cos 2 \frac{\pi x}{c} + \dots$$

$$+ b_1 \sin \frac{\pi x}{c} + b_2 \sin 2 \frac{\pi x}{c} + \dots$$

If the medium is free from dispersion we can determine the shape of the resultant disturbance at a point beyond  $+c$ , if the component waves are travelling in the positive direction, by substituting  $x - vt$  for  $x$ . If we now add the series we find it equal to  $f(x - vt)$ , which shows us that waves of any shape are transmitted in a non dispersive medium without change of form. If, however, the medium is dispersive, the component waves, each one of which is represented by a term in the series, travel with different velocities, and the form of the resultant disturbance changes with the time, *i.e.* it is not propagated with its type unchanged.

Gouy showed that the fringes observed by Fizeau and Foucault could be explained as well on the hypothesis that the original disturbance was a single pulse, or a series of waves of irregular form, *i.e.* not sine waves. A pulse or single irregular wave can be represented by Fourier's theorem, as the resultant of a large number of sine waves which extend to infinity on either side of the pulse. The spectroscope will spread this disturbance out into a spectrum, and at every point of the spectrum we shall have a periodic disturbance. In other words, the spectroscope will sort out the Fourier components into periodic trains of waves, just as if these wave-trains were really present in the incident light. We can perhaps get a clearer idea of the two theories of white light in the following way:

Suppose our source of light to consist of a number of electrons vibrating in simple harmonic motion, but with different periods. The resultant disturbance at any point will have an irregular form, which we can represent by a curve something like that shown in Fig. 310. The regular periodicities due to the electrons are present in the disturbance, and a prism or grating will sort them out into a bright line spectrum of simple harmonic trains, precisely as Fourier's analysis does.



FIG. 310.

We can, however, consider the same type of disturbance as given out by a single electron, if it is constrained to vibrate in an irregular manner, and the spectroscope must necessarily resolve this disturbance

in precisely the same manner as before, since the disturbances are identical.

Suppose now that our disturbance consists of a single pulse, such as is shown in the lower part of Fig. 310. By Fourier's analysis it can be represented as the resultant of a large number of simple harmonic waves, differing in wave-length by infinitesimal steps, and with amplitudes suitably distributed. We can think of this disturbance as originated in two different ways: from a group of continuously vibrating electrons, or from a single electron, executing a half vibration with a motion obeying such a law as to give a wave-form of the requisite shape. In the first case the component wave-trains can be regarded as having an actual existence, the periodicities being actually present both in the source and in the disturbance. In the second, there is no periodicity at the source, and consequently none in the disturbance. Though this way of looking at the matter is not quite rigorous, it may help us in getting a clearer idea of the distinction between the old and new idea regarding white light. In the last two cases considered, the spectroscope will give us a continuous spectrum, in which the intensity at any point is proportional to the square of the amplitude of the corresponding wave-length. In the first case this wave was present in the train, and had a definite origin; in the second, it can only be considered present in a mathematical sense.

Gouy's conception of white light was criticised by Poincaré (*Compt. Rendus*, 120, page 1895), both from a mathematical and experimental standpoint. Since the Fourier components representing the disturbance must extend to infinity in both directions, they ought to appear in the spectroscope not only after the source of light has been extinguished, but even before the source is lighted, an obvious absurdity. Poincaré was of the opinion that the experiment of Fizeau and Foucault indicated the presence of a high degree of regularity in the disturbances constituting white light.

These objections were met by Gouy and also by Schuster (*Comp. Rend.*, 120, pages 915, 987), who had arrived at conclusions similar to those held by Gouy. We find the same idea expressed by Lord Rayleigh in his article on "Wave Theory," written in 1888. "The light," he says, "may be highly composite and homogeneity brought about with the aid of a spectroscope. The analogy is closest if we use a spectroscope to give us a line of homogeneous light in simple substitution for the sodium flame (as a source of light producing interference fringes). Or following Foucault and Fizeau, we may allow the white light to pass (*i.e.* enter the interference apparatus), and subsequently analyze the mixture transmitted by a narrow slit in the screen upon which the interference bands are thrown. *In either case the number of bands observable is limited solely by the resolving power of the spectroscope, and proves nothing with respect to the regularity or otherwise of the vibrations of the original light.*" He shows further that when achromatic bands are formed by using a diffraction spectrum as a source, and duplicating it with Lloyd's mirror, the number of bands possible is still limited by the resolving power of the instrument used to form the spectrum.

If we go back to the source of white light and try to picture the

nature of the disturbances there which would be necessary to give us the two types of radiation which we have discussed, we shall incline towards the ideas of Gouy and Rayleigh. If the light contains periodic trains, which are regular over a length of some thousands of wave-lengths, there must be present in the source electrons or vibrators of some sort vibrating with all possible periods, otherwise there would be dark lines in the spectrum corresponding to the absent frequencies. The smallest particle of solid matter which we can command, when raised to incandescence, gives us a continuous spectrum. The visible region of the spectrum, say 7000 to 4000, comprises 3000 Angstrom units. With the best spectroscopes we can easily resolve to  $\frac{1}{16}$  of a unit, consequently the continuity of the spectrum implies that there must be at least 30,000 electrons, no two of which vibrate in the same period. If the molecules are moving to and fro, the Doppler effect will of course alter the wave-length, consequently we can diminish the necessary number somewhat. From what we know about electrons, however, it is difficult to see how we can have a very large number vibrating regularly in totally different periods, and we naturally incline towards the idea that the disturbances in a source of white light are irregular in character.

**Theory of Damped Vibrations.**—Another idea was put forward by Garbasso (*Arch. de Genève*, vol. 4, p. 105, 1897), who considered white light to be the result of a heavily damped vibration, represented by

$$f(t) = e^{-kt} \sin ht.$$

We have seen that the damping due to radiation is very small, but collision between the molecules, if frequent enough, may accomplish the desired result. In gases the time elapsing between molecular impacts is very large in comparison to the period of the vibration, and we have long trains of approximately homogeneous waves thrown off between collisions. In the case of solids and liquids the conditions are quite different, the molecular excursions being extremely short.

One way of testing the hypothesis is to develop the expression representing the damped vibration by Fourier's theorem, and see whether the intensity distribution in the resulting mathematical spectrum corresponds with the distribution of energy observed in the case of incandescent solids. This was done by Carvallo (*Compt. Rend.*, 130, page 79, 1900).

$$f(t) = e^{-kt} \sin ht \text{ (for } t > 0 \text{ but } f(t) = 0, \text{ for } t < 0).$$

Developed by Fourier's theorem.

$$f(t) = \frac{1}{\pi} \int_0^{\infty} \frac{h \, lq}{\sqrt{(q^2 - h^2 - k^2)^2 + 4k^2q}} \cos \left[ qt - \arctan \frac{2kq}{q^2 - h^2 - k^2} \right] dq.$$

The intensity of a vibration of period  $\frac{2\pi}{q}$  is, according to the formula,

$$y = \frac{h^2 q^2}{(q^2 - h^2 - k^2 + 4k^2 q^2)}.$$

If we put  $k^2 = a^2 h^2$  and  $q^2 = (1 + a^2) h^2 u^2$  the above becomes

$$y = \frac{1}{(1 + a^2) \left(u - \frac{1}{u}\right)^2 + 4a^2}.$$

In this form the properties of the function appear. For  $u = 0$  and  $u = \infty$ ,  $y = 0$ . It attains its maximum  $\frac{1}{4a^2}$ , for  $u = 1$ .

Finally, two values of  $u$  which are equal but of opposite sign, give us the same intensity. The same is true for equal but opposite values of the logarithm of the wave-length.

Carvallo then constructed a curve with the intensities as ordinates and the values of  $\log \lambda$  as abscissae. The lack of agreement between this curve and the curve plotted from the observations of the energy distribution in the spectrum, made by Mouton and by Langley, indicated that the hypothesis of a damped vibration was inadmissible. Another objection was raised by Carvallo, who showed that a grating would yield a band of white light instead of a spectrum if the incident light consisted merely of damped vibrations.

Gouy (*Comp. Rend.*, 130, page 241) comes to a different conclusion, objecting to the treatment of Carvallo, in that he extended his analytical treatment from  $-\infty$  to  $+\infty$ , a condition which could not be realised in experiment. Limiting the number of disturbances falling on the grating to a small number  $n$ , which must be the case when the vibrations are heavily damped, he shows that the disturbance at any point will have the periodicity calculated from the ordinary laws of the grating, and not, as imagined by Carvallo, the same periodicity as the original damped vibration.

Carvallo in his reply (*C.R.*, 130, page 401) proposes an interesting acoustical experiment to settle the question. Let the source of light be represented by a large tuning-fork, driven electrically. The waves from this are to be received by a large concave grating made of broad slats with open spaces between. As long as the vibrations are maintained by the electrical mechanism we should find points of maximum intensity with silent spaces between, at the focus of the grating, corresponding to the spectra produced by a grating when illuminated with monochromatic light. If, however, the current is suddenly turned off, the vibration is damped, and we should, if Carvallo's hypothesis is correct, find a faint sound of varying pitch all along the region between the points previously occupied by the maxima: in other words, a sound spectrum. This effect might be detected by a Helmholtz resonator placed at the proper point in the spectrum. If, however, Gouy's notion is the correct one, the region between the maxima would still be a region of silence, or at least only yield a faint sound corresponding in pitch to the pitch of the fork.

The experiment would be a difficult one to perform on account of the enormous dimensions of the apparatus and the difficulty of protecting the ear from the direct sound of the fork.

**Type of the Impulse constituting White Light.**—If we regard white light as a series of impulses, without regularity, the impulses cannot be

regarded as arbitrary, *i.e.* of any form, for, as Lord Rayleigh has pointed out, there would in this case be no way of distinguishing the radiations corresponding to different temperatures. He considers (*Phil. Mag.*, xxvii., page 460, 1889) the simplest type of impulse that could meet all the requirements of the case, to be the one with which we are familiar in the theory of errors, *viz.* (Fig. 311):

$$y = e^{-c^2x^2}.$$

Such an impulse, he remarks, can be considered as the resultant of a very large number of localized simultaneous impulses, all aimed at a single point ( $x=0$ ), but liable to deviate from it owing to accidental causes. This disturbance he resolves into its elements by means of Fourier's theorem, and then finds the energy carried by each component. By assuming an infinite number of these impulses, of the same form but unequal magnitude, he obtains probable values of the partition of the energy among the various wave-lengths, which agreed fairly well with Weber's law, which at the time best expressed the energy distribution in the spectrum. Wien's law could be satisfied by an impulse of some other definite form.



FIG. 311.

The character of the disturbance is thus fixed by this distribution of energy in the spectrum, and Wien's law marks the limit of our knowledge regarding the nature of white light. Planck definitely states that this will for ever mark the limit. Further analysis, he says, will be based on a reasoning comparable with that involving a contradiction of the second law of thermodynamics, in which the aid of Maxwell's demons was invoked. In the kinetic theory of gases we are obliged to confine our investigations to the average effect of molecular impacts, and we must, in dealing with the present problem, consider only the average effects of the light disturbances, extending as they do over relatively long intervals of time.

As we shall see presently, the dispersion by prisms and gratings can be accounted for without assuming the presence in the light of any periodicity whatever. Up to the present time no experiment has been devised capable of proving or disproving the presence in white light of regular wave-trains.

**Interference Experiments in the Light of the Pulse Hypothesis.**—In the chapter on Interference we have treated all of the problems by tacitly assuming the presence of regular wave-trains. We will now examine a number of cases and see whether the observed effects can be accounted for on the hypothesis that white light consists of irregular pulses. Can we, in other words, account for the colored fringes seen with Fresnel's mirror or two slits, assuming the incident light to consist of a single pulse?

Schuster shows, in a long and interesting paper (*Phil. Mag.*, June 1894), that the above question can be answered in the affirmative. It is easy to see how periodicity can be manufactured by a grating or

prism, but there are a number of cases which at first sight may seem irreconcilable with our hypothesis. If a pulse falls upon a pair of Fresnel mirrors, we shall have at a specified point in space two pulses, the interval between them depending upon the position of the point. If we receive the double disturbance upon the slit of a spectroscope, the prism or grating draws out each pulse into trains of periodic waves, and maxima and minima due to the interference of the two sets of waves appear. But even without the spectroscope a number of colored fringes can be seen, and it may appear impossible to account for these on the hypothesis that we are dealing merely with a pair of pulses.

Schuster shows that the interference in this case is a physiological effect, due to a peculiarity of the eye. The retinal elements can be regarded as tuned to the three primary colors, and we are obliged to consider each element as containing some sort of a vibrating system, which responds to the light waves. If the retinal vibration has a period of its own, which seems probable, since it responds to certain wave-lengths and not to others, it is not difficult to see how interference takes place when two pulses strike the element in succession. The effect on the vibrator will depend on the time elapsing between the two impacts. The first pulse starts the vibration, and the second increases or annuls it according to the state of the vibration when the second pulse arrives. The phenomenon is thus seen to depend upon the fact that a periodic disturbance is set up in the eye, which lasts until the second pulse arrives. Schuster explains the fact that the fringes can be photographed in the same way, the vibrators in this case being located in the molecules of the silver salts.

Schuster's explanation may seem a little fanciful at first sight, but the reasoning is perfectly logical, and the assumptions are not at all improbable. We have a perfectly analogous case in an experiment of Hertz with electrical waves, which may be cited in this connection. The analogy is obvious, though it does not appear to have been noticed.

Hertz found that when his electrical waves were reflected from a wall, the resonator sparked when placed at certain definite distances from the wall, while no sparks were *observed* in intermediate positions. He drew the erroneous conclusion that he was dealing with stationary waves, formed by the interference of the direct and reflected waves. His experiment appeared to indicate that there was a definite periodicity present in his electrical radiation, just as the interference fringes observed with Fresnel's mirrors appear to indicate a certain amount of periodicity in white light.

Other experimenters found, however, that the positions of maximum sparking depended not at all upon the dimensions of the vibrator or source of the radiation, but solely upon the size of the wire loop, which served as a resonator, and that the same effects would be observed if the radiation consisted of a single pulse only. The explanation of the phenomenon is found in the fact that the vibrations of the resonator persist for some time, and the intensity of its sparking depends upon the state of its vibration at the moment when the reflected pulse meets it. At the given point in space the condition of the vibration of the

resonator, when the reflected wave meets it, will obviously depend upon its period, i.e. upon the size of the wire loop. The only difference between this case and the optical one is that in the latter the pulses are travelling in the same direction, while in the former they are going in opposite directions.

The interference fringes observed by the eye or recorded by the photographic plate are thus seen to depend upon a resonance phenomenon. If we could explore the field over which the radiation from the two sources of light is spread, with some instrument not biased by resonance, no trace of the fringes should appear. The smoked strip of the bolometer is such an instrument: it absorbs all wave-lengths equally well, and is free from resonance effects, roughly speaking at least. Now, it is found that when the bolometer is used to explore the region, no recurring maxima and minima are found, the curve having the form shown in Fig. 312. There is a central maximum bordered on each side by a minimum, beyond which points the curve is practically level. The occurrence of the two minima can be explained by the distribution of energy in the spectrum.

"The fact," says Schuster, "that white light shows any objective interference (as with bolometer) without the artificial introduction of regularity is due to the prevalence of certain wave-lengths over others. Whatever regularity there is in the light is intimately connected with the distribution of intensity in the spectrum.

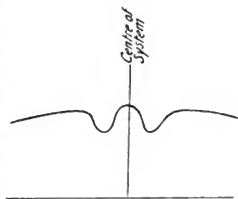


FIG. 312.

We cannot help speculating as to the ultimate cause which renders the regularity of vibration a function of the temperature only, and independent of the natural periods of the molecules. Perhaps the solution of the difficulty will be found in the fact that our observations tell us nothing directly as to the vibrations of the atoms or molecules. What we observe is the disturbance of the medium, and the distribution of energy in the spectrum of an incandescent black body, which is in thermal equilibrium, may indicate a property of the medium rather than that of matter. That is to say, the motion of the vibration in the molecule may be perfectly irregular, but the medium may take up and propagate some vibrations quicker than others. There are many signs tending to show that the time is not far distant when, in order to explain the connection between optical and electrical facts, we must recognize some structural properties of the medium, and the regularity in the radiation of a black body may be intimately connected with such structural properties."

Corbino has made the suggestion that the phenomenon of light beats obtained with white light by any of the methods originated by Righi or by himself is adverse to the hypothesis of Gouy. The different component rays (sinusoidal disturbances) into which a prism or grating decomposes the complex vibration which constitutes white light, having a common origin, ought to be capable of interfering with one another

producing beats, that is, beats should be obtained by uniting two streams of light taken from two adjacent points of a continuous spectrum. If the light from a narrow white source is divided into two streams, which are received upon the slit of a spectroscope, the spectrum is crossed by dark bands. If now the periods of one of the interfering pencils is altered, say by means of Righi's revolving Nicol arrangement, the spectrum of this pencil, according to Gouy's hypothesis, is merely slightly displaced with reference to the spectrum of the other pencil. Each sinusoidal train takes the place of its neighbor, so to speak, and should be capable of interfering with one of equal wave-length in the other set. The fringes should therefore appear exactly as before, that is stationary. If, however, the sinusoidal components do not have a common origin, *i.e.* if they originate at the source independently of one another, each set in the modified pencil can only interfere with the set in the unmodified pencil *which had the same wave-length before the modification was introduced*. These two trains now have different wave-lengths, and should therefore produce beats, or moving fringes, which is the phenomenon actually observed.

**Analysis of White Light by a Grating.**—In considering the action of a diffraction grating when analyzing white light, it will help us to get

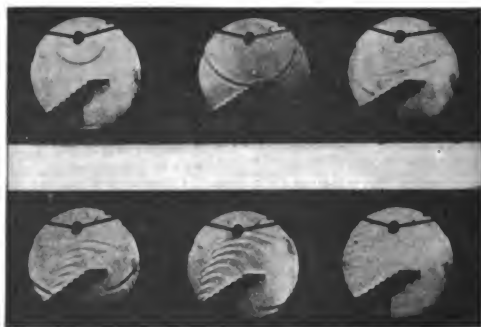


FIG. 313.

a clear idea if we bear in mind a well-known acoustical effect. If a sudden sharp noise, such as is made by clapping the hands together, is reflected from a high flight of steps, the sound comes back to us as a musical note; in other words, the steps impress the element of periodicity upon the reflected disturbance, each step throwing off an echo-wave. These reflected wavelets reach us in succession owing to the fact that the distance from our ear to the successive steps increases in arithmetical progression. The formation of a train of waves by the reflection of the sound-wave from a spark from a flight of steps is shown in Fig. 313. This is a photograph of the actual phenomenon made in the same manner as the photographs, illustrating the reflection of waves from curved surfaces, which we have studied in the chapter on

**Reflection.** The grating acts in the same manner when analyzing white light. This explanation was offered by Young in 1813, but Lord Rayleigh appears to have been the first to make use of the conception in treating grating problems. The ruled lines which constitute the grating prevent the "shadow-producing interference," as imagined by Fresnel, to account for regular reflection, and the secondary wavelets go off in all directions, instead of uniting to form a regularly-reflected wave-front.

These wavelets or impulses will pass by any given point with a periodicity depending on the location of the point. In the last section of the chapter on Diffraction this action of the grating has been explained.

**Analysis by a Prism.**—The mechanism by which a prism converts an impulsive disturbance into a periodic one is not quite as obvious as it is in the case of the grating, where we have a periodic structure.

We can, however, get a clue as to the mode of its action in the following way:

As we know, the phase of the vibration is everywhere the same on a wave-front. If this condition holds, the wave will be propagated parallel to itself, and no lateral effects will be produced. If, however, certain portions are blocked off, as by diffracting screens or gratings, lateral effects are produced, or we have deviations of a portion of the energy, which no longer obeys the laws of rectilinear propagation. The same thing results if, instead of blocking off portions at regular intervals, we change the phase of the vibration periodically: this occurs in the case of laminary gratings in which the retardation effected by the strips alters the phase of the vibration. We shall now show that a prism is capable of impressing a somewhat similar condition on the front of an impulsive disturbance.

In the section on group velocity we have seen that in a dispersing medium the group is propagated with a velocity different from that of the component waves which form it. We will now prove in a very elementary manner that, as the group proceeds, it changes its form, becomes inverted, and eventually reappears in its original form. Take first the simple type of group previously considered, formed by two trains of waves of slightly different wave-lengths.

The two trains are shown in the lower part of Fig. 314, the resultant in the upper part. We will select as a given form of our group the shape which it has at the moment when it has its maximum amplitude above the line of equilibrium: this is the condition shown in the diagram, the maximum amplitude being at 1. The two sets of component waves travel in the direction of the arrow, the shorter ones (solid lines) at the higher velocity. It is clear that the group will not be propagated without change of form, for at an instant later the two sets of waves will be nowhere exactly in step, and we shall have at no place an amplitude as great as that figured above. The waves will eventually get into step again at 2, and we shall again have our maximum amplitude above, and the original form of the group restored. Notice, however, that before this event occurs the waves will be exactly together at the trough immediately to the right of *B*, and we shall have the same maximum amplitude, only in this case it

will be below the line. The form of the group will be the same as before, only it will be *inverted*. This inversion of the form of the group before its re-establishment is of fundamental importance in

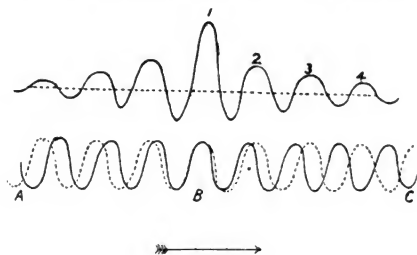


FIG. 314.

considering the action of dispersing media upon white light. If we take less simple groups made up of a large number of component trains, the original form will reappear at stated intervals only in special cases.

If the dispersion is represented by the formula

$$V = a + b\lambda,$$

$\frac{dV}{d\lambda}$  is a constant, and the group-velocity is independent of the wave-

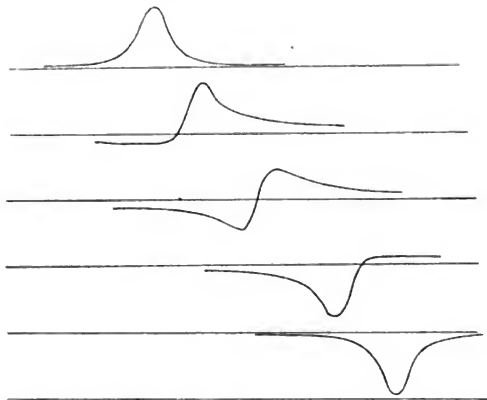


FIG. 315.

length. Our group may have the form shown in Fig. 315, which is taken from Schuster's last paper on the subject (Boltzmann's *Festschrift*, page 569, 1904). The form of this group is given by the equation

$$y = \frac{h^2}{h^2 + x^2},$$

and its successive appearances as it advances through a medium of dispersion  $V = a + b\lambda$  are shown.

As a matter of fact the dispersion formula assumed is not possessed by any known medium, but the problem is simplified by the assumption of a medium of this nature.

It is clear that if the hypothetical medium is formed into a prism, the pulse will leave the second surface with a periodicity impressed upon it, that is to say at certain points it will emerge in its original form, and in other places in its inverted form.

We will now go back to our original simple case of a group formed from two infinite trains of waves, and show that the periodicity impressed upon the group-front will produce a periodic disturbance at the focus of the telescope pointed in a direction parallel to that in which the group is advancing, identical with the periodicity of the component trains. I am indebted to my friend Prof. Ames for the following very simple and easily-intelligible treatment:

"Let us consider the action of a prism upon such a group, and for the sake of simplicity let the group have a plane front and fall perpendicularly upon the face of the prism (Fig. 316). We may choose

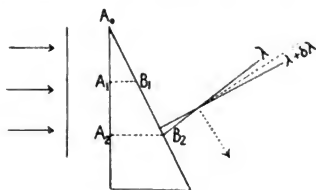


FIG. 316.

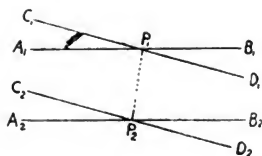


FIG. 317.

any feature of the group by which to recognize it, and note its periodic recurrence, *e.g.* the condition marked by the sum of the two amplitudes of the component trains. As the group advances towards the prism, this 'crest' moves forward with the velocity  $V_c$ , that of waves in the pure ether; when the group enters the prism, it changes its form, the 'crest' recurring at intervals equal to  $X$ ; consequently at certain points  $B_1, B_2$ , etc., on the second face of the prism, such that  $A_1B_1 = X$ ,  $A_2B_2 = 2X$ , etc., the 'crest' will emerge. Thus the vertex  $A_0$ , and the points  $B_1, B_2$ , etc., may serve as centres of secondary disturbances in a Huygens's construction, and a plane drawn tangent to these secondary spheres may be called the 'group-front.' It is apparent, however, that in the time  $T$  required for the 'crest' to reappear at  $B_1$ , after disappearance at  $A_1$ , the component *trains of waves* have advanced a greater distance than  $A_1B_1$ , and have emerged from the prism and passed on as two separate trains in slightly different directions, owing to their different indices of refraction.

"There is thus a periodicity in the group-front, due to the fact that at certain regularly spaced intervals there is the maximum amplitude. This is caused obviously by the superposition of the two crests of the component *trains of waves*, whose wave-fronts cross at a small angle.



group-fronts will be differently refracted, both on entering and on emerging; (2) since the distance required for a certain feature of a group to reappear, *i.e.* the length  $\bar{\lambda}$ , is different for the different groups, they will recur at different intervals, and therefore the complex group itself could not reappear. These complications might be avoided if a dispersive medium could be found for which

$$V - \lambda \frac{dV}{d\lambda} \quad \text{and} \quad \left( V - \lambda \frac{dV}{d\lambda} \right) \frac{1}{\frac{dV}{d\lambda}}$$

are both constant. These conditions are satisfied if the dispersive formula for the medium obeys the relation  $V = A + B\lambda$ , where  $A$  and  $B$  are constants; for, in this case, the group velocity is  $A$ , and the periodic distance  $\bar{\lambda}$  is  $A/B$ ; both of which are independent of  $\lambda$ , and therefore the same for all the component simple groups.

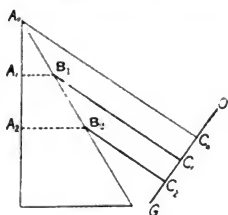


FIG. 319.

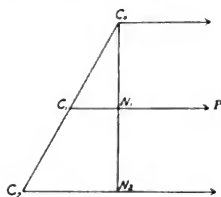


FIG. 320.

“To avoid any refraction of the *wave-fronts* of the ultimate trains of waves on entrance into a prism, we may, as before, consider normal incidence. Then, again, we will have what may be called a ‘group-front,’ for the emerging light, by drawing a plane tangent to secondary spherical disturbances, having  $A_0, B_1, B_2$ , etc., as centres, where  $A_1B_1 = X, A_2B_2 = 2X$ , etc. (Fig. 319). Let the trace of this plane be  $OG$ . It will contain periodicities, for the conditions are the same at  $C_0, C_1, C_2$ , etc.—the points of tangency. As is seen by considering the complex group made up of simple ones, the condition at these points is due to a superposition of trains of waves, and as these advance, the different component simple groups separate out and give rise to different periodicities proceeding in different directions. We may trace these in the following manner (Fig. 320): Let  $C_0C_1C_2 \dots$  be the ‘group-front,’ then the effects propagated in the direction  $C_1P$ —which is taken at random—have the periodicity  $C_1N_1$ , where the line  $C_0N_1N_2 \dots$  is drawn perpendicular to the direction  $C_1P$ ; for  $C_2N_2 = 2C_1N_1$ , etc. We will prove that this periodic distance  $C_1N_1$  is equal to  $\lambda_e$ , where this is the wave-length of the *train of waves* which, after normal incidence on the prism, would on emergence have the *wave-front*  $C_0N_1N_2 \dots$ . The difference in time required for the group-front and the train of waves to traverse the prism along the line  $A_1B_1$  is

$$X \left( \frac{1}{u} - \frac{1}{\bar{V}} \right) \quad \text{or} \quad X \frac{V - u}{u\bar{V}},$$

which, as proved above, equals  $\frac{\lambda}{V}$ , where  $\lambda$  is the wave-length of the train of waves while in the prism. Hence the distance of the wave-front in advance of the group-front, after emergence, along the line  $C_1P$  is  $\frac{V_e\lambda}{V}$  or  $\mu\lambda$ , which equals  $\lambda_e$ . That is, the distance  $C_1N_1$  equals  $\lambda_e$ .

"It is thus seen that if a telescope is pointed in different directions towards the prism, disturbances of different periodicities will be brought to a focus; and further, that the periodicity corresponding to any one direction is exactly that of the train of waves which would be brought to a focus if this train had been incident upon the prism instead of the group. In other words, a complex group gives rise, through the agency of the prism, to periodic effects advancing in different directions, which are identical—with an important limitation to be noted presently—with the effects which could have been produced if a complex train of waves had been incident upon the prism. Accordingly, the fact that a prism produces approximately homogeneous trains of waves when white light falls upon it, is not a proof of the existence in the white light of periodic component trains of waves.

"The 'resolving power' of the prism is evidently proportional to the number of periodicities which occur in the emergent 'group-front,' and if  $\Delta$  is the thickness of the base of the prism, this number equals  $\frac{\Delta}{X}$  or  $\frac{\Delta}{u} \frac{dV}{d\lambda}$ . This limits then the periodic nature of the resolved components.

"In thus explaining how an arbitrary group or pulse may, by means of a prism, produce what to a certain extent may be called trains of waves, a particular kind of dispersive medium has been considered. This is, however, no limitation upon the argument, as Schuster notes, since for a series of simple groups associated with the wave lengths not far removed from any definite value  $\lambda$ , the quantities  $u$  and  $X$  may be considered to have the same values, and so any arbitrary group may be treated as made up of these series of simple groups."

As a matter of fact it is still an open question whether a single luminous pulse would be spread out into a spectrum by a prism or even refracted at all. Dispersion involving resonance of some sort, it is quite possible that the action of a prism upon the first few waves of the train, before the vibration of the electron has been established, may be quite different from the action after a steady state has been established. An experiment to detect such a possible action was carried out a number of years ago at Professor Rowland's suggestion, but the results were negative. The question is being attacked along a different line by the author, but the results up to the present time are also negative. Upon the whole it is perhaps best to assume a certain amount of periodicity in white light. Not in the manner assumed by Newton, nor of the degree supposed to be indicated by interference experiments, but sufficient at least to establish a vibration of the electron.

## CHAPTER XXII.

### THE RELATIVE MOTION OF ETHER AND MATTER.

**Aberration of Light.**—The discovery was made by Bradley in 1728, that the apparent direction of the stars was modified by the motion of the earth through space. To understand just how this results, let us take the case of a gun on shore which has sent its projectile through the hull of a ship. If the ship is at rest, the position of the gun could be determined by sighting through the shot-holes made by the entrance and exit of the ball. If, however, the ship is moving at high speed, it will have advanced a certain distance during the time occupied by the projectile in passing through the hull, and the point of exit will be further aft than in the previous case. A line drawn through the two holes will not, in the present instance, determine the true direction of the gun, as can easily be seen by constructing a diagram. The gun's position, as determined by this method, will appear to have shifted in the direction of the ship's motion, through an angle, the tangent of which is the ratio of the ship's velocity to that of the projectile. This angle is called the angle of aberration. Consider now the case of light waves entering the object glass of a telescope. The lens transforms them into concave waves, and we will assume that the telescope is so pointed that they come to a focus on the cross-hairs of the eye-piece. If the earth were at rest, a line drawn from the point of intersection of the cross-hairs through the center of the lens would give the true direction of the star. But the earth and the telescope are in motion, and while the waves are travelling down the tube, the tube is being carried forward. The focus point will in this case fall a little behind the point at which the rays would have met if the telescope had been at rest, and if the star image is now brought upon the intersection of the cross-hairs it is clear that the telescope is pointing a little ahead of the star's true position. The amount of the shift due to the earth's motion can of course only be determined by extending the observations over an entire year; the total change in the star's position will clearly be double the true angle of aberration, for the shift is in opposite directions when the earth is on opposite sides of its orbit around the sun. The case is analogous to that of a ship steaming around in a circle, the crew of which are endeavouring to locate the position of a gun on shore by sighting through the shot-holes.

Bradley found the total angle of aberration to be 40.89 seconds of arc, or that the actual shift due to the earth's motion in its orbit was 20.44 seconds. The velocity of light in space, which was given by dividing the earth's velocity by the tangent of this angle, agreed well with the value found by Römer from observations of the eclipses of Jupiter's satellites.

The phenomenon of aberration clearly indicates that the medium which is transmitting the undulations must be at rest with respect to the telescope. If the ether in the tube were carried along with it, the point at which the waves came to a focus would be wholly uninfluenced by the motion of the tube, and there would be no aberration.

As we shall see presently, however, certain experiments indicate that the earth carries the ether along with it, a condition which cannot well be reconciled with the phenomenon which we have just considered.

It is probable, however, that the trouble is to be sought for in the theory of the experiment rather than in the theory of aberration.

**Airy's Experiment.**—The angle of aberration being determined by the ratio of the earth's velocity to the velocity of light, we should expect a change if either one of these quantities could be altered. The velocity of light down the tube of the telescope can be diminished by filling the tube with water, and we should consequently expect the angle of aberration to be increased. This experiment was tried by Airy, who found, however, that the angle was the same as when the tube was filled with air.

To explain this we may assume that the water carries the contained ether along with it, not with its full velocity, for in this case there would be no aberration, but with a velocity sufficient to compensate for the change resulting from the diminished velocity of the light. That something analogous to this dragging along of the ether actually occurs, was proved experimentally by Fizeau, and subsequently by Michelson and Morley.

**Fizeau's Experiment.**—Fizeau arranged an apparatus in which two beams of light were caused to traverse a system of tubes through which

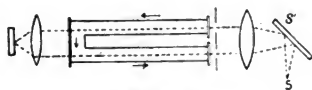


FIG. 321.

water could be forced at a high velocity. A system of interference fringes was formed by the union of the two beams, and the effect of the motion of the fluid upon the position of the fringes was studied. The arrangement of the apparatus is shown in Fig. 321. Light from the slit at *S* after reflection from a plate of glass is made parallel by a collimating lens, and divided into two portions which traverse tubes containing running water. It is clear from the diagram that each interfering beam traverses the same thickness of ponderable medium, for each ray is obliged to pass through the entire tube system. This is accomplished by focusing the rays upon a plane mirror, the effect of which is to interchange the paths. Moreover, it will be seen that one ray is travelling always with the current, the other against it. On emerging from the apparatus the rays are brought to a focus at *S'* behind the plate (a portion at *S* also), where a system of interference

fringes is formed. A shift of the fringes was observed when the water was put in motion, which could be doubled by reversing the direction of the current.

Let  $v_0$  be the velocity of light in vacuo,  $v$  the velocity in water, and  $V$  the velocity of the water. Assume that the ether is carried along by the water with a velocity  $V\theta$ , in which  $\theta$  is a fraction. The velocity of the two interfering beams will be  $v + V\theta$  and  $v - V\theta$ , and if  $l$  is the total length of the water path, the difference in time over the two paths will be:

$$\frac{l}{v - V\theta} - \frac{l}{v + V\theta}.$$

Fizeau observed a measurable displacement with a velocity of seven meters per second. The phase-difference can be determined by the shift of the fringes, from which the value of  $\theta$  in the above equation can be determined. In the case of water,  $\theta = .434$ , that is the motion of the water, apparently gives to the contained ether a velocity very nearly half as great as its own. The general expression for  $\theta$ , as developed by Fresnel, for any moving medium of refractive index  $\mu$  is

$$\theta = \frac{\mu^2 - 1}{\mu^2} = 1 - \mu^{-2}.$$

This experiment was repeated in an improved form by Michelson and Morley (*Am. Journal of Sci.*, xxxi., p. 377 (1886)). In Fizeau's arrangement the distance between the slits which divide the beam into two portions is necessarily large, and the fringes are in consequence extremely close together and require very high magnification, with its accompanying loss of light. Michelson's arrangement permitted the use of an extended source of light such as a gas flame, and any desired distance between the tubes. Light from a source at  $S$  (Fig. 322) is divided at a half-silvered surface at  $A$ , and sent around the water-tube system in opposite directions, as shown in the diagram. With tubes six meters long and a velocity of eight meters per second, the displacement observed on reversing the direction of the current amounted to less than the width of a single fringe. The results obtained were fairly concordant however, the value .434 being found for  $\theta$ . They also experimented with an air current moving with a velocity of 25 meters per second, but the effect in this case was too small to measure.

The expression for  $\theta$  which has been given above was developed by Fresnel from the following considerations: He regarded the refractive

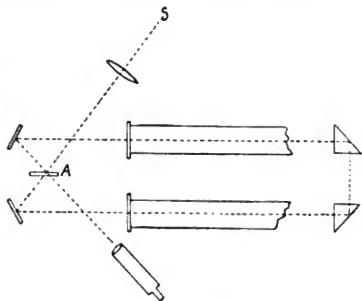


FIG. 322.

index as the square root of the ratio of the ether density in the medium to the ether density in vacuo, the refraction being due to the condensation of the ether within the pores of the medium. Consider a transparent plate, of ref. index  $\mu$ , moving with velocity  $V$ , and let  $D_1$  be the ether density within it, and  $D$  the density in vacuo. Then  $D = \mu^2 D_1$ . If there is no flow of the ether around the edges of the plate, the same amount must enter the front surface in unit-time as leaves the back surface, or  $DV = D_1(V - \theta V)$ , which gives us, if we substitute  $\mu^2 D$  for  $D_1$ ,

$$\theta = 1 - \mu^{-2} = .438 \text{ for water.}$$

This amounts to saying that the condensed ether within the plate is carried forward with a velocity such that the *excess* of the ether in the body over that in the corresponding free space is carried along with the full velocity of the plate. We may, however, regard the condensed ether as a part of the medium, in which case we can say that the ether proper is entirely uninfluenced by the motion of the medium. If  $v_0$  is the velocity of light in a vacuum, and  $\frac{v_0}{\mu}$  the velocity in a medium at rest, the absolute velocity of the light in a medium moving with a velocity  $v$  is

$$\frac{v_0}{\mu} \pm V(1 - \mu^{-2}),$$

the plus or minus sign being used according as the light travels in the same or opposite direction as that in which the medium moves. Commenting on the derivation of the above formulae, Lord Rayleigh remarks: "Whatever may be thought of the means by which it is obtained, it is not a little remarkable that this formula and no other is consistent with the facts of terrestrial refraction if we once admit that the ether in the atmosphere is at absolute rest. It is not probable that the ether in moving refracting bodies can properly be regarded as itself in motion, but if we knew more about the matter we might come to see that the objection is verbal rather than real. Perhaps the following illustration may assist the imagination:

"Compare the ether in vacuum to a stretched string, the transverse vibrations of which represent light. If the string is loaded (say with beads) the velocity of propagation is diminished. This represents the passage of light through stationary refracting media. If now the loads be imagined to run along the string with a velocity not insensible in comparison with that of the waves, the velocity of the latter is modified. It appears that the suggested model would lead to a somewhat different law of velocity from that of Fresnel; but in bringing it forward the object is merely to show that we need not interpret Fresnel's language too literally."

**Retardation by a Moving Plate.**—Let us now investigate, following Lord Rayleigh, the effect of the motion of a plate upon the retardation which it exerts upon light waves passing through in the same (or in the opposite) direction. Let the velocity of the plate be designated as before by  $V$ , its thickness by  $d$ , and its refractive index by  $\mu$ . If the velocity of the ether within the plate is  $\theta V$ , and the velocity of light in

vacuo is  $v_0$ , we have for the absolute velocity of the wave in the plate:

$$\frac{v_0}{\mu} + \theta V.$$

The time  $t$  occupied by the wave in traversing the plate is not found by dividing  $d$  by the velocity as given above, for during the time  $t$  the anterior face of the plate, which the wave reaches last, is carried forward a distance  $Vt$ . The velocity of the wave in the plate multiplied by the time  $t$  is equal to the thickness of the plate plus the distance through which the plate moves in time  $t$ , or

$$\left(\frac{v_0}{\mu} + \theta V\right)t = d + Vt \quad \text{or} \quad \frac{v_0 t}{d} = \frac{\mu}{1 + (\theta - 1)\frac{\mu V}{v_0}}.$$

The time  $t_0$  which would have been occupied in traversing the same distance  $d + Vt$ , had the plate been away, is given by

$$v_0 t_0 = d + Vt,$$

so that

$$\frac{v_0 t_0}{d} = 1 + \frac{\frac{\mu V}{v_0}}{1 + (\theta - 1)\frac{\mu V}{v_0}} \quad \text{or} \quad \frac{v_0(t - t_0)}{d} = \frac{\mu\left(1 - \frac{V}{v_0}\right)}{1 + (\theta - 1)\frac{\mu V}{v_0}} - 1.$$

If we substitute in this Fresnel's value of  $\theta$ , viz.  $1 - \mu^{-2}$ , neglecting as insensible the square of  $\frac{V}{v_0}$ , we find

$$v_0(t - t_0) = (\mu - 1)d\left(1 - \frac{V}{v_0}\right),$$

an equation which gives us the relative retardation between a wave passing through the plate and one passing by its side. The retardation depending upon the sign of  $\frac{V}{v_0}$ , will be altered when the direction of the light is reversed, which can be done by a simple rotation of the apparatus through  $180^\circ$ . If, however, we employ a terrestrial source of light, such as a sodium flame, we must take into account the fact that the source is in motion, and that the waves are consequently shortened or lengthened by Doppler's principle.

If  $V$  is the velocity of the source, the wave-length is changed from  $\lambda$  to  $\lambda\left(1 - \frac{V}{v_0}\right)$  on the side of the source towards which it is moving and to  $\lambda\left(1 + \frac{V}{v_0}\right)$  on the opposite side. We thus see that if we measure the retardation in the above equation in wave-lengths, as we are obliged to do in all experiments, it is independent of  $V$ , that is, no displacement of the fringes is to be expected on rotating the apparatus through  $180^\circ$ . An experiment was devised by Hock in 1869, in which the part of the retardation independent of  $V$  was eliminated. Two beams of light

were passed, the one through a refracting plate, the other through the air: they were then brought to a focus on a mirror, as in Fizeau's experiment, by which the paths were interchanged. It would appear at first sight as if an effect of the motion of the plate should be observed in this case, but Lord Rayleigh shows that if the change in wave-length which occurs at reflection from a moving mirror is taken into account, no results are to be expected.

**The Michelson-Morley Experiment.**—Attempts have been made by Michelson and Morley to detect effects resulting from the relative motion of the earth and the ether. These effects depend upon the square of the ratio of the velocity of the earth in its orbit to the velocity of light, a term which can be neglected in all experiments involving such small velocities as occur in experiments such as that of Fizeau. The theory of these celebrated experiments, about which so much discussion has occurred, is as follows (Michelson and Morley, *Phil. Mag.*, xxiv., page 449):

Consider a system of interference fringes formed by a Michelson interferometer, the three mirrors of which occupy the positions  $A, B, C$  (Fig. 323) at the moment when the incident beam  $SA$  strikes the first plate. While the light is travelling from the mirror  $A$  to the

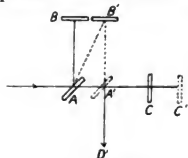


FIG. 323.

mirrors  $B$  and  $C$  and back again to  $A$ , assume the whole apparatus carried forward by the earth in the direction of the incident light to the position  $A'B'C'$ . The ray reflected from  $B$ , which interferes with a given ray reflected from  $C$ , along the line  $A'D$ , we must consider as travelling along  $AB'A'$ , the angle  $BAB'$  being equal to the angle of aberration. It must not be thought, however, that the path of the reflected ray is altered by the motion of the mirror. The change of path merely indicates that the ray which we are utilizing, and which strikes the mirror  $A$  in its second position at the point where the ray  $BA$  would have met it had the apparatus been at rest, is a ray reflected at the angle indicated. The mirror  $B$ , at the moment when reflection occurs at its surface, has only moved one half of the distance between  $A$  and  $A'$ , from which it follows that the angle  $BAB'$  is equal to the angle of aberration, the tangent of which is the ratio of  $\frac{BB'}{2}$  to  $AB$ .

Suppose that the ether remains absolutely at rest, and let  $v$  = the velocity of light, and  $u$  = the velocity of the apparatus, *i.e.* of the earth in its orbit.

Further, let  $T$  = the time occupied by the ray in passing from  $A$  to  $C$ , and  $T'$  = the time in returning from  $C$  to  $A'$ . At the moment of reflection from  $C$  the mirror will occupy a position midway between  $C$  and  $C'$ .

Call  $D$  the distance  $AB$  or  $AC$ , then

$$vT = D + uT,$$

so that

$$T = \frac{D}{v - u}, \quad T' = \frac{D}{v + u},$$

the whole time being given by

$$T + T' = 2D \frac{v}{v^2 - u^2}.$$

The distance traversed in this time is obviously given by multiplying the time of transit by the velocity  $v$ , or

$$\text{Path} = 2D \frac{v^2}{v^2 - u^2} = 2D \left( 1 + \frac{u^2}{v^2} \right),$$

which we obtain by simple division, neglecting  $\frac{u^4}{v^4}$  and terms of higher order.

The length of the other path  $ABA'$  is

$$2D \left( 1 + \frac{u^2}{v^2} \right)^{\frac{1}{2}} = 2D \left( 1 + \frac{u^2}{2v^2} \right) \text{ (approx.)}, \text{ since } \frac{AA'}{AB} = \frac{2u}{v}.$$

It is thus seen that the effect of the motion of the apparatus is to slightly increase both paths, the increment being much greater, however, along the path parallel to the earth's motion. The path-difference which was originally zero is now

$$2D \left( 1 + \frac{u^2}{v^2} \right) - 2D \left( 1 + \frac{u^2}{2v^2} \right) = D \frac{u^2}{v^2}.$$

If now we rotate the whole apparatus through  $90^\circ$ , the path  $ABA'$  will be the one which receives the larger increment, and a shift in the position of the fringes should result.

In the first experiments tried, the expected shift only amounted to about  $\frac{1}{20}$  of the distance between the fringes; moreover, it was found impossible to rotate the apparatus without introducing strains, which caused slight changes in the position of the fringes. As a result no very definite conclusions could be drawn from the observations. The experiment was then repeated with improved apparatus. By means of multiple reflections the path  $D$  was increased to 11 meters. The mirrors, 16 in number, were mounted on a heavy slab of stone which was floated on mercury. The apparatus was kept in slow rotation while the observations were taken, which did away with the strains which always occurred when it was brought to rest. A diagram of the apparatus is shown in Fig. 324, the number of mirrors having been reduced by one half however. The beam of light is divided at the half-silvered plate  $A$ . The compensating plate is located at  $B$ .

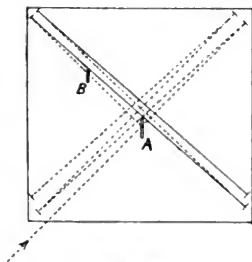


FIG. 324.

The value of  $\frac{u^2}{v^2}$  is  $10^{-8}$ , if the earth's orbital motion is alone considered, while  $D$  measured in wave-lengths of sodium light was  $2 \times 10^7$ . If the ether remains at rest relatively to the earth, we should expect a displacement of the fringes equal to

$$4 \times 10^7 \times 10^{-8} = .4 \text{ of a fringe width.}$$

The actual displacement observed was certainly less than  $\frac{1}{16}$  of the expected, and probably less than  $\frac{1}{40}$ , from which the conclusion was drawn that since the displacement is proportional to the square of the velocity, the relative velocity of the earth and the ether is probably less than one-sixth of the earth's orbital velocity. This amounts to saying that the earth drags the ether in its vicinity along with it, a circumstance which cannot be reconciled with the phenomenon of stellar aberration, to account for which we must assume the ether at rest with respect to the earth.

An explanation of the absence of any fringe-shift in the experiment of Michelson and Morley was suggested simultaneously by Fitzgerald and Lorentz. This explanation was based upon an assumed change in the linear dimensions of matter resulting from its motion through the ether; a contraction of the base upon which the mirrors are supported occurring in the direction of motion would compensate for the increment of optical path due to the motion of the apparatus.

Attempts have been made to detect this hypothetical effect, but thus far all have been unsuccessful. If the effect occurs it might very well happen that its magnitude would vary with different materials. Morley and Miller therefore repeated the experiment under conditions such that the distance between the mirrors could be made to depend upon the length either of a metal rod or a pine stick. The result was the same in each case however.

It occurred to Lord Rayleigh that the contraction in the direction of motion, if it existed, might give rise to double refraction, but he was unable to detect anything of the kind, and a subsequent experiment by Brace, performed with the greatest care, has established conclusively that no trace of double refraction occurs as a result of the motion of transparent media through the ether.

**Lodge's Experiment.**—The experiment of Michelson and Morley indicating that the earth drags the adjacent ether along with it, it occurred to Lodge<sup>1</sup> to investigate directly the effect of moving matter upon the ether. Two steel discs were mounted side by side and close together upon a common axle, and two interfering beams of light were passed in opposite directions around the annular space between the discs, by means of a system of mirrors. The discs could be rotated at a high speed, and if they dragged the ether wholly or in part the effect would be noticeable in a shift of the interference fringes, since one beam of light is travelling in the direction of rotation, the other in the opposite direction. No effect was observed at even the highest possible speeds. Thinking that perhaps the mass of the moving matter entered as a factor, Lodge substituted for the discs an immense spheroid of iron weighing half a ton, provided with a narrow circular crevasse along its equator, around which the luminous beams were reflected. The spheroid could be magnetized by means of a coil of wire, since it appeared possible that magnetization of the moving medium might have some effect. As in the previous case, the results were all negative, proving that at such speeds as can be handled in the laboratory the ether remains practically at rest. This is in agree-

<sup>1</sup> *Phil. Mag.*

ment with all of the other experiments except the one performed by Michelson and Morley.

**Influence of the Earth's Motion on Rotatory Polarization.**—Lorentz developed a formula which apparently indicated that a change of one part in ten thousand in the rotation of the plane of polarization by active substances such as quartz was to be expected when the polarimeter, set parallel to the earth's orbit, was turned through  $180^\circ$ . Larmor, in his *Aether and Matter*, criticised this result, and concluded that no effect was to be expected. Lorentz replied, defending his position (*Proc. Amsterdam Acad.*, May 1902), and maintained that Larmor was in error. The subject was then attacked experimentally by Lord Rayleigh (*Phil. Mag.*, 4, page 215, 1902), who found that the change, if it occurred, was less than  $\frac{1}{1000000}$  part of the total rotation. He used the five quartz blocks which had been prepared for Tait's rotation spectroscope, each block 5 cms. thick, the battery producing a rotation of over 4000 degrees for sodium light. As the difference in the rotations for  $D_1$  and  $D_2$  amounted to  $11^\circ$ , it was impossible to secure complete extinction with sodium light, and the helium tube was consequently employed, which gave an abundance of yellow monochromatic light. The apparatus was mounted on a horizontal stand, which could be rotated on a pivot. No change whatever was observed, however, which was in agreement with the predictions of Larmor. Still more recently Brace (*Phil. Mag.*, Sept. 1905) has shown that any change must be less than  $\frac{1}{100000000}$  of the whole.

**Effect of the Earth's Motion upon the Intensity of Terrestrial Sources.**—Fizeau came to the conclusion that if the ether was at rest with respect to the earth, the earth's orbital motion ought to affect the intensity of the light emitted by terrestrial sources, the light emitted in the direction of the earth's motion being less intense than that emitted in the opposite direction. This conclusion was reached by the following reasoning: Let  $A$  be a lamp (Fig. 325),  $B$

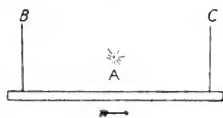


FIG. 325.

and  $C$  two screens, upon which the light falls, the whole apparatus moving with the earth through a stationary ether in the direction of the arrow, with a velocity  $v$ . Let the screens be at distance  $s$  from the lamp. To reach screen  $B$  the light has to traverse the distance  $s$ , but the slightly smaller distance  $\frac{sc}{c+v}$  (in which  $c$  is the velocity of light)

since the screen is approaching the source during the passage of the light. To reach the screen  $C$  the distance traversed is  $\frac{sc}{c-v}$ . The in-

tensity being inversely as the square of the distance, it follows that if  $J_0$  is the intensity at each screen when the apparatus is at rest, or turned perpendicular to the direction of the earth's motion, the intensity when the apparatus is flying through the ether in the direction indicated will be

$$J = J_0 \left( 1 \pm \frac{2v}{c} \right) = J_0 \left( 1 \pm \frac{1}{5000} \right).$$

Fizeau proposed measuring the total intensity of the radiation at two points equidistant from a lamp by means of a pair of thermo-elements opposed to each other, so as to secure compensation and no current. On rotating the apparatus through  $180^\circ$ , a feeble current would result from the slight change of temperature at the two points, due to the exchange of the slightly different intensities of radiation. Strangely enough, the experiment was not tried until 1903, when Nordmeyer (*Ann. der Physik*, 11, 284, 1903), working in Kayser's laboratory at Bonn, conducted a very careful series of experiments embodying the ideas of Fizeau.

The question was carefully considered from a theoretical standpoint by Bucherer working in the same laboratory, his results appearing in a paper immediately preceding the one referred to above.

There are a number of points which must be carefully considered. A change of amplitude results from the difference of path, as shown above; in addition to this we have, however, a change in the wave-length of the emitted light due to Doppler's principle, and, as we have seen in the chapter on Radiation Laws, a decrease of wave-length, the amplitude remaining the same, means an increase in the energy in unit volume of the medium. Moreover, if the intensity is measured by the thermopile, it is represented by the energy absorbed by the blackened surface of this instrument. If the instrument is moving against the light-waves, work must be done owing to the pressure of the radiation, the equivalent of this work appearing as heat in the body receiving the radiation. Bucherer considers all these questions carefully, and comes to the conclusion that Maxwell's theory leads to the conclusion

that the intensity is increased by the amount  $1 + \frac{2v}{c}$ , or in the same amount as found by the more elementary treatment given above. This is a first order effect, while all other experiments pertaining to the relative motion of the earth and the ether depend upon  $\frac{v^2}{c^2}$ , or on

second order effects. Taking into account the pressure of radiation, however, and calculating the amount of energy absorbed by a moving screen, he finds that the common motion of the screen and source is without influence upon the intensity as measured by the heating of the absorbing surface. This result is obtained by first calculating the amount of energy absorbed by a screen moving in the direction in which the light is travelling. The relative velocity is then  $c - v$ , and

the energy absorbed will be the  $\frac{c-v}{c}$  part of that which would be absorbed if the body were at rest. From this is to be subtracted the heat equivalent of the work done by the pressure of the radiation, which is equal to the product of the pressure and the distance moved in unit time. This work can only be done at the expense of the absorbed energy.

Lorentz has also shown in a different manner that the earth's motion is without influence upon terrestrial sources of light.

The experimental investigation was conducted by Nordmeyer in the manner proposed originally by Fizeau, but with all the refinements employed at the present time. The results were purely negative, and

it was established that the intensity was not changed by one part in 300,000 by the rotation of the apparatus, thus confirming the conclusions arrived at by Lorentz and Bucherer.

The experiment cannot, however, be regarded as proving that the ether is at rest with respect to the earth, since the same results are to be expected on the hypothesis of a moving ether. It is worthy of remark, however, that every experiment, with the exception of the one performed by Michelson and Morley, is in accord with the hypothesis of a stationary ether.

**Optics of Moving Media: Lorentz's Treatment.**—A very elegant treatment of the whole subject of the optical properties of moving media has been given by H. A. Lorentz. The assumption is made that in all cases the ether remains absolutely at rest, and all of the phenomena which we have had under discussion are explained by considering the system of electrons as having a motion of translation.

In the treatment of dispersion we have considered the components of current density as made up of two parts, the displacement current in the ether and the convection current of the vibrating electrons, representing the components of current density equations of the form

$$(1) \quad 4\pi j_x = \frac{\partial X}{\partial t} + 4\pi e N \frac{d\xi}{dt},$$

in which  $\xi$  represents the displacement of the electron from its position of equilibrium within the body.

If now the whole system of electrons be set in motion of translation with a constant velocity, of which the components are  $v_x, v_y, v_z$ , the above equation must be modified by the addition of a term representing the convection current due to the motion of translation.

$$(2) \quad 4\pi j_x = \frac{\partial X}{\partial t} + 4\pi e N \frac{d\xi}{dt} + 4\pi e N v_x.$$

In this equation  $\frac{\partial}{\partial t}$  indicates a change with respect to time *at a definite point in space*, while  $\frac{d}{dt}$  indicates a change with time at a definite point *within the body*.

During the time element  $dt$  (in the formation of the differential quotient  $\frac{d}{dt}$ ) the point in question moves a distance, the projections of which on the axes of coordinates are  $v_x dt, v_y dt$ , and  $v_z dt$ .

If  $x, y, z$  are referred to a stationary system of coordinates, we must write for a differentiation with respect to time as expressed by  $\frac{d}{dt}$  (referred to a fixed point in the body),

$$(3) \quad \frac{d}{dt} = \frac{\partial}{\partial t} + v_x \frac{\partial}{\partial x} + v_y \frac{\partial}{\partial y} + v_z \frac{\partial}{\partial z},$$

since the quantity to be differentiated changes by the amount  $v_x dt \frac{\partial}{\partial x}, v_y dt \frac{\partial}{\partial y}, v_z dt \frac{\partial}{\partial z}$  as a result of the motion of the body.

The components of the magnetic current density are represented as before by

$$(4) \quad 4\pi s_z = \frac{\partial a}{\partial t}, \text{ etc.}$$

Our previous equation of motion of the electron was

$$m \frac{\partial^2 \xi}{\partial t^2} + re^2 \frac{\partial \xi}{\partial t} + \frac{4\pi e^2}{\theta} \xi = eX.$$

In the present case the electrons are moving forward as a system, and can be regarded as constituting an electric current with components  $ev_x, ev_y, ev_z$ , on which the magnetic forces of the light waves will act.

Our equation of motion must therefore be modified in a manner analogous to that employed in the treatment of magnetic rotation by the hypothesis of the Hall effect, except that in that case the magnetic forces acting were those of the steady field due to the magnet.

Our equation now takes the form

$$(5) \quad m \frac{\partial^2 \xi}{\partial t^2} + re^2 \frac{\partial \xi}{\partial t} + 4\pi \frac{e^2}{\theta} \xi = eX + \frac{e}{c}(v_x \gamma - v_y \beta).$$

For periodic changes we write as before

$$\frac{d\xi}{dt} = \frac{i}{\tau} \xi, \quad \frac{d^2 \xi}{dt^2} = -\frac{1}{\tau^2} \xi,$$

in which  $\tau' = \frac{T'}{2\pi}$  refers to the period with respect to the moving body and  $\tau$  with respect to a fixed point in space.

In the case of a moving body the periodicity of the vibration with respect to the body will be different from the periodicity with respect to a fixed point in space. If the medium is moving in the direction in which the light is travelling, it is obvious that the waves will pass more slowly by a given point in the medium than a given point in space; in other words, the periodic time will be greater.

Call  $T'$  and  $T$  the periodic times with respect to the moving medium and a fixed coordinate system. We have then  $\tau' = \frac{T'}{2\pi}$  and  $\tau = \frac{T}{2\pi}$ .

Let us now determine the ratio of the two periodicities in terms of the velocity of the light and the velocity of the moving medium.

If  $\omega$  is the velocity of the light in the moving body with reference to a fixed coordinate system, *i.e.* its absolute velocity,  $v_n$  the velocity of the body in the direction of the normal to the wave-front, and  $\lambda$  the wave-length in the moving medium, the periodicity  $T$  with respect to the fixed point in space is  $T' = \frac{\lambda}{\omega}$ , which the periodicity with respect to a point in the medium is  $T' = \frac{\lambda}{\omega - v_n}$ , and the ratio

$$(6) \quad \frac{\tau'}{\tau} = \frac{T'}{T} = \frac{\omega}{\omega - v_n} = 1 + \frac{v_n}{\omega}.$$

since  $v_n$  is small in comparison to  $\omega$ , and can be neglected in the denominator of  $\frac{v_n}{\omega - v_n}$ .

Writing as before (see chapter on Dispersion Theory, page 334),

$$a = \frac{r\theta}{4\pi}, \quad b = \frac{m\theta}{4\pi e^2},$$

we obtain from equation (5),

$$4\pi e\xi \left(1 + i\frac{a}{\tau} - \frac{b}{\tau^2}\right) = \theta \left(X + \frac{v_x\gamma - v_z\beta}{c}\right),$$

an equation analogous to equation (3), page 334.

From this equation is developed by a somewhat lengthy series of transformations (see Drude's *Optik*, pages 423-427) the expression

$$\frac{n^2}{c^2} = \frac{\partial^2 X}{\partial t^2} + 2 \frac{n^2 - 1}{c^2} \frac{\partial}{\partial t} \left( v_x \frac{\partial X}{\partial x} + v_y \frac{\partial X}{\partial y} + v_z \frac{\partial X}{\partial z} \right) = \Delta r.$$

In the process it appears that  $\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y} + \frac{\partial Z}{\partial z}$ , which, in previous cases, has been shown to be equal to zero, is in the present case equal to  $\frac{n^2 - 1}{c^2} \frac{\partial}{\partial t} (v_x X + v_y Y + v_z Z)$ , which indicates that in a moving medium the electric force is no longer propagated in plane transverse waves; in other words, that the electric force is no longer in the wave-front.

It has seemed best to omit the steps by which the above equation is deduced, and simply state the final result. The two modifications which have been introduced into the equations applied to matter at rest, to make them conform to moving matter, should be clear in our minds however.

First, the velocity of translation of the electrons is now sufficiently great to cause them to be acted upon by the magnetic field of the light waves. Were this the case in matter at rest, we should have the refractive index a function of the intensity of the light, as we have seen in the chapter on Magneto-Optics.

Secondly, the change in the period of the light with respect to the moving matter due to Doppler's principle is taken into account.

We can now deduce the velocity of light in a moving medium, such as the water in the tube system in Fizeau's apparatus.

**Velocity of Light in Moving Medium.**—Writing

$$X = Ae^{i \left( t - \frac{p_1 x + p_2 y + p_3 z}{\omega} \right)}$$

and differentiating  $X$  twice with respect to  $t$  and once with respect to  $x$ ,  $y$ , and  $z$ , and substituting in the final equation given above, gives us

$$\frac{n^2}{c^2} - \frac{2(n^2 - 1)}{c^2} \frac{p_1 v_x + p_2 v_y + p_3 v_z}{\omega} = \frac{1}{\omega^2}, \quad \text{or} \quad \frac{n^2}{c^2} \left( 1 - \frac{2n^2 - 1}{n^2} \frac{v_n}{\omega} \right) = \frac{1}{\omega^2},$$

in which  $v_n$  is the velocity of the medium in the direction of the normal to the wave-front.

From the above equation we get

$$\omega^2 = \frac{c^2 n^2 \omega}{n^4 \omega - 2n^4 v_n - n^2 v_n} = \frac{c^2}{n^2} \left( \frac{n^2 \omega}{n^2 \omega - 2n^2 v_n - v_n} \right),$$

which by simple division of the last fraction gives

$$(7) \quad \omega^2 = \frac{c^2}{n^2} \left( 1 + \frac{2(n^2 - 1)v_n}{n^2 \omega} \right),$$

if we neglect  $2n^2 v$  and  $v$  in the denominator of the remainder, which we can do since they are small in comparison to the first term  $n^2 \omega$ . This equation gives us for  $\omega$ , the velocity in the moving medium, with reference to a fixed system of coordinates,

$$\omega = \frac{c}{n} \left( 1 + \frac{n^2 - 1}{n^2} \cdot \frac{v_n}{\omega} \right),$$

the second term of which can be freed from  $\omega$  if we substitute for it its approximate value  $\frac{c}{n}$ , which is justifiable, since  $\frac{v_n}{\omega}$  is a very small fraction. We find thus

$$(8) \quad \omega = \frac{c}{n} + \frac{n^2 - 1}{n^2} v_n,$$

an expression identical with the one developed by Fresnel, with the exception that in the above expression  $n$  is not the refractive index for the absolute period  $T$ , but for the relative period  $T'$ . If we wish to get an expression for  $\omega$  in terms of the refractive index of the medium at rest for the absolute period  $T$ , we proceed as follows:

Referring to eq. (6), we see that

$$T' = T \left( 1 + \frac{v_n}{\omega} \right).$$

If now we designate the refractive index of the medium at rest for the absolute period  $T$  by  $\nu$ , we have

$$n = \nu + \frac{\partial \nu}{\partial T'} \cdot T' \frac{v_n}{\omega} = \nu + \frac{\partial \nu}{\partial \lambda} \lambda \frac{v_n}{\omega},$$

in which  $\lambda = cT$ , the wave-length in vacuo.

Substituting in eq. (8) gives us

$$\omega = \frac{c}{\nu} \left( 1 - \frac{\partial \nu}{\partial \lambda} \cdot \frac{\lambda}{\nu} \frac{v_n}{\omega} \right) + \frac{n^2 - 1}{n^2} v_n,$$

or if we write the approximate values  $n = \nu$ ,  $\omega = \frac{c}{\nu}$  in the members containing  $v_n$

$$\omega = \frac{c}{\nu} + v_n \left( \frac{\nu^2 - 1}{\nu^2} - \frac{\lambda}{\nu} \frac{\partial \nu}{\partial \lambda} \right).$$

In this expression  $\frac{c}{\nu}$  is the velocity of light in the medium at rest for light of absolute period  $T$ , while the second term represents the change in velocity due to the motion of the medium.

This change is slightly larger than in the previous formula, since  $\frac{\partial v}{\partial \lambda}$  is negative in normally dispersing media.

It is worthy of remark, however, that the results obtained by Michelson and Morley are better expressed by the simpler formula, which is identical with the original one of Fresnel. Experiment gave for water the value .434 for the term to be multiplied by  $v_n$ ; the first formula gives .438, the second .451.

In conclusion, it may be remarked that all experimental evidence, with the exception of the Michelson-Morley experiment, is in favor of a stagnant ether. A test of the matter, about which there could be no question as to its interpretation, would be to measure the velocity of light on the earth's surface, first in the direction of its motion through space, and then in the opposite direction, with the accuracy required to show the ether's motion if it exists.

Various methods for accomplishing this have been suggested, but all have been shown to be impracticable.

## APPENDIX A.

### SCREENS FOR SHOWING SUBJECTIVE YELLOW.

THE liquid screens alluded to on page 10 are troublesome to prepare, and not as convenient for purposes of demonstration as stained gelatine films. Moreover, the permanganate screen transmits a good deal of red light, which can be removed by the addition of a little soluble prussian blue. A good substitute for this screen can be made by combining a piece of rather dense orange glass, such as is used in dark-room lamps, with a plate coated with a film of gelatine stained with soluble prussian blue. It will be found advisable to flow a good size plate with the stained gelatine, and dry it in a slightly inclined position, so that the thickness of the film will vary. It will then be easy to pick out a portion which, when combined with the orange glass, gives a yellow matching the subjective yellow of the balsam screen described on page 10. The screens should be matched by holding them before a gas flame.

The gelatine solution is made by dissolving about 5 grams of gelatine in 100 c.c. of hot water, and adding a few drops of a strong solution of soluble prussian blue. If the dye is precipitated, as is sometimes the case, boil the solution and filter if necessary.

## APPENDIX B.

### REFLECTION OF HEAT WAVES FROM ROUGH SURFACES.

It has been shown in Chapter II. that a rough surface may regularly reflect the long waves while diffusing the shorter ones. Lord Rayleigh has made some interesting experiments upon the reflection of heat waves from ground-glass surfaces, too rough to give any trace of regular reflection with visible light. The ground surface was silvered and the radiations of Welsbach lamp reflected from it. In some cases two reflecting surfaces were used. It was found that the radiation, freed by the process from the shorter waves, was reflected almost as well by a third ground and silvered surface as by a polished silver mirror. The method is analogous to that originated by Rubens and Nichols for isolating long heat waves by repeated reflection from quartz or rock-salt surfaces (see article on "Polish" previously alluded to).

## APPENDIX C.

## THEORY OF LIPPMANN'S COLOR PHOTOGRAPHS.

THE elementary theory given at the end of the chapter on Interference is by no means complete. It accounts perfectly for the reproduction of pure spectrum colors, but does not well account for compound colors such as occur in nature. Suppose we illuminate the film with monochromatic light, say with the light of the soda flame. It is clear that if only a few reflecting laminae are formed the color of the light selectively reflected will be far from monochromatic. If the number of the laminae be increased the reflected light will become purer, as in the case of the opal (see p. 134).

If we examine the light reflected from a Lippmann photograph of the spectrum with a spectroscope we shall find that each element of it reflects light which embraces a considerable range of wave-lengths. This being the case, it seems very remarkable that the very impure colors which occur in nature are reproduced with any fidelity at all. Lippmann assumed that every elementary wave in a heterogeneous mass built up its own system of laminae, each system reflecting the color peculiar to its interval, regardless of the presence of the others. If this were the case a photograph taken of a source emitting a number of monochromatic radiations (*e.g.* a sheet of paper illuminated with a mercury arc-lamp) should reflect light showing the mercury lines when examined with a spectroscope.

Lehmann has found, however, that only two, or at most three, separate colors can be simultaneously reflected from a given spot on the film.<sup>1</sup>

He has, moreover, determined the distribution of density of the reduced silver in the film when it is illuminated by heterogeneous waves, and finds a periodic structure which, if represented by a curve, has a decreasing amplitude similar to a damped vibration. Fewer laminae are therefore formed than in the case of monochromatic illumination. Lehmann verified these conclusions by making sections of the films and examining them under the microscope. Some very interesting results were obtained in the case of films illuminated with light made up of two different wave-lengths. The laminae were found to be periodically in and out of step, the analogy with "beats" being obvious. The periodic structure produced in this way was coarse enough to be easily seen with low powers, in thin sections of the film. Lehmann has also made some interesting experiments on the production of Lippmann colors of the second and third orders, analogous to the Newton's colors. A film acted on by red light of wave-length 800 should not only reflect light of this color, but also violet light of wave-length 400. This experiment could not be made to work, owing to the strong absorbing power which the developed films have for

<sup>1</sup> H. Lehmann, *Phys. Zeit.*, i., p. 17, 1905.

violet light. The phenomenon was observed, however, by breathing on certain films, the moisture increasing the distance between the laminae to such an extent that colors of the second and sometimes the third order appeared.

An interesting theoretical treatment of Lippmann's photographs by Wiener will be found in *Wied. Annalen*, 69, p. 488, 1899.

## APPENDIX D.

### SELECTIVE EMISSION AND ABSORPTION OF INCANDESCENT SALTS.

SOME extremely interesting experiments have been recently made by Lenard (*Annalen der Physik*, 17, p. 197, 1905) with beads of the fused salts of the alkali metals supported upon platinum wires in the Bunsen flame. The emitted light was found to be strongly colored, the color depending upon the metal. The sulphates of K, Rb, and Cs emitted green light, while sodium sulphate shone with a bluish tinge. An examination of the absorption spectra of the fused salts showed that the color of the transmitted light was complementary to that of the emitted light, as should follow from Kirchhoff's law. The salts were colorless when cold, however, showing that some sort of dissociation resulted from the high temperature, metallic ions being set free which had the property of absorbing and emitting radiations of the same frequency. The failure of the ions to show color when the salts are dissolved in water is ascribed by Lenard to a loading of the ions with water. In the case of most of the salts examined the color was found to depend upon the metal, *i.e.* upon the cations: the borates and phosphates were marked exceptions, however, the color being chiefly due to the anions, the nature of the metal being immaterial. In the same paper will be found a number of very interesting conclusions regarding the emission of colored light by flames, and the probable centers from which radiate the different types of spectra.

## APPENDIX E.

### DOPPLER'S PRINCIPLE.

THE formula given on page 20 for the change in the period applies to the case of a stationary source of light and a moving observer as well as to a moving source. The two conditions are represented by the same formula only when the velocity of translation is small

in comparison to the velocity with which the waves travel, which is usually the case. The number of waves of frequency  $N$ , coming from a fixed source, which in one second pass an observer moving towards the source with a velocity  $v$ , is  $N + \frac{v}{\lambda}$  or  $\frac{V+v}{\lambda}$ , in which  $V = \text{vel. of light}$ . If, however, the observer is fixed, and the source moves with velocity  $v$ , the wave-length is changed from  $\lambda = \frac{V}{N}$  to  $\lambda = \frac{V-v}{N}$ , and the number which pass the observer per second is the velocity  $V$  divided by this number or  $\frac{V}{V-v} N$ .

A very complete treatment of the principle will be found in Kayser's *Spectroscopy*, vol. ii.

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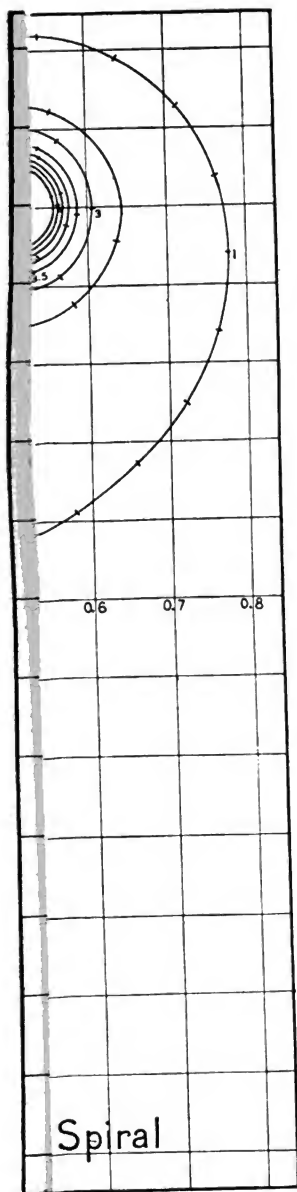
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